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Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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TOPOLOGIZATION OF HECKE PAIRS AND HECKE C^*-ALGEBRAS

HELGE GLÖCKNER AND GEORGE A. WILLIS

ABSTRACT. Let (G, S) be a Hecke pair, *i.e.*, G is a group and S an *almost normal* subgroup, meaning that every double coset SgS is the union of finitely many left cosets of S. We show that there exists a homomorphism ϕ from G to a totally disconnected, locally compact group $\widetilde{G}$ such that $\widetilde{S} := \overline{\phi(S)}$ is a compact, open subgroup of $\widetilde{G}$, and such that the Hecke algebras $\mathcal{H}(G, S)$ and $\mathcal{H}(\widetilde{G}, \widetilde{S})$ are isomorphic. This "topologization" construction is then used to solve a problem in the theory of Hecke C^*-algebras.

INTRODUCTION

A *Hecke pair* (G, S) consists of a group G and an *almost normal* subgroup S of G, meaning that each double coset SgS of S in G is a union of finitely many left cosets of S. For example, (G, S) is a Hecke pair when S is a compact, open subgroup of the topological group G. This example is effectively the general case because to each Hecke pair (G, S) may be associated a *topologised* Hecke pair $(\widetilde{G}, \widetilde{S})$ and a homomorphism $\phi : G \to \widetilde{G}$, where $\widetilde{G}$ is a topological group and $\widetilde{S}$ a compact, open subgroup, such that $\phi(G)$ is dense in $\widetilde{G}$ and $\phi(S)$ is dense in $\widetilde{S}$. (The construction of $\widetilde{G}$ is described in Section 2 and is based on work of G. Schlichting, [16].) For many purposes the topologised Hecke pair $(\widetilde{G}, \widetilde{S})$ can replace the original one. Several advantages in making this replacement are here described.

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The *Hecke algebra* associated with the Hecke pair (G,S) is the free complex vector space $\mathcal{H}(G,S)$ over the set $S\backslash G/S$ of double S-cosets equipped with a unital $*$-algebra structure which is defined in Section 1. Such Hecke algebras have been studied in [3], [5], [6], [7], [11], [13] and [17]. They generalise the Hecke algebras used in the study of representations of finite groups. Unitary representations of G containing a topologically cyclic set of S-fixed vectors correspond to representations of $\mathcal{H}(G,S)$.

The product and involution in $\mathcal{H}(G,S)$ are defined in terms of S-coset representatives and abuse of notation is required for the product to be manageable. However, $\mathcal{H}(G,S)$ is isomorphic to $\mathcal{H}(\widetilde{G},\widetilde{S})$ and this Hecke algebra is isomorphic to the subalgebra $\mathbf{1}_S * C_c(\widetilde{G}) * \mathbf{1}_S$ of the algebra $C_c(\widetilde{G})$ of continuous functions with compact support in $\widetilde{G}$ with convolution product. The Hecke algebra product thus becomes familiar under this point of view. The same approach is adopted in the thesis of K.Tzanev, [17], where also completions of $\mathcal{H}(G,S)$ under certain norms are identified with subalgebras of $L^1(\widetilde{G})$ and $C^*(\widetilde{G})$.

The thesis of R. Hall, [7], investigates the question of when $\mathcal{H}(G,S)$ has an enveloping C^*-algebra, see [15, Definition 10.1.10]. (*-algebras having an enveloping C^*-algebra are called *G*-algebras* in [15].) It is shown that under a suitable positivity condition, there is an enveloping C^*-algebra, called the *Hecke C^*-algebra* $C^*(G,S)$, and there is a category equivalence between the category of unitary representations (π,V) of G topologically generated by the subspace V^S of S-fixed vectors on the one hand, and the category of (non-degenerate) $*$-representations of $C^*(G,S)$ on the other hand (results motivated partly by studies in [1] and [2]). The topologised Hecke pair $(\widetilde{G},\widetilde{S})$ can be used to clarify some points, in particular [7, Lemma 3.24], in Hall's argument establishing the category equivalence. The present paper is based on the manuscript [6], which makes more detailed comments on Hall's thesis.

The topological group $\widetilde{G}$ appearing in the topologised Hecke pair is totally disconnected and locally compact. All such groups have a base of neighbourhoods of the identity consisting of compact open (and therefore almost normal) subgroups. There is likely therefore to be a strong interaction between the study of general Hecke algebras and the structure theory of totally disconnected, locally compact groups as developed in [18] and [19]. One sign of this potential interaction is the simple Hecke algebra proof given in Section 7 of a new result in the structure theory of totally disconnected, locally compact groups.

1. Preliminaries

Throughout this article, (G, S) denotes a Hecke pair, as defined in the Introduction.

1.1. Let $\mathbb{C}[G/S]$ denote the free complex vector space over the set G/S. As a vector space, the Hecke algebra $\mathcal{H} := \mathcal{H}(G, S)$ is the vector subspace of $\mathbb{C}[G/S]$ generated by the characteristic functions

$$[SgS] := \mathbf{1}_{SgS/S}$$

for $g \in G$, where $SgS/S := \{xS : x \in G, xS \subseteq SgS\}$. Thus $\mathcal{H} = \mathbb{C}[G/S]^S$ is the subspace of S-fixed vectors of $\mathbb{C}[G/S]$ under the natural representation

$$\lambda \colon G \to \operatorname{End}_{\mathbb{C}}(\mathbb{C}[G/S])$$

of G on $\mathbb{C}[G/S]$ by left translation, given by $g.f := \lambda(g).f = f(g^{-1}\bullet)$ for $g \in G$, $f \in \mathbb{C}[G/S]$. Note that $g.[hS] = [ghS]$ for all $g, h \in G$, where $[xS] := \mathbf{1}_{\{xS\}}$ for $x \in G$. The complex vector space $\mathcal{H}$ becomes a unital $*$-algebra with identity element $[S]$ if we define multiplication of $f, g \in \mathcal{H}$ via

$$(f * g)(x) := \sum_{\gamma \in G/S} f(\gamma)\, g(\gamma^{-1}x)$$

for $x \in G$ and an involution $* \colon \mathcal{H} \to \mathcal{H}$, $f \mapsto f^*$ via

$$f^*(x) := \overline{f(x^{-1})} \quad \text{for } x \in G;$$

see [7], Section 2.2, cf. also [10]. Here f, g, $f * g$ and f^* are considered as S-biinvariant functions in $\mathbb{C}[G]$, and summation over "$\gamma \in G/S$" is not meant literally but means that γ ranges through a set of representatives for the left cosets of S in G. Similar abuses of notation are used throughout the following. Note that $[SgS]^* = [Sg^{-1}S]$ for all $g \in G$.

1.2. The complex vector space $\mathbb{C}[G/S]$ becomes a right $\mathcal{H}$-module via

$$(f * h)(x) := \sum_{\gamma \in G/S} f(\gamma)\, h(\gamma^{-1}x) \quad (x \in G),$$

for $f \in \mathbb{C}[G/S]$, $h \in \mathcal{H}$ (with the abuse of notation just explained: γ is supposed to range over a set of representatives for the left cosets of S in G). The following formulas are useful: we have

$$[xS] * [SyS] = \sum_{\gamma \in SyS/S} [x\gamma S]$$

and

$$[SxS] * [SyS] = \sum_{\gamma \in SxS/S} \sum_{\eta \in SyS/S} [\gamma\eta S],$$

for all $x, y \in G$ ([7], p. 15). The right action of $\mathcal{H}$ commutes with the left action of G (described in **1.1**), whence $\mathbb{C}[G/S]$ is a $\mathbb{C}[G]$-$\mathcal{H}$-bimodule. In fact, $\mathcal{H}^{\mathrm{op}} \cong \mathrm{End}_G(\mathbb{C}[G/S])$ is the full algebra of G-module endomorphisms (via $\mathcal{H}^{\mathrm{op}} \ni h \mapsto (\bullet) * h)$), cf. [10], Proposition 3.9.

1.3. We need further notation. We define S-biinvariant functions

$$L\colon G \to \mathbb{N}, \quad R\colon G \to \mathbb{N}, \quad \text{and} \quad \Xi\colon G \to \mathbb{Q}^+$$

via

$$L(g) := \#(SgS/S), \quad R(g) := L(g^{-1}) = \#(S\backslash SgS), \quad \text{and} \quad \Xi(g) := \frac{R(g)}{L(g)}$$

for $g \in G$. Then $\Xi : G \to (\mathbb{Q}^+, \cdot)$ is a homomorphism of groups ([7], Theorem 2.7). Given $g \in G$, let us write $S_g := S \cap gSg^{-1}$; this is the stabilizer of $[gS]$ in S under the left action of S on $\mathbb{C}[G/S]$. Thus

$$S/S_g \to SgS/S, \quad xS_g \mapsto xgS$$

is a bijection. By the preceding, $[S : S_g] = L(g) < \infty$.

1.4. In order to link unitary representations of G and $*$-representations of the Hecke algebra, it turns out to be useful to modify the action of G on $\mathbb{C}[G/S]$ defined in **1.1**. Using the homomorphism Ξ from **1.3**, we define

$$x \diamond f := \Xi(x)^{-\frac{1}{2}} \cdot (x.f)$$

for $x \in G$, $f \in \mathbb{C}[G/S]$. Clearly $\mathbb{C}[G/S]$ is a $\mathbb{C}[G]$-$\mathcal{H}$-bimodule also with respect to the modified action $\diamond$.

Suppose that G is a totally disconnected, locally compact group and S a compact, open subgroup of G. Clearly S is almost normal, whence (G, S) is a Hecke pair. We normalize Haar measure μ on G via $\mu(S) = 1$. Then the linear mapping $\mathcal{H}(G, S) \to L^1(G)$, determined by $[SgS] \mapsto \mathbf{1}_{SgS} = L(g) \cdot \mathbf{1}_S * \delta_g * \mathbf{1}_S$ for $g \in G$, is an injective algebra homomorphism, which allows $\mathcal{H}(G, S)$ to be identified with the subalgebra of $L^1(G)$ spanned by the characteristic functions of double cosets. Furthermore, it is easily verified that $\Xi(g) = \Delta(g)^{-1}$ for $g \in G$, where Δ is the modular function on G (as defined in [8], Theorem 15.11). As we shall presently see, up to isomorphism, every Hecke algebra can be obtained from a Hecke pair of the special type just described.

2. The topologization construction

In this section, we adapt ideas by G. Schlichting (cf. [16]) to the Hecke algebra setting. We show that every Hecke algebra $\mathcal{H}(G, S)$ is isomorphic to the Hecke algebra $\mathcal{H}(\widetilde{G}, \widetilde{S})$ of some suitable locally compact, totally disconnected group $\widetilde{G}$ and compact, open subgroup $\widetilde{S} \leq \widetilde{G}$.

2.1. Let (G, S) be an arbitrary Hecke pair. We consider G/S as a discrete topological space and equip the group $\mathrm{Sym}(G/S)$ of all permutations of G/S with the compact-open topology, which makes it a topological group.[1] Then

$$H := \{\pi \in \mathrm{Sym}(G/S) \colon \pi(SgS/S) = SgS/S \text{ for all } g \in G\}$$

is a totally disconnected, compact subgroup of $\mathrm{Sym}(G/S)_{c.o.}$ in the induced topology, since the mapping

$$H \to \prod_{X \in \{SgS/S \colon g \in G\}} \mathrm{Sym}(X), \quad \pi \mapsto (\pi|_X)$$

is an isomorphism of topological groups onto a product of finite groups.

[1]By definition, a basis of identity neighbourhoods of $\mathrm{Sym}(G/S)_{c.o.}$ is given by finite intersections of stabilizers of points, *i.e.*, $\{\pi \in \mathrm{Sym}(G/S) : (\forall c \in F)\ \pi(c) = c\}$, where F ranges through the finite subsets of G/S.

2.2. The left multiplication action of G on G/S gives rise to a permutation representation

$$\Psi\colon G \to \mathrm{Sym}(G/S), \quad \Psi(g)(xS) := gxS.$$

Since S acts on G/S with finite orbits SgS/S, we have $\Psi(S) \subseteq H$. Thus

$$\widetilde{S} := \overline{\Psi(S)} \subseteq H$$

is a totally disconnected, compact group. We define

$$\widetilde{G} := \langle \widetilde{S} \cup \Psi(G) \rangle \leq \mathrm{Sym}(G/S)$$

as an abstract group.

Next, we put a suitable topology on $\widetilde{G}$.

Lemma 2.3. *There exists a unique group topology on $\widetilde{G}$ which makes $\widetilde{S}_{c.o.}$ a compact, open subgroup of $\widetilde{G}$.*[2] *Equipped with this topology, $\widetilde{G}$ is a totally disconnected, locally compact group which contains* $\mathrm{im}\,\Psi$ *as a dense subgroup. The topology on $\widetilde{G} \leq \mathrm{Sym}(G/S)$ is finer than the compact open topology* (*or equal to it*)*; therefore the natural action*

$$\widetilde{G} \times G/S \to G/S, \quad (\tilde{g}, xS) \mapsto \tilde{g}(xS)$$

is continuous.

Proof. Let $\mathcal{U}_1(\widetilde{S})$ be the filter of identity neighbourhoods of the compact group $\widetilde{S}_{c.o.}$. We claim that $\mathcal{U}_1(\widetilde{S})$ is the filter basis for the filter of identity neighbourhoods of a group topology on $\widetilde{G}$ (which clearly is uniquely determined by this property). Since $\widetilde{S}$ already is a topological group, we only need to verify the following

Claim. *For every $\tilde{g} \in \widetilde{G}$ and $U \in \mathcal{U}_1(\widetilde{S})$, there exists $V \in \mathcal{U}_1(\widetilde{S})$ such that $\tilde{g} V \tilde{g}^{-1} \subseteq U$.*

A simple inductive argument shows that the claim will hold provided that it holds for all elements $\tilde{g}$ in the symmetric generating set $\widetilde{S} \cup \mathrm{im}\Psi$ of G. For $\tilde{g} \in \widetilde{S}$, the assertion is obvious. Hence assume that $\tilde{g} = \Psi(g)$ for some $g \in G$. Clearly, we only need to prove the claim for neighbourhoods U in a subbasis (as we may form finite intersections of the corresponding V's). As $\widetilde{S}$ is equipped with the

[2] It is part of this requirement that $\widetilde{G}$ induces the compact-open topology on $\widetilde{S}$.

compact-open topology, we may therefore assume that $U = \widetilde{S}_{xS}$ is the stabilizer of some left coset xS in $\widetilde{S}$. We define V as the intersection of stabilizers

$$V := \widetilde{S}_{g^{-1}xS} \cap \widetilde{S}_{g^{-1}S}$$

and readily observe that

$$(gvg^{-1}).xS = g.(v.g^{-1}xS) = g.(g^{-1}xS) = xS$$

for all $v \in V$, *i.e.*, gvg^{-1} stabilizes xS. It remains to show that $gvg^{-1} \in \widetilde{S}$: then $gvg^{-1} \in \widetilde{S}_{xS} = U$ indeed for all $v \in V$ and thus $gVg^{-1} \subseteq U$, as required. Since $\Psi(S)$ is dense in $\widetilde{S}$ and V is open, there exists a net (s_α) in S such that $v_\alpha := \Psi(s_\alpha) \to v$ and $v_\alpha \in V$ for all α. Thus $s_\alpha \in S_{g^{-1}x} \cap S_{g^{-1}}$. We remark first that

$$(\forall \alpha) \quad \widetilde{g} v_\alpha \widetilde{g}^{-1} \in \widetilde{S} \tag{1}$$

since $\widetilde{g} v_\alpha \widetilde{g}^{-1} = \Psi(g s_\alpha g^{-1})$, where $s_\alpha \in S_{g^{-1}} = S \cap g^{-1}Sg$ and therefore $g s_\alpha g^{-1} \in gSg^{-1} \cap S \leq S$. As $\mathrm{Sym}(G/S)_{c.o.}$ is a topological group in which $\widetilde{S}$ is closed (being compact), we deduce from (1) that $\tilde{g} v \tilde{g}^{-1} = \lim \tilde{g} v_\alpha \tilde{g}^{-1} \in \widetilde{S}$, where the limit is computed in $\mathrm{Sym}(G/S)_{c.o.}$. Thus $\tilde{g} v \tilde{g}^{-1} \in \widetilde{S}$ indeed. The remainder is obvious. ☐

Theorem 2.4. *Given a Hecke pair* (G, S), *define* $\widetilde{G}$ *and* $\widetilde{S}$ *as before. Then the following holds* (*where the bar indicates closures in* $\widetilde{G}$):

(a) *The mapping*

$$\Gamma \colon G/S \to \widetilde{G}/\widetilde{S}, \quad gS \mapsto \Psi(g)\widetilde{S} = \overline{\Psi(gS)}$$

is a bijection and therefore induces an isomorphism of vector spaces

$$\Theta \colon \mathbb{C}[\widetilde{G}/\widetilde{S}] \to \mathbb{C}[G/S], \quad f \mapsto f \circ \Gamma.$$

Here

$$(\forall \tilde{g} \in \widetilde{G}) \quad \Theta(\,[\tilde{g}\widetilde{S}\,]\,) = [\tilde{g}(S)]. \tag{2}$$

(b) *The mapping*

$$S\backslash G/S \to \widetilde{S}\backslash\widetilde{G}/\widetilde{S}, \quad SgS \mapsto \widetilde{S}\Psi(g)\widetilde{S} = \overline{\Psi(SgS)}$$

is a bijection.

(c) *The mapping* Θ *maps* $\mathcal{H}(\widetilde{G},\widetilde{S})$ *onto* $\mathcal{H}(G,S)$ *and induces an isomorphism of* $*$*-algebras* $\mathcal{H}(\widetilde{G},\widetilde{S}) \to \mathcal{H}(G,S)$.

Proof. (a) Since $\widetilde{G}$ acts transitively on G/S, the mapping

$$\beta\colon \widetilde{G}/\widetilde{G}_S \to G/S, \quad \tilde{g}\,\widetilde{G}_S \mapsto \tilde{g}(S) \tag{3}$$

(where $\tilde{g} \in \widetilde{G}$) is a bijection. The stabilizer $\widetilde{G}_S$ of S in $\widetilde{G}$ is open, since the topology on $\widetilde{G}$ is finer than the compact-open topology. Hence $\widetilde{G}_S \cap \Psi(G)$ is dense in $\widetilde{G}_S$. But $\widetilde{G}_S \cap \Psi(G) = \Psi(G_S) = \Psi(S)$, a dense subset of the closed subset $\widetilde{S}$ of $\widetilde{G}$. Stabilizers being closed, we deduce that $\widetilde{G}_S = \widetilde{S}$. Then clearly $\beta \circ \Gamma = \mathrm{id}$, whence $\Gamma = \beta^{-1}$ is a bijection. Then clearly Θ is a bijection as well. Given $\tilde{g} \in \widetilde{G}$, we compute for all $x \in G$

$$\begin{aligned}\Theta([\tilde{g}\widetilde{S}])(xS) &= [\tilde{g}\widetilde{S}](\Psi(x)\widetilde{S}) = \delta_{\tilde{g}\widetilde{S},\Psi(x)\widetilde{S}} \\ &= \delta_{\tilde{g}(S),\Psi(x)(S)} = \delta_{\tilde{g}(S),xS} = [\tilde{g}(S)](xS),\end{aligned}$$

using that Γ is a bijection to obtain the third equality; here $\delta_{\cdot,\cdot}$ denotes Kronecker's delta. Thus $\Theta([\tilde{g}\widetilde{S}]) = [\tilde{g}(S)]$ indeed.

(b) Let $\tilde{g} \in \widetilde{G}$. By Part (a) of the theorem, $\tilde{g}\widetilde{S} = \Psi(g)\widetilde{S}$ for some $g \in G$, whence $\widetilde{S}\tilde{g}\widetilde{S} = \widetilde{S}\Psi(g)\widetilde{S}$. Hence the specified mapping is surjective. If $\widetilde{S}\Psi(g)\widetilde{S} = \widetilde{S}\Psi(h)\widetilde{S}$ for certain $g, h \in G$, then $\Psi(g)\widetilde{S} \cap \widetilde{S}\Psi(h) \neq \emptyset$; the latter set being open, the density of $\Psi(G)$ in $\widetilde{G}$ implies the existence of elements $k \in G$, $\tilde{x}, \tilde{y} \in \widetilde{S}$ such that $\Psi(k) = \Psi(g)\tilde{x} = \tilde{y}\,\Psi(h)$. By the preceding formula, $\tilde{x}, \tilde{y} \in \widetilde{S} \cap \Psi(G) = \Psi(S)$. Pick $x, y \in S$ such that $\tilde{x} = \Psi(x)$ and $\tilde{y} = \Psi(y)$. Then $\Psi(gx) = \Psi(yh)$ and hence $gxS = \Psi(gx)(S) = \Psi(yh)(S) = yhS$, which entails $SgS = ShS$.

(c) Let $g \in G$ and $T \subseteq G$ be a set of representatives for the left cosets of S contained in SgS. Then

$$\Psi(SgS) = \bigcup_{\gamma \in T} \Psi(\gamma)\Psi(S) \subseteq \bigcup_{\gamma \in T} \Psi(\gamma)\widetilde{S} =: C,$$

where the second union is disjoint by Part (a). Note that C contains $\Psi(SgS)$ as a dense subset and is compact, hence closed. Thus $\widetilde{S}\Psi(g)\widetilde{S} = C$ by Part (b). We deduce that, for every set T of representatives for the left cosets of S in SgS, the set $\Psi(T)$ is a set of representatives for the left cosets of $\widetilde{S}$ contained in $\widetilde{S}\Psi(g)\widetilde{S}$; the finite sets T and $\Psi(T)$ have the same cardinality. Note that $\Theta([\Psi(g)\widetilde{S}]) = [gS]$ for all $g \in G$. It therefore follows from the preceding observation that $\Theta([\widetilde{S}\Psi(g)\widetilde{S}] = [SgS]$ for all $g \in G$. Hence Θ maps $\mathcal{H}(\widetilde{G}, \widetilde{S}) \subseteq \mathbb{C}[\widetilde{G}/\widetilde{S}]$ onto $\mathcal{H}(G, S) \subseteq \mathbb{C}[G/S]$ indeed. Given $g, h \in G$, we have

$$[\widetilde{S}\Psi(g)\widetilde{S}] * [\widetilde{S}\Psi(h)\widetilde{S}] = \sum_{\gamma \in SgS/S} \sum_{\eta \in ShS/S} [\Psi(\gamma)\Psi(\eta)\widetilde{S}]$$

in $\mathcal{H}(\widetilde{G}, \widetilde{S})$, by the above observation. Thus

$$\Theta([\widetilde{S}\Psi(g)\widetilde{S}] * [\widetilde{S}\Psi(h)\widetilde{S}]) = \sum_{\gamma \in SgS/S} \sum_{\eta \in ShS/S} [\gamma\eta S] = [SgS] * [ShS].$$

We have proved that Θ restricts to an isomorphism $\mathcal{H}(\widetilde{G}, \widetilde{S}) \to \mathcal{H}(G, S)$ of complex algebras; clearly it respects the involutions. $\square$

It is shown in [17, Théorème 1.17] that $\mathcal{H}(\widetilde{G}, \widetilde{S})$ is isomorphic to $m * C_c(\widetilde{G}) * m$, where m is the normalised Haar measure on $\widetilde{S}$ and $C_c(\widetilde{G})$ has the convolution product. It is shown in the same theorem in [17] that cutting down $L^1(\widetilde{G})$ and both the full and the reduced C^*-algebras of $\widetilde{G}$ by m yields subalgebras isomorphic to natural completions of $\mathcal{H}(\widetilde{G}, \widetilde{S})$.

Extension of actions from G to $\widetilde{G}$

A calculation similar to the preceding one shows that $\Theta(f) * \Theta(h) = \Theta(f * h)$ for all $f \in \mathbb{C}[\widetilde{G}/\widetilde{S}]$ and $h \in \mathcal{H}(\widetilde{G}, \widetilde{H})$. We now make $\mathbb{C}[G/S]$ a left $\widetilde{G}$-module via

$$\tilde{g}.f := \Theta(\tilde{g}.\Theta^{-1}(f)) \tag{4}$$

for $\tilde{g} \in \widetilde{G}$, $f \in \mathbb{C}[G/S]$. Note that for each $f \in \mathbb{C}[G/S]$, $\tilde{g} \in \widetilde{G}$, and $h \in \mathcal{H}(G, S)$, we have

$$\begin{aligned}(\tilde{g}.f) * h &= \Theta(\Theta^{-1}(\tilde{g}.f) * \Theta^{-1}(h)) \\ &= \Theta((\tilde{g}.\Theta^{-1}(f)) * \Theta^{-1}(h)) = \Theta(\tilde{g}.(\Theta^{-1}(f) * \Theta^{-1}(h))) \\ &= \Theta(\tilde{g}.(\Theta^{-1}(f * h)) = \tilde{g}.(f * h).\end{aligned}$$

We have shown:

Proposition 2.5. *The action of $\widetilde{G}$ defined in Equation (4) makes $\mathbb{C}[G/S]$ a $\mathbb{C}[\widetilde{G}]$-$\mathcal{H}(G,S)$-bimodule. The same conclusion holds for the modified action defined via*

$$\tilde{g} \diamond f := \widetilde{\Xi}(\tilde{g})^{-\frac{1}{2}} \cdot (\tilde{g}.f) \tag{5}$$

for $\tilde{g} \in \widetilde{G}$, $f \in \mathbb{C}[G/S]$, where $\widetilde{\Xi}$ is the homomorphism $\widetilde{G} \to \mathbb{Q}^+$ as defined in Section 1. □

The following information concerning the actions of $\widetilde{G}$ on $\mathbb{C}[G/S]$ is useful.

Lemma 2.6. *Let $\tilde{g} \in \widetilde{G}$. Then*

$$(\forall x \in G) \quad \tilde{g}.[xS] = [\tilde{g}(xS)]. \tag{6}$$

In particular, $\Psi(g).f = g.f$ for all $g \in G$ and $f \in \mathbb{C}[G/S]$, and every $f \in \mathbb{C}[G/S]$ has open stabilizer in $\widetilde{G}$ with respect to the action defined in Equation (4). Furthermore, every $f \in \mathbb{C}[G/S]$ has open stabilizer in $\widetilde{G}$ under the action defined in Equation (5).

Proof. Let $\tilde{g} \in \widetilde{G}$. Then for all $x \in G$, we have $\Theta([\Psi(x)\widetilde{S}]) = [xS]$ by Equation (2) and thus $\Theta^{-1}([xS]) = [\Psi(x)\widetilde{S}]$. Therefore $\tilde{g}.[xS] = \Theta(\tilde{g}.\Theta^{-1}([xS])) = \Theta([\tilde{g}\Psi(x)\widetilde{S}]) = [\tilde{g}(xS)]$ indeed, using Equation (2) once more. The topology on $\widetilde{G}$ being finer than the compact-open topology, every coset $xS \in G/S$ has open stabilizer in $\widetilde{G}$. It now readily follows from Equation (6) that every $f \in \mathbb{C}[G/S]$ has open stabilizer in $\widetilde{G}$. The character $\widetilde{\Xi}^{-\frac{1}{2}}$ being locally constant, the final assertion easily follows. □

3. Hermitian representations of G and $\mathcal{H}(G,S)$

Let (G,S) be a Hecke pair, and $\mathcal{H} := \mathcal{H}(G,S)$ be the associated Hecke algebra. In this section, we describe functors from $\mathcal{H}$-modules to G-modules and vice versa, which take hermitian modules to hermitian modules. The eventual goal is to link unitary representations of G and $*$-representations of $\mathcal{H}$. However, this will only be possible under an additional hypothesis, a certain positivity condition due to R. Hall [7], which we discuss in the following section. We hope that additional transparency is gained by splitting up the general case and the specific consequences of the positivity condition.

The proofs in the slightly more general setting directly parallel Hall's (who assumes the positivity condition from the start).

3.1. Given a representation (π, V) of G, the formula

$$\check{\pi}(f).v := \sum_{\gamma \in G/S} f(\gamma)\, \Xi(\gamma)^{\frac{1}{2}}\, \pi(\gamma).v \tag{7}$$

for $f \in \mathcal{H}$ and $v \in V^S$ defines a representation $R(\pi, V) := (\check{\pi}, V^S)$ of the Hecke algebra $\mathcal{H}$ on the subspace $V^S := \{v \in V : \pi(s).v = v \text{ for all } s \in S\}$ of S-fixed vectors of V (using the standard abuse of notation: γ is intended to range over a set of representatives for the left cosets of S in G). Given a left G-module homomorphism $\Phi\colon V_1 \to V_2$, we define $R(\Phi) := \Phi|_{V_1^S}^{V_2^S} \colon V_1^S \to V_2^S$.

3.2. Given a left $\mathcal{H}$-module W, we let $I(W)$ denote the left G-module

$$I(W) := \mathbb{C}[G/S] \otimes_{\mathcal{H}} W,$$

where $\mathbb{C}[G/S]$ is considered as a $\mathbb{C}[G]$-$\mathcal{H}$-bimodule as in **1.4**. Explicitly, the corresponding representation $\check{\rho}$ of G on $I(W)$ is given by

$$\check{\rho}(g).\sum_{i=1}^{n} f_i \otimes w_i := \Xi(g)^{-\frac{1}{2}} \cdot \sum_{i=1}^{n} (g.f_i) \otimes w_i. \tag{8}$$

If $\Psi\colon W_1 \to W_2$ is a left $\mathcal{H}$-module homomorphism, we define

$$I(\Psi) := \mathrm{id} \otimes \Psi \colon \mathbb{C}[G/S] \otimes_{\mathcal{H}} W_1 \to \mathbb{C}[G/S] \otimes_{\mathcal{H}} W_2.$$

Recall from [7] that these definitions make sense; they define functors $R : {}_G\mathbf{Mod} \to {}_{\mathcal{H}}\mathbf{Mod}$ and $I : {}_{\mathcal{H}}\mathbf{Mod} \to {}_G\mathbf{Mod}$, respectively, where ${}_G\mathbf{Mod}$ denotes the category of left G-modules, and ${}_{\mathcal{H}}\mathbf{Mod}$ the category of left $\mathcal{H}$-modules (and module homomorphisms). We remark that if W is a left $\mathcal{H}$-module, then $R(I(W)) = (\mathbb{C}[G/S] \otimes_{\mathcal{H}} W)^S = [S] \otimes W$ (cf. [7, Lemma 3.6], noting that $\Xi|_S = 1$), which generates $I(W) = \mathbb{C}[G/S] \otimes_{\mathcal{H}} W$ as a G-module.

Definition 3.3. Let V be a complex vector space, equipped with a hermitian form $\langle \bullet, \bullet \rangle \colon V \times V \to \mathbb{C}$ (a "hermitian space"). Let G be a group and A a $*$-algebra. A representation $\pi : G \to \mathrm{GL}(V)$ of G on V is called *hermitian* if

$$\langle \pi(g).v, w \rangle = \langle v, \pi(g^{-1}).w \rangle$$

for all $g \in G$ and $v, w \in V$. A representation $\rho\colon A \to \mathrm{End}_{\mathbb{C}}(V)$ is *hermitian* if

$$\langle \rho(a).v, w\rangle = \langle v, \rho(a^*).w\rangle$$

for all $a \in A$ and $v, w \in V$. In the following, we shall call a hermitian space $(V, \langle \bullet, \bullet\rangle)$ a *semi-inner product space* if $\langle \bullet, \bullet\rangle$ is positive semidefinite; we call it an *inner product space* or *pre-Hilbert space* if the hermitian form is positive definite.[3]

Note that a representation of a group on a complex Hilbert space is hermitian if and only if it is unitary; a representation of a $*$-algebra on a complex Hilbert space is hermitian if and only if it is a $*$-representation.

Lemma 3.4. *If π is a hermitian representation of G on a hermitian space $(V, \langle \bullet, \bullet\rangle)$, then the representation $\breve{\pi}$ of $\mathcal{H}$ on $(V^S, \langle \bullet, \bullet\rangle|_{V^S \times V^S})$ is hermitian.*

Proof. For every $g \in G$ and $v, w \in V^S$, we have

$$\breve{\pi}([SgS]) = \sum_{\gamma \in SgS/S} \Xi(\gamma)^{\frac{1}{2}} \pi(\gamma)|_{V^S} = \sum_{s \in S/S_g} \Xi(sg)^{\frac{1}{2}} \pi(sg)|_{V^S}$$

with S_g as in **1.3**. Hence, noting that $\Xi(s) = 1$ and $\pi(s).w = w$ for all $s \in S$,

$$\begin{aligned} \langle \breve{\pi}([SgS]).v, w\rangle &= \sum_{s \in S/S_g} \Xi(s)^{\frac{1}{2}} \Xi(g)^{\frac{1}{2}} \langle \pi(s)\pi(g).v, w\rangle \\ &= \sum_{s \in S/S_g} \Xi(g)^{\frac{1}{2}} \langle \pi(g).v, \pi(s^{-1}).w\rangle \\ &= \Xi(g)^{\frac{1}{2}} \sum_{s \in S/S_g} \langle \pi(g).v, w\rangle = \Xi(g)^{\frac{1}{2}} L(g) \langle \pi(g).v, w\rangle \\ &= (R(g)L(g))^{\frac{1}{2}} \langle \pi(g).v, w\rangle . \end{aligned}$$

Similarly, $\langle v, \breve{\pi}([SgS]^*).w\rangle = (R(g)L(g))^{\frac{1}{2}} \langle \pi(g).v, w\rangle$; the claim follows. $\square$

[3]Our terminology deviates from Hall's here, who calls semi-inner product spaces pre-Hilbert.

Definition 3.5. [Canonical $\mathcal{H}$-valued hermitian map] We define

$$\langle\!\langle \bullet, \bullet \rangle\!\rangle : \mathbb{C}[G/S] \times \mathbb{C}[G/S] \to \mathcal{H}$$

via

$$\langle\!\langle f, g \rangle\!\rangle(x) := \frac{1}{L(x)} \sum_{\gamma \in G/S_x} \overline{f(\gamma)}\, g(\gamma x)\, \Xi(\gamma) \qquad (x \in G).$$

Lemma 3.6. *The canonical map $\langle\!\langle \bullet, \bullet \rangle\!\rangle$ has the following properties:*

P1 *For all $f, g \in \mathbb{C}[G/S]$ and $h \in \mathcal{H}$, we have $\langle\!\langle f, g * h \rangle\!\rangle = \langle\!\langle f, g \rangle\!\rangle * h$. Thus the sesquilinear map $\langle\!\langle \bullet, \bullet \rangle\!\rangle$ is $\mathcal{H}$-linear in the second argument.*

P2 *$\langle\!\langle \bullet, \bullet \rangle\!\rangle$ is hermitian indeed, that is, $\langle\!\langle g, f \rangle\!\rangle = \langle\!\langle f, g \rangle\!\rangle^*$ for all $f, g \in \mathbb{C}[G/S]$.*

P3 *We have $\langle\!\langle f, g \rangle\!\rangle = \langle\!\langle \Xi(x)^{-\frac{1}{2}}\, x.f,\ \ \Xi(x)^{-\frac{1}{2}}\, x.g \rangle\!\rangle$, for all $f, g \in \mathbb{C}[G/S]$ and $x \in G$.*

Proof. This is Lemma 3.17 in [7]. For **P1**, see also [6], Lemma 3.6. The lemma becomes much clearer when G is totally disconnected and S is a compact open subgroup. In that case $\mathbb{C}[G/S]$ may be identified with the subspace $C_c(G) * \mathbf{1}_S$ of $C_c(G)$ and $\langle\!\langle f, g \rangle\!\rangle = f^* * g$. Properties **P1-3** are then obvious. □

Proposition 3.7 (Canonical hermitian form on $I(W)$)**.** *Let ρ be a hermitian representation of $\mathcal{H}$ on a hermitian space $(W, \langle \bullet, \bullet \rangle)$. Then there exists a unique hermitian form*

$$\langle \bullet, \bullet \rangle : (\mathbb{C}[G/S] \otimes_{\mathcal{H}} W)^2 \to \mathbb{C}$$

on $I(W)$ such that

$$\langle f \otimes v, g \otimes w \rangle = \langle v, \rho(\langle\!\langle f, g \rangle\!\rangle).w \rangle \tag{9}$$

for all $f, g \in \mathbb{C}[G/S]$, $v, w \in W$. This form makes $I(W)$ a hermitian G-module.

Proof. Properties **P1** and **P2** ensure that a hermitian form $\langle \bullet, \bullet \rangle$ satisfying Equation (9) exists, making repeated use of the universal property of tensor products of bimodules. Property **P3** implies that $\check{\rho}$ is a hermitian representation of G on the hermitian space $(\mathbb{C}[G/S] \otimes_{\mathcal{H}} W, \langle \bullet, \bullet \rangle)$. (See also [7], §3.4.1 for the case of semi-inner product spaces). □

Remark 3.8. If V is a hermitian G-module and W a hermitian $\mathcal{H}$-module, we shall always equip the subspace V^S with the inherited hermitian form (as described in Lemma 3.4) and $\mathbb{C}[G/S]\otimes_{\mathcal{H}}W$ with the canonical hermitian form, as described in Proposition 3.7.

Lemma 3.9. *For every hermitian $\mathcal{H}$-module W, the mapping*

$$\xi_W\colon W \to R(I(W)) = (\mathbb{C}[S/G]\otimes_{\mathcal{H}} W)^S, \quad w \mapsto [S]\otimes w$$

is an isomorphism of $\mathcal{H}$-modules and an isometry.

Proof. As $\Xi|_S \equiv 1$, [7, Lemma 3.6] shows that ξ_W is an isomorphism of vector spaces. It is easy to check that ξ_W is a homomorphism of left $\mathcal{H}$-modules. As $\langle\!\langle [S],[S]\rangle\!\rangle = [S]$, the definition of the canonical hermitian form on $I(W)$ yields $\langle [S]\otimes v, [S]\otimes w\rangle = \langle v,w\rangle$ for all $v,w\in W$. Thus ξ_W is an isometry (cf. [7], proof of Lemma 3.24 for the semi-inner product case). □

4. Hall's positivity condition

In this section, we recall a certain positivity condition formulated by R. Hall, which, as we shall explain in detail, ensures that the canonical hermitian form on $I(W)$ is positive semidefinite whenever (ρ, W) is a $*$-representation of $\mathcal{H}(G,S)$ on a complex Hilbert space.

Definition 4.1. Let us say that a Hecke pair (G,S) satisfies the *positivity condition* if $\langle\!\langle f,f\rangle\!\rangle \in \mathcal{H}_+$ for all $f\in\mathbb{C}[G/S]$, where $\mathcal{H}_+$ is the convex cone in $\mathcal{H} := \mathcal{H}(G,S)$ generated by elements of the form $h^* * h$, $h\in\mathcal{H}$ ([7, p. 31]).

Example 4.2. $G := \mathrm{SL}_2(\mathbb{Q}_p)$, together with an Iwahori subgroup S, satisfies the positivity condition ([7, Theorem 1.3]), whereas the positivity condition fails for the Hecke pair $(\mathrm{SL}_n(\mathbb{Q}_p), \mathrm{SL}_n(\mathbb{Z}_p))$ (*loc. cit.*, Proposition 3.19). Furthermore, the Hecke algebra $\mathcal{H}(\mathrm{SL}_n(\mathbb{Q}_p), \mathrm{SL}_n(\mathbb{Z}_p))$ does not have an enveloping C^*-algebra (*loc. cit.*, Proposition 2.21).

Proposition 4.3. *Suppose that (G,S) is a Hecke pair satisfying the positivity condition. Then the canonical hermitian form on $\mathbb{C}[G/S]\otimes_{\mathcal{H}}W$ is positive semidefinite for every $*$-representation ρ of $\mathcal{H} = \mathcal{H}(G,S)$ on a complex Hilbert space W.*

Proof. Suppose that (ρ, W) is $*$-representation of $\mathcal{H}$ on a complex Hilbert space. Hall already observed that the positivity condition implies that

$$\langle f \otimes w, f \otimes w \rangle = \langle w, \rho(\langle\!\langle f, f \rangle\!\rangle).w \rangle \geq 0$$

for all simple tensors $f \otimes w \in \mathbb{C}[G/S] \otimes_{\mathcal{H}} W$ ([7], lines preceding Definition 3.15). In order to deduce that

$$\langle v, v \rangle = \sum_{i,j=1}^{n} \langle f_i \otimes w_i, f_j \otimes w_j \rangle \geq 0 \tag{10}$$

holds for arbitrary tensors $v = \sum_{i=1}^{n} f_i \otimes w_i \in \mathbb{C}[G/S] \otimes_{\mathcal{H}} W$, we employ a lemma drawing on ideas from the theory of Hilbert C^*-modules:

Lemma 4.4. *Let $\rho \colon A \to B$ be a $*$-algebra homomorphism from a unital $*$-algebra A to a unital C^*-algebra B, with dense image. Let V be a right A-module, and consider B as a right A-module via $b.a := b\,\rho(a)$ for $a \in A$, $b \in B$. Suppose that $K \colon V \times V \to B$ is a hermitian mapping which is A-linear in the second argument (thus $K(y, x) = K(x, y)^*$, $K(x, y + y_1) = K(x, y) + K(x, y_1)$, and $K(x, y.a) = K(x, y).a$ for all $x, y, y_1 \in V$, $a \in A$). If $K(x, x) \in B_+$ for all $x \in V$, then*

$$(K(x_i, x_j))_{i,j=1,\ldots,n} \in M_n(B)_+$$

for all finite sequences $x_1, \ldots, x_n \in V$.

Proof. Let $x_1, \ldots, x_n \in V$. Identifying $M_n(B)$ with $\mathcal{L}(B^n)$, [12, Lemma 4.1] shows that $(K(x_i, x_j))_{i,j=1}^{n} \in M_n(B)_+$ if and only if

$$\sum_{i,j=1}^{n} b_i^* K(x_i, x_j) b_j \in B_+ \tag{11}$$

for all $b_1, \ldots, b_n \in B$. As we assume that $\mathrm{im}(\rho)$ is dense in B, due to closedness of B_+ in the C^*-algebra B clearly Equation (11) holds for all $b_1, \ldots, b_n \in B$ if and only if it holds for all $b_1, \ldots, b_n \in \mathrm{im}(\rho)$. However, for all $a_1, \ldots, a_n \in A$ we have

$$\sum_{i,j=1}^{n} \rho(a_i)^* K(x_i, x_j) \rho(a_j) = \sum_{i,j=1}^{n} K(x_i.a_i, x_j.a_j) = K(s, s) \geq 0$$

in B, where $s := \sum_{i=1}^{n} x_i.a_i$. $\square$

Proof of Proposition 4.3, completed: Given a $*$-representation ρ of $\mathcal{H}$ on a Hilbert space W, we let A be the closure of $\rho(W)$ in $(\mathcal{L}(W), \|.\|)$. Then A is a C^*-algebra, $\rho(W)$ being a $*$-subalgebra of $\mathcal{L}(W)$. We define a hermitian mapping

$$K := \rho \circ \langle\!\langle \bullet, \bullet \rangle\!\rangle \colon \mathbb{C}[G/S] \times \mathbb{C}[G/S] \to A.$$

Note that $K(f, g * h) = \rho(\langle\!\langle f, g\rangle\!\rangle * h) = K(f,g)\rho(h)$ for all $f, g \in \mathbb{C}[G/S]$ and $h \in \mathcal{H}$. Thus K and ρ (co-restricted to A) satisfy the hypothesis of Lemma 4.4. Given

$$v = \textstyle\sum_{i=1}^n f_i \otimes w_i \in \mathbb{C}[G/S] \otimes_{\mathcal{H}} W,$$

the lemma yields $(K(f_i, f_j)) \in M_n(A)_+ \subseteq M_n(\mathcal{L}(W))_+ \cong \mathcal{L}(W^n)_+$. Thus $\langle v, v\rangle = \sum_{i,j=1}^n \langle w_i, K(f_i, f_j).w_j\rangle \geq 0$, as required. □

Proposition 4.5. *Suppose that (G, S) is a Hecke pair and ρ a hermitian representation of the associated Hecke algebra $\mathcal{H}$ on a pre-Hilbert space W such that the canonical hermitian form on $\mathbb{C}[G/S] \otimes_{\mathcal{H}} W$ is positive semidefinite. Then $\rho(f)$ is a bounded operator for all $f \in \mathcal{H}$, and*

$$\|\rho([SgS])\| \leq (R(g)L(g))^{\frac{1}{2}} \tag{12}$$

for all $g \in G$.

Proof. The hermitian representation ρ of $\mathcal{H}$ gives rise to a hermitian representation $\check{\rho}$ of G on $\mathbb{C}[G/S] \otimes_{\mathcal{H}} W$ (see Equation (8)), which in turn gives rise to a hermitian representation $\delta := \check{\check{\rho}}$ of $\mathcal{H}$ on the subspace $[S] \otimes W$ (see Equation (7)). By Lemma 3.9, it suffices to show the assertions for δ in place of ρ. Given $\gamma \in G$, the fact that $\check{\rho}$ is hermitian implies that

$$\langle \check{\rho}(\gamma).v, \check{\rho}(\gamma).v\rangle = \langle v, \check{\rho}(\gamma^{-1})\check{\rho}(\gamma).v\rangle = \langle v, v\rangle$$

for all $v \in \mathbb{C}[G/S] \otimes_{\mathcal{H}} W$; thus $\check{\rho}(\gamma)$ is an isometry. Given $x \in G$, we have

$$\delta([SxS]) = \sum_{\gamma \in SxS/S} \Xi(x)^{\frac{1}{2}}\, \check{\rho}(\gamma)|_{[S]\otimes W}$$

and therefore $\|\delta([SxS])\| \leq L(x) \cdot \Xi(x)^{\frac{1}{2}} = (R(x)L(x))^{\frac{1}{2}}$, making use of the triangle inequality for the semi-inner product space $\mathbb{C}[G/S] \otimes_{\mathcal{H}} W$. The assertions follow. □

Suppose that A is a $*$-algebra such that

$$\|a\|_{C^*} := \sup_{\rho} \|\rho(a)\| < \infty \text{ for all } a \in A,$$

where ρ ranges through the $*$-homomorphisms from A to unital C^*-algebras. (Such a $*$-algebra is called a G^*-algebra in [15].) Then the completion of A with respect to the norm $\|a\|_{C^*}$ is a C^*-algebra, denoted $C^*(A)$, and this algebra together with the natural $*$-homomorphism $\eta : A \to C^*(A)$ is called the *enveloping C^*-algebra* of A, see [15, Definition 10.1.10]. The enveloping C^*-algebra has the universal property that for every $*$-algebra homomorphism ρ from A to a C^*-algebra B, there exists a unique $*$-algebra homomorphism $\rho' : C^*(A) \to B$ such that $\rho' \circ \eta = \rho$, [15, Theorem 10.1.11]. The *Hecke C^*-algebra* associated to a Hecke pair (G, S) is defined as the universal enveloping C^*-algebra $C^*(G, S) := C^*(\mathcal{H}(G, S))$, whenever it exists.

We readily deduce from Proposition 4.3 and Proposition 4.5 (see also [7, Corollary 1.2], where the category equivalence is invoked for the proof):

Proposition 4.6. *If a Hecke pair (G, S) satisfies the positivity condition, then the Hecke C^*-algebra $C^*(G, S)$ exists, and*

$$\|f\|_{C^*} \leq \sum_{\gamma \in S\backslash G/S} (R(\gamma)L(\gamma))^{\frac{1}{2}} |f(\gamma)| = \sum_{\gamma \in G/S} \Xi(\gamma)^{\frac{1}{2}} |f(\gamma)|$$

for all $f \in \mathcal{H}(G, S)$. □

There is also a lower bound for the quasi-norm $\|.\|_{C^*}$ on the Hecke algebra. Indeed:

Proposition 4.7. *Let (G, S) be any Hecke pair. Then*

$$\|f\|_{C^*} \geq \max\left\{ \sqrt{\sum_{\gamma \in S\backslash G/S} R(\gamma) |f(\gamma)|^2}, \sqrt{\sum_{\gamma \in S\backslash G/S} L(\gamma) |f(\gamma)|^2} \right\},$$

for every $f \in \mathcal{H}(G, S)$. In particular, $\|f\|_{C^*} \neq 0$ whenever $0 \neq f \in \mathcal{H}(G, S)$, and $\|[SgS]\|_{C^*} \geq \max\{L(g)^{\frac{1}{2}}, R(g)^{\frac{1}{2}}\}$, for every $g \in G$.

Proof. The unitary representation λ of G on $\ell^2(G/S)$ given by $(g.f)(xS) := f(g^{-1}xS)$ for $g, x \in G$, $f \in \ell^2(G/S)$ gives rise to a $*$-representation $\check{\lambda}$ of $\mathcal{H}(G,S)$ on $\ell^2(G/S)^S$. For $f \in \mathcal{H}(G,S)$, we compute in $\ell^2(G/S)$:

$$\begin{aligned} \|\check{\lambda}(f).[S]\|^2 &= \sum_{\gamma,\eta \in G/S} \Xi(\gamma)^{\frac{1}{2}}\, \Xi(\eta)^{\frac{1}{2}}\, \overline{f(\gamma)}\, f(\eta)\, \langle [\gamma S], [\eta S]\rangle \\ &= \sum_{\gamma \in G/S} \Xi(\gamma)\, |f(\gamma)|^2 = \sum_{\gamma \in S\backslash G/S} \Xi(\gamma)\, L(\gamma)\, |f(\gamma)|^2 \\ &= \sum_{\gamma \in S\backslash G/S} R(\gamma)\, |f(\gamma)|^2, \end{aligned}$$

using that

$\langle [\gamma S], [\eta S]\rangle = \delta_{\gamma S, \eta S}$. Thus $\|f\|_{C^*}^2 \geq \sum_{\gamma \in S\backslash G/S} R(\gamma)\, |f(\gamma)|^2$. The assertion follows since $\|f\|_{C^*} = \|f^*\|_{C^*}$. □

5. Functors involving the positivity condition

Given a Hecke pair (G,S), we let $\mathcal{U}(G,S)$ be the category of unitary representations (π, V) of G on complex Hilbert spaces V such that the Hilbert G-module V is generated topologically by its subspace V^S of S-fixed vectors, *i.e.*, $V = \overline{\operatorname{span} \pi(G).V^S}$. We take intertwining partial isometries as the morphisms (see [4], Chapter 5, §4.2). $\mathcal{U}(\mathcal{H})$ denotes the category of unital $*$-representations (ρ, W) of $\mathcal{H} := \mathcal{H}(G,S)$ on complex Hilbert spaces W, with intertwining partial isometries as morphisms. In this section, we recall R. Hall's definition of functors $\overline{R}\colon \mathcal{U}(G,S) \to \mathcal{U}(\mathcal{H})$ and $\overline{I}\colon \mathcal{U}(\mathcal{H}) \to \mathcal{U}(G,S)$, assuming henceforth that (G,S) satisfies the positivity condition. We also explain the effect of $\overline{I}$ on morphisms (Lemma 5.3), which is not described in [7].

5.1. Given a unitary representation (π, V) of G with V generated topologically by its S-fixed vectors, we define $\overline{R}(\pi, V)$ as $\overline{R}(\pi, V) = V^S$, equipped with the $*$-representation $\check{\pi}$ and the inner product inherited from V, see Equation (7). If $\phi\colon (\pi_1, V_1) \to (\pi_2, V_2)$ is a partial isometry intertwining the representations, we set $\overline{R}(\phi) := \phi|_{V_1^S}^{V_2^S}$.

5.2. Given a $*$-representation (ρ, W) of $\mathcal{H}$, let us define a unitary representation $\overline{I}(\rho, W)$ of G. First, we note that $\|\check{\rho}(g).v\|^2 = \langle \check{\rho}(g).v, \check{\rho}(g).v\rangle = \langle v, \check{\rho}(g^{-1})\check{\rho}(g).v\rangle = \langle v, v\rangle = \|v\|^2$ for all $g \in G$ and all v in the semi-inner product space $P := I(W) = \mathbb{C}[G/S] \otimes_{\mathcal{H}} W$. In other words, the hermitian representation $\check{\rho}$ (defined in Equation (8)) is a representation by isometries of P. In particular, $\|\check{\rho}(g).v\| = \|v\| = 0$ for all $v \in N := P^{\perp}$, the subspace of zero-length vectors in P. Hence, if $q : P \to P/N$ denotes the quotient map, there is a unique linear map $A \colon P/N \to P/N$ satisfying $A \circ q = q \circ \check{\rho}(g)$, and A is a surjective isometry. Let H_W be the Hilbert space completion of P/N. Then A has a continuous extension $\overline{A} \colon H_W \to H_W$, which is an isometry with dense image and thus a unitary operator. By abuse of notation, we write $\check{\rho}(g) := \overline{A}$ also for this operator. It is readily verified that $\check{\rho} \colon G \to U(H_W)$ is a unitary representation of G. We set $\overline{I}(\rho, W) := (\check{\rho}, H_W)$.

The definition of the induction functor $\overline{I}$ on morphisms requires a little bit of work.

Lemma 5.3. *Suppose that (ρ_1, W_1) and (ρ_2, W_2) are $*$-representations of $\mathcal{H}$ on Hilbert spaces, and assume that $\psi : W_1 \to W_2$ is a partial isometry which intertwines the representations. Set $P_j := I(W_j) = \mathbb{C}[G/S] \otimes_{\mathcal{H}} W_j$ for $j = 1, 2$, $N_j := P_j^{\perp}$, and let $q_j : P_j \to P_j/N_j$ be the quotient map. Then the left G-module homomorphism $\overline{\psi} := I(\psi) = \mathrm{id} \otimes \psi \colon P_1 \to P_2$ is a contraction and therefore $q_2 \circ \overline{\psi}$ factors over q_1 to a continuous linear map*

$$P_1/N_1 \to P_2/N_2. \tag{13}$$

The continuous extension

$$\overline{I}(\psi) := \overline{\overline{\psi}} \colon H_{W_1} \to H_{W_2}$$

of the mapping (13) *is a partial isometry which intertwines the unitary representations $(\check{\rho}_1, H_{W_1})$ and $(\check{\rho}_2, H_{W_2})$.*

Proof. We have $\psi \circ \rho_1(g) = \rho_2(g) \circ \psi$ for all $g \in G$ and therefore $\rho_1(g^{-1}) \circ \psi^* = \psi^* \circ \rho_2(g^{-1})$. Thus ψ^* intertwines the representations ρ_2 and ρ_1, and thus $\Pi := \psi^* \circ \psi \colon W_1 \to W_1$ is an orthogonal projection and a self-intertwiner of ρ_1. Note that $\rho_1(g)$ leaves $\ker \Pi$ (the 0-eigenspace of Π) and $\operatorname{im} \Pi$ (the 1-eigenspace of Π) invariant for each $g \in G$, since $\rho_1(g)$ commutes with Π.

Abbreviate $\overline{\Pi} = I(\Pi) = \mathrm{id} \otimes \Pi$. Given $v = \sum_{i=1}^{n} f_i \otimes w_i \in \mathbb{C}[G/S] \otimes_{\mathcal{H}} W_1$, we calculate

$$\begin{aligned}
\|v\|^2 &= \|\overline{\Pi}.v + (\mathbf{1} - \overline{\Pi}).v\|^2 \\
&= \|\overline{\Pi}.v\|^2 + \|(\mathbf{1} - \overline{\Pi}).v\|^2 + \sum_{i,j=1}^{n} \langle \Pi.w_i, (\mathbf{1} - \Pi).\rho_1(\langle\!\langle f_i, f_j \rangle\!\rangle).w_j \rangle \\
&\qquad + \sum_{i,j=1}^{n} \langle (\mathbf{1} - \Pi).w_i, \Pi.\rho_1(\langle\!\langle f_i, f_j \rangle\!\rangle).w_j \rangle \\
&= \|\overline{\Pi}.v\|^2 + \|(\mathbf{1} - \overline{\Pi}).v\|^2;
\end{aligned}$$

so $\overline{\Pi}\colon P_1 \to P_1$ is a contraction. As

$$\begin{aligned}
\|\overline{\psi}.v\|^2 &= \sum_{i,j} \langle \psi.w_i, \rho_2(\langle\!\langle f_i, f_j \rangle\!\rangle).\psi.w_j \rangle \\
&= \langle \psi^*\psi.w_i, \rho_1(\langle\!\langle f_i, f_j \rangle\!\rangle).w_j \rangle \\
&= \langle (\psi^*\psi)^2.w_i, \rho_1(\langle\!\langle f_i, f_j \rangle\!\rangle).w_j \rangle \\
&= \|\overline{\Pi}.v\|^2 \le \|v\|^2,
\end{aligned}$$

the mapping $\overline{\psi}$ is a contraction as well. By the preceding, continuous linear mappings $\overline{\overline{\psi}} \in \mathcal{L}(H_{W_1}, H_{W_2})$ and $\overline{\overline{\Pi}} \in \mathcal{L}(H_{W_1})$ can be defined as described in the lemma. Now

$$\begin{aligned}
\langle \overline{\psi}.(f \otimes w_1), g \otimes w_2 \rangle &= \langle f \otimes (\psi.w_1), g \otimes w_2 \rangle \\
&= \langle \psi.w_1, \rho_2(\langle\!\langle f, g \rangle\!\rangle).w_2 \rangle \\
&= \langle w_1, \rho_1(\langle\!\langle f, g \rangle\!\rangle).\psi^*.w_2 \rangle \\
&= \langle f \otimes w_1, \overline{\psi^*}.(g \otimes w_2) \rangle
\end{aligned}$$

holds for all $f, g \in \mathbb{C}[G/S]$, $w_1, w_2 \in W_1$, which entails $\overline{\overline{\psi}}^* = \overline{\overline{\psi^*}}$. But then $\overline{\overline{\psi}}^* \, \overline{\overline{\psi}} = \overline{\overline{\psi^* \, \psi}}$ and therefore

$$(\overline{\overline{\psi}}^* \, \overline{\overline{\psi}})^2 = \overline{\overline{(\psi^* \, \psi)^2}} = \overline{\overline{\psi^* \, \psi}} = \overline{\overline{\psi}}^* \, \overline{\overline{\psi}},$$

using that all of these operators are continuous and coincide on the dense subspace P_1/N_1 of H_{W_1}. Thus $\overline{\overline{\psi}}^* \, \overline{\overline{\psi}}$ is an orthogonal projection, whence $\overline{\overline{\psi}}$ is a partial isometry indeed. $\square$

It is easily verified that $\overline{R} : \mathcal{U}(G, S) \to \mathcal{U}(\mathcal{H})$ and $\overline{I} : \mathcal{U}(\mathcal{H}) \to \mathcal{U}(G, S)$ are functors.

6. Hall's category equivalence

If (G, S) is a Hecke pair satisfying the positivity condition, then the functors $\overline{R}$ and $\overline{I}$ give rise to a category equivalence between the categories $\mathcal{U}(G, S)$ and $\mathcal{U}(\mathcal{H})$ (see [7]). In this section, we use the topologization construction developed in Section 2 to justify *loc. cit.*, Lemma 3.24, which is used essentially in *loc. cit.* to establish the category equivalence:

[7, Lemma 3.24]: *Let (G, S) be a Hecke pair satisfying the positivity condition, and (ρ, W) be a $*$-representation of $\mathcal{H}(G, S)$. Then $W \cong (H_W)^S$ as Hilbert spaces.*

For convenience, let us fix some notation. Throughout the following, (G, S) denotes a Hecke pair satisfying the positivity condition. If (ρ, W) is a $*$-representation of $\mathcal{H} := \mathcal{H}(G, S)$ on a Hilbert space W, we abbreviate $P := I(W) = \mathbb{C}[G/S] \otimes_{\mathcal{H}} W$, let $N := P^{\perp}$ be the radical of the semi-inner product $\langle \bullet, \bullet \rangle$ on P, and $P^S \leq P$ be the subspace of S-fixed vectors. $H_W := \overline{P/N}$ denotes the Hilbert space completion of P/N, and $\pi := \check{\rho}$ the hermitian representation of G on P associated with ρ. The unitary representation $G \to U(H_W)$ induced by π will be denoted by π as well. Finally, we let $q\colon P \to P/N$ be the canonical quotient map. The goal of this section is to prove the following proposition, corresponding to [7, Lemma 3.24]:

Proposition 6.1. *$(H_W)^S = q(P^S)$ holds, and the mapping $W \to \overline{R}(\overline{I}(W)) = (H_W)^S$, $w \mapsto q([S] \otimes w)$ is an isometric isomorphism.*[4]

Proof. It is shown in [7], proof of Lemma 3.24 that $W \cong P^S$ as Hilbert spaces via $w \mapsto [S] \otimes w$ (see also Lemma 3.9 above). We complete the proof in two main steps:

Step 1. If $v \in P$, then $q(v) \in (H_W)^S$ if and only if $\check{\rho}(s).v \in v + N$ for all $s \in S$, which is a *weaker condition* than being S-fixed. In a first step, we show that if $q(v) \in P/N$ is S-fixed for some $v \in P$, then the coset $v + N$ always contains an S-fixed vector.

Step 2. In general, the pre-Hilbert space P/N is a proper subspace of H_W, so there might be S-fixed vectors $v \in H_W$ which are not contained in P/N. We show that this is not the case, *i.e.*, that

[4] And an isomorphism of $\mathcal{H}$-modules.

$(H_W)^S \subseteq q(P)$. It is here where the topologization construction comes in.

Step 1 doesn't pose major difficulties.

Lemma 6.2. *Define* $P^S := \{v \in P : \pi(s).v = v \ \text{ for all } s \in S\}$, *and* $X := \{v \in P : \pi(s).v \in v + N \ \text{ for all } s \in S\}$. *Then* $q(P^S) = q(X)$.

Proof. Let $v \in X$, say $v = \sum_{i=1}^n [x_i S] \otimes w_i$ where $x_1, \ldots, x_n \in G$ and $w_1, \ldots, w_n \in W$. The subgroup $T := \bigcap_{i=1}^n S_{x_i}$ has finite index in S, and fixes v. Being a convex combination of elements in the affine subspace $v + N$ of P, the element

$$v' := \frac{1}{[S:T]} \sum_{s \in S/T} \pi(s).v$$

is contained in $v + N$. Furthermore, v' is S-invariant. Thus $v' \in P^S$ and $q(v') = q(v)$. □

Step 2 of the proof is more involved, and will be accomplished in a series of steps.

6.3. First, we observe that $\widetilde{W} := q(P^S)$ is closed in H_W. In fact, $P^S \cong W$ is already a Hilbert space and $q|_{P^S}^{\widetilde{W}}$ is a surjective isometry; hence $\widetilde{W}$ is a Hilbert space and therefore closed in H_W, being complete.

6.4. Next, we associate a totally disconnected, locally compact group $\widetilde{G}$ and open compact subgroup $\widetilde{S} \leq \widetilde{G}$ to the Hecke pair (G, S), as described in Section 2. We make $\mathbb{C}[G/S]$ a $\mathbb{C}[\widetilde{G}]$-$\mathcal{H}(G,S)$-bimodule by means of the action $(\tilde{g}, f) \mapsto \tilde{g} \diamond f$ of $\widetilde{G}$ described in Proposition 2.5. We obtain a representation $\tilde{\pi}$ of $\widetilde{G}$ on $P = \mathbb{C}[G/S] \otimes_{\mathcal{H}} W$ via

$$\tilde{\pi}(\tilde{g}).(\sum_{i=1}^n f_i \otimes w_i) := \sum_{i=1}^n (\tilde{g} \diamond f_i) \otimes w_i$$

for $\tilde{g} \in \widetilde{G}$, $f_1, \ldots, f_n \in \mathbb{C}[G/S]$, $w_1, \ldots, w_n \in W$.

Lemma 6.5. *The representation* $\tilde{\pi}$ *of* $\widetilde{G}$ *on* P *is hermitian with respect to the positive semidefinite hermitian form* $\langle \bullet, \bullet \rangle$.

Proof. Let $\tilde{g} \in \widetilde{G}$ and $v_1, v_2 \in P = \mathbb{C}[G/S] \otimes_{\mathcal{H}} W$. Define $\Psi \colon G \to \widetilde{G}$ as in Section 2 and let $(g_\alpha)_{\alpha \in A}$ be a net in G such that $\Psi(g_\alpha) \to \tilde{g}$ in $\widetilde{G}$. We readily deduce from Lemma 2.6 that there exists $\alpha_0 \in A$ such that $\tilde{\pi}(\tilde{g}).v_1 = \pi(g_\alpha).v_1$ and $\tilde{\pi}(\tilde{g}^{-1}).v_2 = \pi(g_\alpha^{-1}).v_2$ for all $\alpha \geq \alpha_0$. For any such α, we compute $\langle \tilde{\pi}(\tilde{g}).v_1, v_n \rangle = \langle \pi(g_\alpha).v_1, v_2 \rangle = \langle v_1, \pi(g_\alpha^{-1}).v_2 \rangle = \langle v_1, \tilde{\pi}(\tilde{g}).v_2 \rangle$. We deduce that the representation $\tilde{\pi}$ is hermitian. □

6.6. The hermitian representation $\tilde{\pi}$ on the semi-inner product space P induces a hermitian representation of the group $\widetilde{G}$ on the inner product space P/N, which via continuous extension of operators gives rise to a unitary representation $\widetilde{G} \to U(H_W)$, which we also denote by $\tilde{\pi}$ (cf. **5.2**).

6.7. If V is a complex Hilbert space, it is well-known that the weak operator topology and strong operator topology coincide on $U(V)$ and make it a topological group. Furthermore, the action $U(V) \times V \to V$, $(A, v) \mapsto A(v)$ is continuous ([14], Corollary A.I.4).

Lemma 6.8. *The representation $\tilde{\pi} \colon \widetilde{G} \to U(H_W)$ is continuous.*

Proof. Since every $f \in \mathbb{C}[G/S]$ has open stabilizer in $\widetilde{G}$ by Lemma 2.6, clearly every $v \in P = \mathbb{C}[G/S] \otimes_{\mathcal{H}} W$ has open stabilizer in $\widetilde{G}$. This implies that the orbit map

$$\widetilde{G} \to H_W, \quad \tilde{g} \mapsto \tilde{\pi}(\tilde{g}).q(v) \tag{14}$$

is locally constant and therefore continuous. Now $U(H_W) \subseteq \mathcal{L}(H_W)$ being bounded, the strong operator topology coincides on this set with the initial topology induced by the mappings $U(H_W) \to H_W$, $A \mapsto A.v$, where v ranges through any dense vector subspace D of H_W. We choose $D = P/N$ and deduce from the continuity of the mappings (14) for all $v \in P$ that $\tilde{\pi} \colon \widetilde{G} \to U(H_W)$ is continuous indeed. □

By **6.7** and Lemma 6.8, the action $\widetilde{G} \times H_W \to H_W$, $(\tilde{g}, v) \mapsto \tilde{\pi}(\tilde{g}).v$ of the locally compact group $\widetilde{G}$ on the Hilbert space H_W is continuous (although point stabilizers might not be open here). We introduced $(\widetilde{G}, \widetilde{S})$ with the aim of achieving such a situation, since now the following standard fact applies (cf. [9], Proposition 1.11):

Lemma 6.9. *Let C be a compact group and $C \times X \to X$, $(c, x) \mapsto c.x$ be a continuous action of C on a topological space X. Suppose that $x_0 \in X$ is fixed by the C-action. Then, for every neighbourhood U of x_0 in X, the C-invariant set $V := \bigcap_{c \in C} c.U \subseteq U$ is a neighbourhood of x_0 in X. If X is a topological vector space, C acts by linear maps, and U is a convex neighbourhood of x_0 here, then V is convex.* □

To complete the proof of Proposition 6.1, we have to show that $(H_W)^S = \widetilde{W} := q(P^S)$.

Let $v \in H_W$ be an S-fixed vector. It easily follows from Lemma 2.6 that $\tilde{\pi}(\Psi(g)) = \pi(g)$ for all $g \in G$. Thus v is $\Psi(S)$-fixed. The latter set being a dense subgroup of $\widetilde{S}$, we deduce from the continuity of the action of $\widetilde{G}$ on H_W that v is $\widetilde{S}$-fixed. Let $U \subseteq H_W$ be any convex neighbourhood of v. By Lemma 6.9, U contains an $\widetilde{S}$-invariant convex neighbourhood V of v. Due to the density of P/N in H_W, there exists $w \in V \cap P/N$. Then w has open stabilizer $\widetilde{T}$ in $\widetilde{S}$, as observed above. The convex combination

$$b := \frac{1}{[\widetilde{S} : \widetilde{T}]} \sum_{x \in \widetilde{S}/\widetilde{T}} \tilde{\pi}(x).w$$

is contained in $V \cap (P/N)$ and is $\widetilde{S}$-fixed, hence also S-fixed. Lemma 6.2 shows that $b \in \widetilde{W}$. We have proved that $\widetilde{W}$ is dense in $(H_W)^S$. The set $\widetilde{W} \subseteq (H_W)^S$ being closed in H_W (**6.3**), we deduce that $(H_W)^S = \widetilde{W}$. □

We conclude this section with the formulation of Hall's category equivalence:

Theorem 6.10. *Let (G, S) be a Hecke pair satisfying the positivity condition. Then the functors $\overline{I}$ and $\overline{R}$ define an equivalence between the categories $\mathcal{U}(G, S)$ and $\mathcal{U}(\mathcal{H})$.* □

Proof. See [7, Theorem 3.25] and its proof. Proposition 4.3, Lemma 5.3, and Proposition 6.1 above provide additional details. □

Of course, the categories of (non-degenerate) $*$-representations of $\mathcal{H}$ and its universal enveloping C^*-algebra $C^*(G, S)$ being equivalent, Theorem 6.10 entails a category equivalence between $\mathcal{U}(G, S)$ and the category of non-degenerate $*$-representations of the Hecke C^*-algebra $C^*(G, S)$.

7. Submultiplicativity of the scale function

The scale function $s\colon G \to \mathbb{N}$ on a totally disconnected, locally compact group G (introduced first in [18]) can be defined via

$$s(g) := \min\,[U : U \cap gUg^{-1}] \tag{15}$$

for $g \in G$, where U runs through the compact, open subgroups of G. A compact, open subgroup U is *tidy for g* if the minumum in (15) is attained at U (see [19]). In this section, we use a simple Hecke algebra argument to prove that the scale function is submultiplicative on group elements possessing a joint tidy subgroup.

Lemma 7.1. *Given a Hecke pair (G, S), let $\Lambda\colon \mathcal{H}(G, S) \to \mathbb{C}$ be the unique linear map such that $\Lambda([SgS]) = L(g)$ for all $g \in G$. Then Λ is an algebra homomorphism.*

Proof. (cf. [10], Corollary 4.5). The assertion follows from the observation that Λ maps

$$[SgS] * [ShS] = \sum_{\gamma\in SgS/S}\ \sum_{\eta\in ShS/S} [\gamma\eta S] = \sum_{\gamma,\eta} \frac{1}{L(\gamma\eta)}[S\gamma\eta S]$$

to $\sum_{\gamma,\eta} \frac{1}{L(\gamma\eta)} L(\gamma\eta) = L(g)L(h) = \Lambda([SgS])\,\Lambda([ShS])$, for all $g, h \in G$. □

We deduce:

Proposition 7.2. *Let G be a totally disconnected, locally compact group, and $s\colon G \to \mathbb{N}$ be its scale function. If $x, y \in G$ possess a joint tidy subgroup, then $s(xy) \leq s(x)s(y)$.*

Proof. Let S be a compact, open subgroup of G which is tidy for both x and y. Then

$$L(x) = [SxS : S] = [S : S \cap xSx^{-1}] = s(x)$$

and similarly $L(y) = s(y)$. As $[SxS] * [SyS] = [SxyS] + r$, where r is a sum of characteristic functions of double cosets and thus $\Lambda(r) \geq 0$, we obtain

$$\begin{aligned} s(x)s(y) &= L(x)L(y) = \Lambda([SxS])\Lambda([SyS]) \\ &= \Lambda([SxS] * [SyS]) = \Lambda([SxyS]) + \Lambda(r) \\ &\geq \Lambda([SxyS]) = L(xy) = [S : S \cap (xy)S(xy)^{-1}] \\ &\geq s(xy). \end{aligned}$$

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TU Darmstadt, FB Mathematik AG 5, Schlossgartenstr. 7, 64289 Darmstadt, Germany
E-mail address: gloeckner@mathematik.tu-darmstadt.de

Department of Mathematics, University of Newcastle, Callaghan, NSW 2308, Australia
E-mail address: george@frey.newcastle.edu.au