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**A CATEGORY OF TOPOLOGICAL GROUPS
SUITABLE FOR A STRUCTURE THEORY OF
LOCALLY COMPACT GROUPS**

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ABSTRACT. This survey outlines an approach to a projected monograph on the “Structure of pro-Lie groups and Locally Compact Groups” [3] which may be considered a sequel to the book “The Structure of Compact Groups” [1]. In the focus is the category of projective limits of finite dimensional Lie groups. In a nontrivial fashion, every member G of this category has a special presentation as a projective limit of quotients G/N which are finite dimensional Lie groups. The category of these limits is complete, and all of its objects have a good Lie theory in terms of certain Lie algebras which are well behaved projective limits of finite dimensional ones.

0. LOCALLY COMPACT GROUPS: PRO AND CON

The success story of locally compact groups has two roots: The existence of Haar integral on a locally compact group G and the successful resolution of Hilbert’s Fifth Problem with the proof that connected locally compact groups can be approximated by Lie groups.

Haar measure is the key to the representation theory of compact and locally compact groups on Hilbert space, and the wide field of harmonic analysis with ever so many ramifications (including e.g. abstract probability theory on locally compact groups). A theorem of A. Weil’s shows that, conversely, a complete topological group

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with a left- (or right-) invariant measure is locally compact. Thus the category of locally compact groups is that which is exactly suited for real analysis which rests on the existence of an invariant integral. One cannot expect to extend that aspect of locally compact groups to larger classes. This situation may be considered as very satisfactory.

The situation is different if one thinks of locally compact groups as a class of topological groups extending the class of finite dimensional real Lie groups. Indeed every locally compact group G has a Lie algebra $\mathfrak{L}(G)$ which is in general infinite dimensional, and it has an exponential function. But from the view point of Lie theory, the category of locally compact groups has two major drawbacks: —The topological abelian group underlying the Lie algebra $\mathfrak{L}(G)$ fails to be locally compact unless $\mathfrak{L}(G)$ is finite dimensional. In other words, the very Lie theory making the structure theory of locally compact groups interesting leads us outside the class. —The category of locally compact groups is not closed under the forming of products, even of copies of \mathbb{R} ; it is not closed under projective limits of projective systems of finite dimensional Lie groups, let alone under arbitrary limits. In other words, the category of locally compact groups is badly incomplete.

Let us denote the category of all (Hausdorff) topological groups and continuous group homomorphisms by TOPGR . It will turn out that the full subcategory proLIEGR of TOPGR consisting of all projective limits of finite dimensional Lie groups avoids both of these difficulties. This would perhaps not yet be a sufficient reason for advocating this category if it were not for two facts: Firstly, while not every locally compact group is a projective limit of Lie groups, every locally compact group has an open subgroup which is a projective limit of Lie groups, so that, in particular, every connected locally compact group is a projective limit of Lie groups. Secondly, the category proLIEGR is astonishingly well-behaved. Not only is it a complete category, it is closed under passing to closed subgroups and to those quotients which are complete, and it has a good Lie theory. It is perhaps a bit surprising that this class of groups has been little investigated in a systematic fashion.

We submit that a general structure theory of locally compact groups should be based on a good understanding of the category proLIEGR .

1. THE STRATEGY FOR A STRUCTURE THEORY OF LOCALLY COMPACT GROUPS

Let us discuss the lines along which a structure theory of locally compact groups itself might be organized. I believe that the basic strategy has to be a reduction to certain well established theories, namely, the reduction to the theories of

- (1) Lie groups,
- (2) compact groups, and
- (3) totally disconnected groups.

The first and second are highly developed and are very well documented in books on various levels. The third one has experienced substantial progress in the last decade through the work of George Willis and Helge Glöckner. The Willis theory proved its value by providing solutions to problems which resisted solution for some time and by providing alternative proofs to results with difficult proofs in topological dynamics.

That a reduction to these basic theories is possible is well exemplified by a series of results, practically all of them classical.

Let us begin with a reminder of two almost elementary observations:

Remark 1.1. In any topological group G , the identity component G_0 is a closed fully characteristic subgroup (i.e. one that is invariant under all continuous endomorphisms) and G/G_0 is a totally disconnected subgroup.

This is a first, although not yet very explicit reduction of the structure theory to that of connected and that of totally disconnected group. This reduction has the expected functorial and universal properties. A *profinite group* is a projective limit of finite groups (see [8], [9], [1], pp. 22, 23).

Remark 1.2. A locally compact totally disconnected group has arbitrarily small compact open subgroups. A compact totally disconnected group has arbitrarily small open normal subgroups and thus is profinite.

As a consequence it was observed at an early stage that profinite groups played a role:

Remark 1.3. A locally compact group has open subgroups H such that H/G_0 is compact.

It is still elementary that G and $H \times G/H$ are homeomorphisms where G/H is a discrete space. Topologically interesting things take place in H .

A topological group G for which the factor group G/G_0 is compact is called *almost connected*.

Now let us turn to deeper results.

One piece of information derives from a basic result proved around 1948 by Iwasawa in [6], p. 547, Theorem 11:

Iwasawa's Local Splitting Theorem. *Let G be a locally compact connected group. Then G has arbitrarily small neighborhoods which are of the form NC such that N is a compact normal subgroup and C is an open n -cell which is a local Lie group commuting elementwise with N and is such that $(n, c) \mapsto NC : N \times C \rightarrow NC$ is a homeomorphism.*

The full power of Iwasawa's Theorem became available only after Hilbert's Fifth Problem was settled a few years later by A. M. Gleason, D. Montgomery and after H. Yamabe proved that every almost connected locally compact group can be approximated by Lie groups.

In [2] the following generalisation of Iwasawa's Theorem is proved, which carries it beyond the connected case:

Theorem 1.4 *Let G be a locally compact group. Then for every identity neighborhood U there is a compact subgroup N contained in U , a (simply) connected finite dimensional Lie group L , and an open and continuous morphism $\varphi : N \times L \rightarrow G$ with discrete kernel such that $\varphi(n, 1) = n$ for all $n \in N$.*

We can no longer assert normality of N in G , but from the very statement of the theorem we see that N is normal in the open subgroup $\varphi(N \times L) \cong (N \times L)/D$ with a discrete normal subgroup D of $N \times L$. Thus we can assert in full generality that

every locally compact group is locally isomorphic to a direct product of a compact subgroup and a Lie group.

A second theorem which likewise goes back to Iwasawa is the following:

Theorem 1.5. *Let G be an almost connected locally compact group. Then there exists a maximal compact subgroup C such that the following conditions hold:*

- (i) *All maximal compact subgroups are conjugates.*
- (ii) *There are one-parameter groups $X_j: \mathbb{R} \rightarrow G$, $j = 1, \dots, n$ such that*

$$((t_1, \dots, t_n), c) \mapsto X_1(t_1) \cdots X_n(t_n)c: \mathbb{R}^n \times C \rightarrow G$$

is a homeomorphism.

As a consequence we know that every locally compact group G is homeomorphic to $\mathbb{R}^n \times C \times D$ for a compact subgroup C of G and some discrete space D . The topologist draws the conclusion that all local properties are entirely settled inside the compact subgroup C . For instance for compact groups we know that C and $C_0 \times C/C_0$ are homeomorphic (see [1], p. 541, Corollary 10.37), and more than that: if C'_0 denotes the algebraic commutator group of C_0 , then C'_0 is closed by Gotô's Theorem (see [1], p. 440, Theorem 9.2) and C is homeomorphic to $C'_0 \times C_0/C'_0 \times C/C_0$. Now C'_0 is a semisimple compact connected group whose structure is well understood and which is arcwise and locally arcwise connected (see [1], p. 441 ff.); moreover, C_0/C'_0 is isomorphic to a compact connected abelian subgroup of C_0 by the Borel-Scheerer-Hofmann Splitting Theorem (see [1], p. 469, Theorem 9.39). We can draw from Theorem 2 the following conclusion:

Theorem 1.6. *Every locally compact group is homeomorphic to a product space $\mathbb{R}^n \times S \times A \times T \times D$ where S is a semisimple compact connected subgroup of G , where A is a compact connected abelian subgroup of G , and T is a compact totally disconnected group and D is a discrete space. The group S is arcwise connected and locally connected.*

We cannot assert that T is isomorphic to a subgroup of G ; but by Dong Hoon Lee's Supplement Theorem there is a compact totally disconnected subgroup of G from which T arises as a quotient (see [1], p. 470, Theorem 9.41).

2. TOPOLOGICAL GROUPS WITH LIE ALGEBRAS

We have seen that it makes good sense to envision one main task of a structure theory of locally compact groups the reduction to the theory of finite dimensional Lie groups and to the theory of compact groups. Thus one must first find a formulation for the exact nature of the link between Lie group theory and topological group theory on a general level. This calls for a precise explanation which topological groups possess a Lie algebra and an exponential function. The space of all one parameter subgroups $X: \mathbb{R} \rightarrow G$ endowed with the topology of uniform convergence on compact sets is denoted $\mathfrak{L}(G)$. Accordingly \mathfrak{L} is a limit preserving functor from the category of topological groups to the category of pointed topological spaces. For suitably good specimen of topological groups, the assignment \mathfrak{L} has much better properties, as we shall outline in the following definition. For a real number r we set $\text{square}(r) = r^2$.

Definitions 2.1. Let G be a topological group. Then it is said that G has a Lie algebra or, equivalently, that G is a *topological group with a Lie algebra* if the following conditions hold:

- (i) For all $X, Y \in \mathfrak{L}(G)$, the following limits exist pointwise:

$$X + Y \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \left(\left(\frac{1}{n} \cdot X \right) \left(\frac{1}{n} \cdot Y \right) \right)^n, \quad (4)$$

$$[X, Y] \circ \text{square} \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \text{comm} \left(\frac{1}{n} \cdot X, \frac{1}{n} \cdot Y \right)^{n^2} \quad (5)$$

and $X + Y, [X, Y] \in \mathfrak{L}(G)$.

- (ii) *Addition* $(X, Y) \mapsto X + Y: \mathfrak{L}(G) \times \mathfrak{L}(G) \rightarrow \mathfrak{L}(G)$ and *bracket multiplication* $(X, Y) \mapsto [X, Y]: \mathfrak{L}(G) \times \mathfrak{L}(G) \rightarrow \mathfrak{L}(G)$ are continuous.

- (iii) With respect to scalar multiplication \cdot , addition $+$ and bracket multiplication $[\cdot, \cdot]$ the set $\mathfrak{L}(G)$ is a real Lie algebra.

In particular, if G has a Lie algebra, then $\mathfrak{L}(G)$ is a *topological Lie algebra*. Note that a topological group G has a Lie algebra if and only if G_0 has a Lie algebra.

We denote the full subcategory of TOPGR consisting of all connected topological groups by CONNGR, and the full subcategory of TOPGR containing all topological groups having a Lie algebra by LIEGR. The category of all topological Lie algebras with continuous Lie algebra homomorphisms is called LIEALG.

Theorem 2.2. *The category LIEGR is closed in TOPGR under the formation of all limits and passing to closed subgroups. It is therefore a complete category. The functor $\mathfrak{L}: \text{LIEGR} \rightarrow \text{LIEALG}$ is continuous, that is, preserves all limits.*

Proof. See [3]. □

For the concept of weakly complete vector spaces see [1], p. 319 ff. One way of explaining a weakly complete vector space is declaring a topological vector space as weakly complete if there is an isomorphism of topological vector spaces to some product vector space \mathbb{R}^X ; two equivalent ways of saying the same thing are firstly, that a weakly complete vector space is a topological vector space which is naturally isomorphic to the projective limit of all of its finite dimensional quotients, and, secondly, that it is the dual of a vector space in the topology of pointwise convergence.

Definition 2.3. A Lie algebra is said to be *profinite dimensional* if it is a projective limit of finite dimensional real Lie algebras. The underlying vector space of a profinite dimensional Lie algebra is a weakly complete vector space.

Using the continuity of the functor \mathfrak{L} , it is not hard to see the important fact that *every topological group G which is the limit of any projective system of finite dimensional real Lie groups has a Lie algebra, and indeed a profinite dimensional one.* As a consequence, all locally compact groups have a Lie algebra.

3. PROJECTIVE LIMITS

Projective limits are clearly fundamental to the structure theory of compact and locally compact groups. In the theory of locally compact groups it has been customary to handle projective limits of Lie groups. Usually one thinks that a topological group G is a projective limit of Lie groups if it has arbitrarily small **compact** normal subgroups N such that G/N is a finite dimensional Lie group. Such a group is necessarily locally compact. At the root of this opinion is the theory of compact groups, which reaches back to the twenties of the last century (for a recent presentation see [1]), to Iwasawa's fundamental paper of 1949 [6], and to Yamabe's article [11] in which he showed that every locally compact group G such that G/G_0 is compact is a projective limit of Lie groups *in*

this sense. This point of view was made popular through the enormously influential book by Montgomery and Zippin [7]. However, we shall deal with arbitrary *projective systems* such as

$$\{f_{jk}: G_k \rightarrow G_j \mid j \leq k, (j, k) \in J \times J\}$$

for a directed index set J and for topological groups G_j and consider the limit

$$G = \lim_{j \in J} G_j = \{(g_j)_{j \in J} \in \prod_{j \in J} G_j : (\forall j \leq k) g_j = f_{jk}(g_k)\}.$$

Any such limit is called the *projective limit* of the system. In our context, the groups G_j will be finite dimensional Lie groups. But before we turn to these groups, we record some fundamental facts on projective limits of topological groups.

Theorem 3.1. (Fundamental Theorem on Projective Limits) *Let $G = \lim_{j \in J} G_j$ be a projective limit of a projective system*

$$\mathcal{P} = \{f_{jk}: G_k \rightarrow G_j \mid (j, k) \in J \times J, j \leq k\}$$

of topological groups and let \mathcal{U}_j denote the filter of identity neighborhoods of G_j , \mathcal{U} the filter of identity neighborhoods of G , and \mathcal{N} the set $\{\ker f_j \mid j \in J\}$. Then

- (i) \mathcal{U} has a basis of identity neighborhoods $\{f_k^{-1}(U) \mid k \in J, U \in \mathcal{U}_k\}$.
- (ii) \mathcal{N} is a filter basis of closed normal subgroups converging to 1.

If $M \supseteq N$ in \mathcal{N} and if $\nu_{MN}: G/N \rightarrow G/M$ is defined by $\nu_{MN}(gN) = gM$, then

$$\{\nu_{MN}: G/N \rightarrow G/M \mid (M, N) \in \mathcal{N} \times \mathcal{N}, M \supseteq N\}$$

is a projective system of topological groups, and there is a unique isomorphism $\eta: \lim_{N \in \mathcal{N}} G/N \rightarrow G$ such that the following diagram commutes with $j \leq k$, $M = \ker f_j$, $N = \ker f_k$, and with the morphisms $f'_j: G/\ker f_j \rightarrow G_j$ induced by the limit map $f_j: G \rightarrow G_j$:

$$\begin{array}{ccccccc} \dots & G/M & \xleftarrow{\nu_{MN}} & G/N & \xleftarrow{\nu_N} & \lim_{P \in \mathcal{N}(G)} G/P & \\ & f'_j \downarrow & & \downarrow f'_k & & \downarrow \eta & \\ \dots & G_j & \xleftarrow{\quad} & G_k & \xleftarrow{f_k} & G & \end{array}$$

The limit maps ν_N are quotient morphisms.

- (iii) Assume that all bonding maps $f_{jk}: G_j \rightarrow G_k$ are quotient morphisms and that all limit maps f_j are surjective. Then the limit maps $f_j: G \rightarrow G_j$ are quotient morphisms.
- (iv) Set $H_j = \overline{f_j(G)}$ for each $j \in J$ and let $f'_{jk}: H_k \rightarrow H_j$ the morphisms induced by f_{jk} for $j \leq k$. Then

$$\{f'_{jk}: H_k \rightarrow H_j \mid (j, k) \in J \times J, j \leq k\}$$

is a projective system of topological groups and $G = \lim_{j \in J} H_j$. The limit maps $f'_j: G \rightarrow H_j$ are corestrictions of the f_j and they have dense images.

- (v) Assume that all G_j are complete, then so is G .
- (vi) Let G be a complete topological group and \mathcal{N} a filter basis of closed normal subgroups converging to the identity. Then $\gamma_G: G \rightarrow G_{\mathcal{N}}$, $\gamma(g) = (gN)_{N \in \mathcal{N}(G)}$ is an isomorphism. That is, $G \cong \lim_{N \in \mathcal{N}} G/N$.

Proof. A detailed proof is given in [3]. □

In Lie theory it is a known fact that closed subgroups of a Lie group are Lie groups again. In order to deal with such a result for projective limits of Lie groups, the following theorem is important.

Theorem 3.2. (The Closed Subgroup Theorem for Projective Limits) *Assume that \mathcal{N} is a filterbasis of closed normal subgroups of the complete topological group G and assume that $\lim \mathcal{N} = 1$. Let H be a closed subgroup of G . For $N \in \mathcal{N}$ set $H_N = \overline{HN}$. Then the following conclusions hold:*

- (i) *The isomorphism $\gamma_G: G \rightarrow \lim_{N \in \mathcal{N}} G/N$ maps H isomorphically onto $\lim_{N \in \mathcal{N}} H_N/N$.*
- (ii) *Under the present hypotheses,*

$$H \cong \lim_{N \in \mathcal{N}} H/(H \cap N) \cong \lim_{N \in \mathcal{N}} HN/N \cong \lim_{N \in \mathcal{N}} \overline{HN}/N.$$

- (iii) *The limit maps $\mu_M: \lim_{N \in \mathcal{N}} HN/N \rightarrow HM/M$, $M \in \mathcal{N}$, are quotient morphisms.*
- (iv) *The standard morphisms $H/(H \cap N) \rightarrow HN/N$ are isomorphisms of topological groups.*

Proof. For the details of the proof we must refer to [3] or [4]. □

In dealing with connected topological groups it is often useful to shift between the categories TOPGR and CONNGR .

Lemma 3.3. *If $D: J \rightarrow \text{CONNGR}$ is a diagram and $\lim^{\text{TOPGR}} D$ denotes the limit of D in TOPGR , then $(\lim^{\text{TOPGR}} D)_0$ is the limit of D in CONNGR with the restrictions of the limit maps giving the limit cone. Thus*

$$\lim^{\text{CONNGR}} D = (\lim^{\text{TOPGR}} D)_0.$$

Proof. See [4]. □

Lemma 3.4. *The functor $G \mapsto G_0$ which associates with a topological group its identity component and with a morphism $f: G \rightarrow H$ its restriction and corestriction $f_0: G_0 \rightarrow H_0$ from the category of topological groups to the category of connected topological groups is right adjoint to the inclusion functor of the latter into the former category. It preserves limits, that is if $D: J \rightarrow \text{TOPGR}$ is a diagram, and $D_0: J \rightarrow \text{CONNGR}$ is defined by $D_0(j) = D(j)_0$, then*

$$\lim D_0 = (\lim D)_0.$$

Proof. We refer to [4]. □

4. THE FUNDAMENTAL CATEGORY OF PRO-LIE GROUPS

The first definition of this principal section outlines the optimal kind of representing a topological group as a projective limit of simpler ones.

For a topological group G we let $\mathcal{N}(G)$ denote the set of all normal subgroups N of G such that G/N is a finite dimensional Lie group. Notice that $G \in \mathcal{N}(G)$, but $\{1\} \in \mathcal{N}(G)$ if and only if G is a finite dimensional Lie group. If $M, N \in \mathcal{N}(G)$ then $G/(M \cap N)$ is injectively mapped into the Lie group $G/M \times G/N$, but that does not imply that $G/(M \cap N)$ is a Lie group, and so we do not know that $M \cap N \in \mathcal{N}(G)$, that is that $\mathcal{N}(G)$ is a filter basis. We note in passing that we do have this information if G is locally compact, because then $G/(M \cap N)$ is a locally compact group without small subgroups.

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Definition 4.1. A topological group G is said to be a proto-Lie group if $\mathcal{N}(G)$ is a filter basis which converges to 1. If, in addition, G is a complete topological group, then G is called a pro-Lie group. The full subcategory of the category TOPGR of topological groups consisting of all pro-Lie groups and all morphisms of topological groups between them is called proLIEGR .

A pro-Lie group is the most lucid and desirable type of projective limit of Lie groups; indeed for any proto-Lie group the quotient maps $G/N \rightarrow G/M$, $N \subseteq M$ in $\mathcal{N}(G)$ for a projective system of finite dimensional Lie groups and

$$g \mapsto (gN)_{N \in \mathcal{N}(G)} : G \rightarrow \lim_{N \in \mathcal{N}(G)} G/N$$

is a dense embedding; if G is a pro-Lie group, then it is an isomorphism. The limit maps are none other than the quotient morphisms $G \rightarrow G/N$. By what we saw in Section 1, every almost connected locally compact group is a pro-Lie group, and every locally compact group has an open subgroup with this property. In fact every locally compact abelian group and every compact group is a pro-Lie group.

However, it is anything but clear that a projective limit of an arbitrary projective system

$$\{f_{jk} : G_k \rightarrow G_j \mid j \leq k, (j, k) \in J \times J\}$$

of finite dimensional Lie groups G_j is a pro-Lie group. Yet this indeed the case:

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Theorem 4.2.

- (i) *Every projective limit of finite dimensional Lie groups is a pro-Lie group.*
- (ii) *The category proLIEGR is closed in TOPGR under the formation of all limits and under the passing to closed subgroups. In particular, it is a complete category containing all finite dimensional Lie groups.*

- (iii) *Every quotient of a projective limit of finite dimensional Lie groups is a proto-Lie group. The group $G = \mathbb{R}^{2^{\aleph_0}}$ is a pro-Lie group which has a totally disconnected subgroup N such that G/N is incomplete and thus fails to be a pro-Lie group.*
- (iv) *Every pro-Lie group G has a Lie algebra $\mathfrak{L}(G)$ and the Lie algebra functor $\mathfrak{L}: \text{proLIEGR} \rightarrow \text{LIEALG}$ is continuous, that is, preserves all limits.*
- (v) *If $f: G \rightarrow H$ is an open proLIEGR morphism then $\mathfrak{L}(f): \mathfrak{L}(G) \rightarrow \mathfrak{L}(H)$ is surjective and open.*

Proof. The proofs are given in [3], [4], and [5]. □

The proof of (i) is harder than one thinks. If $G = \lim_{j \in J} G_j$ is a projective limit of Lie groups, by the Fundamental Theorem on Projective Limits 3.1(i), the group G is a projective limit of the quotients $G/\ker f_j$, where $f_j: G \rightarrow G_j$ denotes the limit morphism, and there is an injective morphism $G/\ker f_j \rightarrow G_j$. But that does not allow us to conclude right away that $G/\ker f_j$ is a Lie group. Extra ideas are needed, and they come from Lie theory.

The basis of the proof of (ii) consists of the easier conclusion that the category of all projective limits of Lie groups is complete, and a harder proof, based on the Closed Subgroup Theorem 3.2, which shows that a closed subgroup of a pro-Lie group is a pro-Lie group.

The proof that a quotient of a pro-Lie group is a proto-Lie group is not exactly hard, but is not trivial either. The construction of an example showing that a pro-Lie group may have incomplete quotients is much harder and is based on new results on compact abelian groups [5] which say that the exponential function of a compact connected abelian group is open onto its image if and only the group is locally connected. The image $\exp \mathfrak{L}(G)$ is the arc component G_a of the identity. There are locally connected compact connected abelian groups which fail to be arcwise connected (e.g. the character group of the discrete group \mathbb{Z}^N); if G is one of these then $\mathfrak{L}(G)$ provides us with an example of an abelian pro-Lie group with a quotient G_a that cannot be complete.

Conclusion (iv) is comparatively easy, given all of the other information and enough category theoretical background [3].

Assertion (v) again is difficult to prove. It involves the lifting of one-parameter subgroups which requires the Axiom of Choice.

Condition (v) shows us that the one-parameter subgroups of a pro-Lie group algebraically generate a dense subgroup of the identity component. A pro-Lie group G is totally disconnected iff $\mathfrak{L}(G) = \{0\}$ iff it is protodiscrete, i.e. has a filterbasis of open normal subgroups converging to 1. For any pro-Lie group, G/G_0 is protodiscrete, but we do not know whether it has to be complete. In the context of Condition (v) we recall that \mathfrak{L} preserves kernels by (iv) since a kernel is an equalizer and thus a limit.

One reason why the Fundamental Theorem is so important for the structure theory of projective limits of Lie groups is the following proposition:

Proposition 4.3. *If G is a pro-Lie group, then the topological Lie algebra $\mathfrak{L}(G)$ has a filter basis of ideals \mathfrak{i} of $\mathfrak{L}(G)$ converging to 0 such that $\mathfrak{L}(G)/\mathfrak{i}$ is finite dimensional that is, $\mathfrak{L}(G)$ is profinite dimensional.*

Proof. See [3]. □

Thus the abelian group underlying $\mathfrak{L}(G)$ is back in proLIEGR . The duality theory of weakly complete vector spaces [1] applies to $\mathfrak{L}(G)$ and shows that the adjoint module $\mathfrak{L}(G)'$ is a vector space in which every finite subset is contained in a finite dimensional $\mathfrak{L}(G)$ -submodule. This aspect of Lie theory has not even been observed for compact groups. When applied to this case this fact yields a new approach to some of the basic theorems of the structure theory of compact groups [1, 3].

5. A POSTSCRIPT

The theory of locally compact groups is a theory of the 20th century. Since we are in the 21st century, one expects some words of justification for an attempt to present it in a book now.

Firstly, a comparison of the situation of mathematics and the mathematicians in 1901 and 2001 very quickly reveals significant psychological and sociological differences. At the beginning of the 20th century mathematicians looked back on the enormous progress mathematics made in the 19th century: the entire foundations of analysis that we teach in our analysis courses for instance and no less the roots of abstract algebra stem from the 19th century;

at that time mathematicians were in a position to have a good command of everything that had been achieved and the best of them had an excellent vision on what should be accomplished in the 20th century; Hilbert's famous address to the international congress of 1900 is a much studied example of a programmatic proclamation. Accordingly, the active researchers looked forward. Our situation is more complex: Naturally we are looking forward and know that new challenges lie ahead. But we also know that an enormous quantity of mathematical insights have been created in the twentieth century—a quantity so enormous that no individual is capable of a global overview. We had to become specialists, and even in our own field it has become impossible to remain familiar with all relevant aspects. As mathematics progresses, entire areas that were active for decades fade out of focus in a natural way; teachers make sure that dissertations are written in topical areas as one moves with the tides. All of this is quite natural and commendable, but on the risk side of this state of affairs we notice that information well present and well understood tends to vanish from conscious memory of living mathematicians. Information is stored in books or other archival media, but information being present in the minds of living individuals is another matter. I think that we have an extra obligation to keep alive some of the mathematics which was mainstream mathematics in the 20th century but which may no longer be quite in the focus of the forward looking. This obligation distinguishes us from our colleagues one hundred years ago.

Secondly and specifically concerning the theory of locally compact groups, ongoing research and a general interest represented by internet inquiries indicate that information on compact and locally compact groups continues to be needed. After the friendly reception which the book [1] on compact groups, and the books on profinite groups [9] and [8] have received, one expects that there is a demand for a comprehensive source on the structure of locally compact groups. Such a source is surprisingly absent while sources on harmonic analysis and representation theory of locally compact groups abound.

Thirdly, the success of the theory of Lie groups and Lie algebras and their representations continues unabated over some 120 years and now takes a direction towards one aspect that was already Sophus Lie's original vision (too early in the 19th century to be really

successful at that time), namely infinite dimensional Lie theory. As is usual in this area, there are many aspects to infinite dimensional Lie theory. The theory of compact and locally compact groups offers one of these aspects and indeed one which has not been very overtly recognized. We hoped to demonstrate in [1] that infinite dimensional Lie theory is an important aspect even in the theory of compact groups. In this regard, the theory of locally compact groups is indeed much in line with current trends.

On the balance, therefore, an attempt to complement a source book on compact groups by one on locally compact groups appears to be justified.

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