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APPLICATIONS OF INDEPENDENT FAMILIES

WANJUN HU*

ABSTRACT. We study maximal independent families (hereafter: mifs) and their applications to topological questions. We prove that if there exists either an (ω, ω_1)-mif of size 2^{ω_1} with open density ω, or an (ω, ω_1)-mif of size $\leq 2^{\omega}$, then there exists an ω-resolvable, not maximally resolvable, Tychonoff space.

1. INTRODUCTION

Following [7], a family $\mathcal{I} \subseteq \mathcal{P}(X)$ is called a $(\theta, \kappa)-$*independent family* on X if for every two disjoint subfamilies $\{A_\alpha : \alpha < \theta_1\}$ and $\{B_\beta : \beta < \theta_2\}$ where $\theta_1, \theta_2 < \theta$, the intersection $\bigcap\{A_\alpha : \alpha < \theta_1\} \cap (\bigcap\{X \setminus B_\beta : \beta < \theta_2\}$ has size κ. Such a family is called *separated* if for every two points $s, t \in \kappa$ there exists $I \in \mathcal{I}$ such that $s \in I$ and $t \notin I$.

Let $< X, \mathcal{T} >$ be a topological space. It is *crowded* if it has no isolated points. X is called *κ-resolvable* if there exists a partition of X into κ-many dense subsets. If X has no two disjoint dense subsets, then X is called *irresolvable* [9]. Following notation in [6], if every nonempty open subspace (respectively, crowded subset) is irresolvable, then X is *open-hereditarily* (respectively, *hereditarily*) *irresolvable*, in short OHI (respectively, HI). The concept of hereditarily irresolvable spaces was first defined as SI spaces in [9]. The *dispersion character* $\Delta(X)$ is defined as $min\{|U| : U \neq \emptyset, U \in \mathcal{T}\}$. X is *maximally resolvable*

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if X is $\Delta(X)$-resolvable. The density of X is denoted by $d(X)$. The *open density* of X is denoted by $od(X)$ and is defined as $min\{d(U) : \emptyset \neq U \in \mathcal{T}\}$.

Independent families have been studied in [8], [11], [14]. In [8], it is proved that on every infinite cardinal κ, there exist an (ω, κ)-independent family of size 2^κ. In [14], K. Kunen proved that the existence of a maximal (ω_1, κ)-independent family on some cardinal $\kappa > \omega$ is equiconsistent with the existence of a measurable cardinal. A maximal (ω_1, κ)-independent family on κ induces a zero-dimensional Baire irresolvable topology on κ. (See also [15]). On the other hand, Kunen and Tall showed in [16] that it is consistent that there exists no Baire irresolvable spaces.

Independent families have also been used to construct irresolvable spaces. In [6], E.K. van Douwen used maximal independent families to construct $n-$resolvable, not $n+1$-resolvable, Tychonoff spaces. In [7], a different construction is given by F. Eckertson. In the same paper, F. Eckertson studied the relation of two concepts of maximality: a maximal (ω, κ)-independent family and a maximal independent (maximal (ω, ω)-independent) family. In [4], it is proved that for every $\kappa \geq \omega$: (1) every (ω, κ)-independent family on κ with open density κ is contained in some maximal (ω, κ)-independent family which is also maximally independent; (2) there exists a maximal (ω, κ)-independent family on κ of size 2^κ that is also maximally independent.

Independent families relate to the following question, which is asked first in [2] and then in [3]: "Is there an ω-resolvable Tychonoff space which is not maximally resolvable?" This question was discussed in several places: [7], [3]. Eckertson in [7] showed that, assuming the existence of "a crowded, HI, strong P_κ space", there exists such a space. In [10], it was proved that, assuming Luzin's Hypothesis, $2^\omega = 2^{\omega_1}$, there exists such a space of size ω_1. It is still unknown to us whether such a space exists in ZFC.

In this paper, we study maximal independent families of various sizes. We show, in section 2 and section 3, that: (1) For any infinite cardinal κ, if there is a maximal (ω, κ)-independent family (briefly, an (ω, κ)-mif) of size 2^κ with open density $< \kappa$, then there exists an ω-resolvable Tychonoff space of size κ which is not maximally resolvable; and (2) If there exists an (ω, ω_1)-mif on ω_1 of size τ such that $log(\tau) < \omega_1$, then there also exists an ω-resolvable Tychonoff space which is not maximally resolvable.

2. (ω,κ)-MIF WITH OPEN DENSITY $< \kappa$

The following result from [7] will be useful to us.

Theorem 2.1. *Let i_κ be the smallest cardinal τ such that there exists a maximal (ω,κ)-independent family of size τ on κ. Then the following hold:*

(1) *If $\kappa = log\ i_\kappa$, then every (ω,κ)-mif on κ is also maximally independent.*
(2) *If $log\ i_\kappa < \kappa$, then there exists an (ω,κ)-mif on κ which is not maximally independent.*

Each (ω,κ)-independent family $\mathcal{I}$ on a set X induces on X the topology with the subbase

$\{A_1 \cap ... \cap A_n \cap (X \setminus B_1) \cap ... \cap (X \setminus B_m) : n, m < \omega, A_i, B_j \in \mathcal{I}\}$.

We use the same symbol $\mathcal{I}$ to denote that topology. The following results were proved in [4].

Theorem 2.2.

(1) *For any cardinal λ such that $log(2^\kappa) \leq \lambda \leq \kappa$, there exists a dense hereditarily irresolvable subset of $\{0,1\}^{2^\kappa}$ of size and open density λ.*
(2) *Let $\mathcal{I}$ be a separated (ω,κ)-independent family of size τ on κ. The topology induced by $\mathcal{I}$ is homeomorphic to a dense subset of the Cantor cube $\{0,1\}^\tau$.*

Very little is known, even if ZFC is augmented with additional axioms, about the (possible) cardinalities of maximal (ω,κ)-independent families on a cardinal $\kappa \geq \omega$. For example, the following question is not known to us.

Problem 2.3. Is it true in ZFC that every (ω,κ)-mif on κ is of size 2^κ?

However, the question whether every (ω,κ)-mif of size 2^κ is also maximally independent has the following answer.

Theorem 2.4. *Let τ, κ be two infinite cardinals such that $\tau < \kappa$. The following are equivalent:*

(1) $2^\tau = 2^\kappa$.
(2) *There exists an (ω,κ)-mif $\mathcal{I}$ of size 2^κ on κ which is not maximally independent and for which od($\mathcal{I}$)$= \tau$.*

Proof. (1) $\rightarrow$ (2). By Theorem 2.2, there exists in the Cantor cube $\{0,1\}^{2^\kappa}$ an HI dense subset D_1 of size and open density κ, and an HI dense subset D_2 of size and open density τ . Since each irresolvable dense subset of size and open density λ in $\{0,1\}^{2^\kappa}$ induces a separating (ω, λ)-mif on λ, we have a separating (ω, κ)-mif $\mathcal{I}$ on the set D_1 of size 2^κ such that $od(\mathcal{I}) = \kappa$, and a separating (ω, τ)-mif $\mathcal{J}$ on the set D_2 of size 2^κ such that $od(\mathcal{J}) = \tau$.

List $\mathcal{I}_1$ as $\{I_\alpha : \alpha < 2^\kappa\}$. Choose an element $J \in \mathcal{I}_2$, and list $\mathcal{I}_2 \setminus \{J\}$ as $\{J_\alpha : \alpha < 2^\kappa\}$. We define a new independent family on $D_1 \cup D_2$ by $\mathcal{K}= \{I_\alpha \cup J_\alpha : \alpha < 2^\kappa\}$.

Obviously, $\mathcal{K}$ is an (ω, κ)-independent family of size 2^κ on the set $D_1 \cup D_2$. It is easy to check that $\mathcal{K}$ is in fact an (ω, κ)-mif.

Certainly, the family $\{J\} \cup \mathcal{K}$ is an independent family on $\kappa \cup \tau$, which implies that the family $\mathcal{K}$ is not maximally independent and $od(\mathcal{K}) \leq \tau$. Since $od(\mathcal{I})= \kappa$ and $od(\mathcal{J})= \tau$, it follows that $od(\mathcal{K})= \tau$.

(2) $\rightarrow$ (1). Since $od(\mathcal{I}) = \tau < \kappa$, there exists a non-empty basic open subset U of the space $< \kappa, \mathcal{I} >$ with a dense subset E of size $\tau = od(\mathcal{I}) < \kappa$. Since there are at least 2^κ-many distinct nonempty clopen sets inside U, the subspace topology on E has cardinality at least 2^κ. Since $|E| = \tau$, we then have $2^\kappa \leq 2^\tau$ and hence $2^\tau = 2^\kappa$. □

Corollary 2.5. *The following are equivalent.*

1. *Luzin's Hypothesis,* $2^\omega = 2^{\omega_1}$.
2. *There exists an* (ω, ω_1)*-mif* $\mathcal{I}$ *on* ω_1 *of size* 2^{ω_1} *which is not maximally independent and for which* $od(\mathcal{I}) = \omega$.

Problem 2.6. Is it true that Luzin's Hypothesis is equivalent to this statement: there exists an (ω, ω_1)-mif of size 2^ω on ω_1?

Following the notation in [5], we use the symbol $S(X)$ to denote the smallest cardinal θ such that every pairwise disjoint family of nonempty open sets of the topological space X has size less than κ. In the book [12], the same cardinal is denoted by $\hat{c}(X)$.

Let us call a space $< X, \mathcal{T} >$ *κ-condensed* if X has a family $\mathcal{K}$ of nowhere dense subsets such that $|\mathcal{K}|= \kappa$ and every nowhere dense subset of X is contained in some element of $\mathcal{K}$. It is shown in [10] that if a space $< X, \mathcal{T} >$ has weight $\leq 2^\tau$ for some cardinal τ, then it is 2^τ-condensed.

In [10], the following result was proved.

Theorem 2.7. *Let $< X, \mathcal{T} >$ be a Tychonoff space. Let τ be an infinite cardinal such that $\tau < cf(S(< X, \mathcal{T} >))$. Suppose that X is 2^τ-condensed and that X has a partition consisting of τ-many OHI dense subsets. Then $\mathcal{T}$ has a Tychonoff expansion $\mathcal{U} \supset \mathcal{T}$ with dispersion character $\geq od(< X, \mathcal{T} >) \cdot \tau$ which is τ-resolvable but not $S(< X, \mathcal{T} >)$-resolvable.*

In particular, the following theorem holds.

Theorem 2.8. *Assuming Luzin's Hypothesis. There exists an ω-resolvable Tychonoff space of size ω_1 which is not maximally resolvable.*

Hence, we have the following result.

Theorem 2.9. *If there exists an (ω, κ)-mif on κ of size 2^κ with open density $< \kappa$, then there exists an ω-resolvable Tychonoff space which is not maximally resolvable.*

3. (ω, κ)-mif of small size

As usual, for any cardinal κ, we let $log(\kappa) := min\{\theta : 2^\theta \geq \kappa\}$. Let us call an (ω, κ)-independent family $\mathcal{I}$ on κ of *small size* if $log(|\mathcal{I}|) < \kappa$. In this section, we prove that if there exists an (ω, ω_1)-mif of small size, then there exists an ω-resolvable Tychonoff space which is not maximally resolvable.

It is not known to us that whether there exists an (ω, κ)-mif of small size. In particular, we do not know whether there exists a dense irresolvable subset of size and open density ω_1 in the Cantor cube $\{0, 1\}^c$. Note that it is showed, in [13] under the assumption of Martin's Axiom, and later in [1] in ZFC, that there exist irresolvable dense subsets in $\{0, 1\}^c$ and hence an irresolvable dense subset of size ω_1 by augumenting a discrete spaces.

Lemma 3.1. *Let D be a dense irresolvable subset of $\{0, 1\}^\tau$ with $\Delta(D) = \kappa$. Then there exists a hereditarily irresolvable dense subset of $\{0, 1\}^\tau$ such that $od(D) \geq \kappa$.*

Proof. Since D is an irresolvable dense subset of $\{0, 1\}^\tau$, there is an (ω, κ)-mif $\mathcal{I}$ on D which is also maximally independent.

Since D is irresolvable, there is a non-empty open and hereditarily irresolvable subset U of D. Without loss of generality, we can assume that U is a basic open set, i.e., a set of the form $U = I_1 \cap ... \cap I_n \cap D \setminus J_1 \cap ... \cap D \setminus J_m$ for some $\{I_1, ..., I_n, J_1, ..., J_m\} \subseteq \mathcal{I}$. Certainly, $\Delta(U) = \kappa$. It remains to see that $od(U) = \kappa$.

Suppose $od(U) = \theta < \kappa$. Then for some basic open set O of U, there is a dense subset D of O such that $|D| = d(O) = \theta < \kappa$. Since the dispersion character of O is κ, the set D cannot contain any open subset of O. Hence the set $O \setminus D$ is also dense in O. This implies that O is not irresolvable, contrary to the assumption that $U \supseteq O$ is a hereditarily irresolvable space. □

From this lemma, we have the following direct corollary.

Corollary 3.2. *Let $\mathcal{I}$ be a separated (ω, κ)-independent family such that the induced topology is open-hereditarily irresolvable. Then $od(\mathcal{I})= \kappa$.*

Lemma 3.3. *Suppose $\mathcal{I}$ is an (ω, κ)-mif of size τ on κ. If $\mathcal{I}$ is not maximally independent, then there exists a cardinal $\theta < \kappa$ such that $\tau \leq 2^\theta \leq 2^\kappa$.*

Proof. Since $\mathcal{I}$ is not maximally independent, there exists a subset $D \subseteq \kappa$ such that $\{D\} \cup \mathcal{I}$ is an independent family properly containing $\mathcal{I}$. Since $\mathcal{I}$ is already maximally (ω, κ)-independent, there exists $\{I_1, ..., I_n, J_1, ..., J_m\}$ such that the set $W = I_1 \cap ... \cap I_n \cap (\kappa \setminus J_1) \cap ... \cap (\kappa \setminus J_m) \cap D$ has size θ for some cardinal $\theta < \kappa$. The fact that $\mathcal{I}$ is an independent family implies that on the subset $W \subset \kappa$, the induced topology inherited from $\mathcal{I}$ has weight τ. Since $|W| = \theta$, we have $\tau \leq 2^\theta$. □

Theorem 3.4. *Assume the existence of an (ω, ω_1)-mif of small size. Then there exists an ω-resolvable Tychonoff space which is not maximally resolvable.*

Proof. By Theorem 2.1, we know that on ω_1 there exists a separated (ω, ω_1)-mif $\mathcal{I}$ of size τ which is not maximally independent.

By Lemma 3.3, we know that there exists a cardinal $\theta < \omega_1$ such that $\tau \leq 2^\theta = 2^\omega \leq 2^{\omega_1}$. Let $D_1 \subseteq \kappa$ be such that $\{D_1\} \cup \mathcal{I}$ is an independent family properly containing $\mathcal{I}$.

Since $\mathcal{I}$ is a maximal (ω, ω_1)-independent family, the space $< \omega_1, \mathcal{I} >$ is not an ω_1-resolvable space.

We use the following process to define for $n < \omega$ a dense subset E_n of the space $< \omega_1, \mathcal{I} >$ and an independent family $\mathcal{I}_n$ on E_n.

For $n < \omega$, repeat the following process until impossible. Let $E_1 = \omega_1 \setminus D_1$. On E_1, the family $\mathcal{I}$ induces an (ω, ω_1)-mif $\mathcal{I}_1$. Certainly, $\mathcal{I}_1$ is still maximally (ω, ω_1)-independent, since E_1 is a dense subset of $< \omega_1, \mathcal{I} >$. If $\mathcal{I}_1$ is already maximally independent, then stop. If $\mathcal{I}_1$ is not maximally independent, we let $D_2 \subset E_1$ be such that $\mathcal{I}_1 \cup \{D_2\}$ is an independent family properly containing $\mathcal{I}_1$. We let $E_2 = E_1 \setminus D_2$, and let $\mathcal{I}_2$ be the independent family of $\mathcal{I}_1$ restricted to E_2. The same reason shows that $\mathcal{I}_2$ is also maximally (ω, ω_1)-independent.

If E_n and $\mathcal{I}_n$ have been defined for all $n < \omega$, then the family $\{D_n : n < \omega\}$ is a disjoint family of dense subsets of the space $< \omega_1, \mathcal{I} >$. In this case, the space $< \omega_1, \mathcal{I} >$ is an ω-resolvable Tychonoff space which is not maximally resolvable.

We will show, in the following, that if the process stops at a step n for some $n < \omega$, i.e, $\mathcal{I}_n$ is already maximally independent, then we can construct an ω-resolvable, not maximal resolvable, Tychonoff space. Clearly, in this case, the space $< E_n, \mathcal{I}_n >$ is an irresolvable Tychonoff space, and $|\mathcal{I}_n| = |\mathcal{I}| = \tau \leq 2^{\omega} \leq 2^{\omega_1}$.

By a standard result from the subject of resolvable space (see [9]), the space $< E_n, \mathcal{I}_n >$ has an basic open set $U \subset E_n$ such that U with the topology inherited from $\mathcal{I}_n$ is hereditarily irresolvable. On the set U, the family $\mathcal{I}_n$ induces an independent family $\mathcal{J}$. Hence $< U, \mathcal{J} >$ is a hereditarily irresolvable Tychonoff space with dispersion character ω_1. Certainly, $|\mathcal{J}| = |\mathcal{I}_n| = \tau$. By Theorem 2.2(2), the space $< U, \mathcal{J} >$ is homeomorphic to a dense subset D of the Cantor cube $\{0, 1\}^{\tau}$. Certainly, $|D| = |U| = |E_n| = \omega_1$. Since D with the topology inherited from $\{0, 1\}^{\tau}$ is hereditarily irresolvable and has dispersion character ω_1, we can use the argument from Lemma 3.1 to show that $od(D) = \omega_1$.

Let $< D >$ be the subgroup generated by D, and let $X = \bigcup_{n<\omega} x_n D$ be such that the infinitely many cosets $x_n < D >$ are distinct. Then X with the topology inherited from $\{0, 1\}^{\tau}$ is a Tychonoff space which is the union of ω-many HI, dense subspaces with size and open density ω_1. Since $\tau \leq 2^{\omega}$, the space X is 2^{ω}-condensed. Hence, by Theorem 2.7, the space has a Tychonoff expansion that is ω-resolvable but not maximally resolvable. $\square$

We have shown that if either there exists an (ω, ω_1)-mif of size 2^{ω_1} with open density ω, or there exists an (ω, ω_1)-mif $\mathcal{I}$ of small size, i.e. $|\mathcal{I}| \leq 2^\omega$, then there exists an ω-resolvable, not maximally resolvable, Tychonoff space of size ω_1. As to the remaining case and the relation between maximal (ω, κ)-independent families and examples of ω-resolvable but not maximally resolvable spaces, the following question remains open.

Problem 3.5. Assume that every (ω, ω_1)-mif on ω_1 is of open density ω_1 and of size τ such that $log(\tau) = \omega_1$. Is there an ω-resolvable, but not maximally resolvable, Tychonoff space?

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Department of Mathematics and Computer Science, Wesleyan University, Middletown, CT 06459, USA
E-mail address: whu01@wesleyan.edu