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ON CONTINUOUS IMAGES OF RADON-NIKODÝM COMPACT SPACES THROUGH THE METRIC CHARACTERIZATION

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ABSTRACT. Through basic properties relating to fragmentation, lower semicontinuity and evaluations along finite paths, we show a necessary and sufficient condition for the invariance of the RN compact spaces under continuous mappings. We observe a simple proof for this invariance in the case of 0-dimensional images. We apply the characterization theorem to conditions of 0-dimensionality and of metrizability of the closure of the set of nontrivial fibers.

1. INTRODUCTION AND PRELIMINARIES

The class of *Radon-Nikodým (RN) compact* spaces originated from studies in Functional Analysis 20 years ago, when they were defined as spaces homeomorphic to compact subsets of $(B^*, weak^*)$, where B^* is a dual Banach space with the Radon-Nikodým property. A Banach space X has the *Radon-Nikodým property* if for every finite measure space $(\Omega, \mathcal{F}, \mu)$ and for every X -valued measure of bounded variation $F: \mathcal{F} \rightarrow X$ which is absolutely continuous with respect to μ , there is a Bochner integrable function $G: \Omega \rightarrow X$ such that $F(A) = \int_A G d\mu$ for any $A \in \mathcal{F}$.

The RN compacts were also introduced by Reynov in [8] under the name of *compacts of RN-type*, which were defined as Hausdorff spaces homeomorphic to RN sets in dual Banach spaces. For a

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weak*-compact subset K of a dual Banach space, K is an *RN-set* if it has the property that each finite Radon measure on $(K, weak^*)$ is supported on a norm- σ -compact subset of K .

Several characterizations of RN compacts followed since then. Namioka developed in [7] a self-contained extensive treatment of these spaces, where one of the characterizations of RN compactness became the property of a compact space to admit a lower semicontinuous fragmenting metric. The notion of fragmentability was explicitly defined by Jayne and Rogers ([6]) and its application to RN compactness became an important tool in the study of these spaces by freeing them of the Banach space context.

The question which remains open is whether RN compactness is preserved by continuous images.

The present work first introduces the notions and results that are being used and further describes basic properties relating to fragmentation, lower semicontinuity and evaluations along finite paths - these properties also motivate investigations that shall be described in a work in preparation, where we relate to studies of Arvanitakis ([2]), Fabian, Heisler and Matoušková ([4]) and Reznichenko ([1]). We apply the above to the setting where an RN compact space X is mapped onto a compact Hausdorff space Y by a continuous map f and, for some compact subset K of Y , the preimage of each point of $Y \setminus K$ is a singleton. In Theorem 3.3, we give a necessary and sufficient condition for Y to be RN compact in terms of the existence of lower semicontinuous fragmenting metrics on X and K that are related in a certain way through f . This theorem is applied to obtain the result that, under our setting, Y is RN compact if K can be chosen to be 0-dimensional or metrizable.

Let us first review some definitions and results related to the subject. Our spaces are assumed Hausdorff. The unit interval is denoted by I .

Definition 1.1. Let $h: X^2 \rightarrow [0, \infty)$ be a function. For a nonempty subset A of X we define $h\text{-diam}(A) = \sup\{h(a, a') : a, a' \in A\}$ and for nonempty subsets A_1, A_2 of X we define $h\text{-dist}(A_1, A_2) = \inf\{h(a_1, a_2) : a_1 \in A_1, a_2 \in A_2\}$.

A function $h: X^2 \rightarrow [0, \infty)$ is *fragmenting* if it has the property that for any nonempty subset A of X and for any positive ε there is

an open set O such that $A \cap O \neq \emptyset$ and $h\text{-diam}(A \cap O) \leq \varepsilon$. Equivalently we may require A to be closed. A space X is *fragmentable* if there exists a fragmenting metric $\rho: X^2 \rightarrow [0, \infty)$.

Definition 1.2. Given a function $h: X^2 \rightarrow [0, \infty)$ and $A_1, A_2 \subseteq X$, we say that h *separates* A_1 and A_2 if $h\text{-dist}(A_1, A_2) > 0$. The function h is *separating* if $h^{-1}(\{0\}) = \Delta_X$ and a family of functions $\{h_i: X^2 \rightarrow [0, \infty)\}_{i \in \mathcal{I}}$ is *separating* if $\bigcap_{i \in \mathcal{I}} h_i^{-1}(\{0\}) = \Delta_X$.

Definition 1.3. Given $y_1, y_2 \in Y$, a *path* p in Y between y_1 and y_2 is a finite sequence, $p = (y_1, z_1, \dots, z_n, y_2)$ with all z_j in Y . In this case we define $|p| = n$. For a symmetric function $g: Y^2 \rightarrow [0, \infty)$ we define the *g-length* of p by

$$\ell_g(p) = g(y_1, z_1) + \sum_{j=1}^{n-1} g(z_j, z_{j+1}) + g(z_n, y_2).$$

A path p is said to *reduce to* another path q between y_1 and y_2 (relative to g) if $\ell_g(q) \leq \ell_g(p)$ and $|q| < |p|$. For $n_0 \in \mathbb{N}$ we denote by $g^{\Delta_{n_0}}$ and by g^Δ the function and respectively the pseudometric defined for $y_1, y_2 \in Y$ by

$$g^{\Delta_{n_0}}(y_1, y_2) = \inf\{\ell_g(p) : p \text{ path between } y_1 \text{ and } y_2, |p| \leq n_0\},$$

$$g^\Delta(y_1, y_2) = \inf\{\ell_g(p) : p \text{ path between } y_1 \text{ and } y_2\}.$$

Definition 1.4. For two subsets A and B of X^2 we denote by $A \circ B = \{(a, b) \in X^2 : \text{there is } c \in X \text{ such that } (a, c) \in A, (c, b) \in B\}$. We denote the n -fold composition of A by $A^{\circ n}$. If $A^{\circ 2} = A$ we call A *idempotent*.

A subset E of X^2 is an *almost neighbourhood* of Δ_X if for any nonempty $S \subseteq X$ there is an open subset O of X such that $S \cap O \neq \emptyset$ and $S^2 \cap O^2 \subseteq E$.

For a space X we let \mathfrak{U}_X be the family of all open symmetric neighbourhoods of Δ_X in X^2 .

We shall make use of following theorem from the theory of uniform spaces (consequence of 8.1.10 in [3]):

Theorem 1.5. For a sequence $\mathcal{V} = \{V_n\}_{n \in \omega}$ of members of \mathfrak{A}_X with $V_0 = X^2$ and $V_{i+1}^{\circ 3} \subseteq V_i$ for $i \in \mathbb{N}$ we let $g_{\mathcal{V}}: X^2 \rightarrow I$ be the function

$$g_{\mathcal{V}}(x, y) = \begin{cases} \frac{1}{2^n} & \text{if } (x, y) \in V_n \setminus V_{n+1} \\ 0 & \text{if } (x, y) \in \bigcap_{n \in \omega} V_n. \end{cases}$$

The pseudometric $\rho_{\mathcal{V}}$ on X defined by $\rho_{\mathcal{V}} = g_{\mathcal{V}}^{\Delta}$ is continuous with the property that

$$\{(x, y) : \rho_{\mathcal{V}}(x, y) < \frac{1}{2^{i+1}}\} \subseteq V_i \subseteq \{(x, y) : \rho_{\mathcal{V}}(x, y) \leq \frac{1}{2^i}\} \text{ for } i \in \omega.$$

The property written as Lemma 6.5 in [7] will be applied:

Lemma 1.6. Let X be a compact space and C, D closed subsets of X^2 such that $D^{\circ n} \subseteq C$. For any open neighbourhood U_C of C in X^2 there is an open neighbourhood U_D of D such that $U_D^{\circ n} \subseteq U_C$. If D is symmetric, then U_D can be chosen to be symmetric.

From the work of Ribarska [9] and Namioka [7], we emphasize the following characterizations that we shall use:

Theorem 1.7. For a space X the following are equivalent:

- (i) X is fragmentable;
- (ii) X admits a fragmenting separating function;
- (iii) There is a sequence $\{A_n\}_{n \in \mathbb{N}}$ of almost neighbourhoods of Δ_X such that $\bigcap_{n \in \mathbb{N}} A_n = \Delta_X$.

Theorem 1.8. For a compact space X the following are equivalent:

- (a) X is RN compact;
- (b) X admits a lower semicontinuous (lsc) fragmenting metric;
- (c) There is a sequence $\{C_n\}_{n \in \mathbb{N}}$ of closed almost neighbourhoods of Δ_X such that $\bigcap_{n \in \mathbb{N}} C_n = \Delta_X$ and $C_{n+1}^{\circ 2} \subseteq C_n$ for $n \in \mathbb{N}$.

A direct proof for (c) \Rightarrow (b) is implicit in the proof of Theorem 6.6 of [7] and we make it explicit in Lemma 1.10.

Definition 1.9. Let $\mathcal{D} = \{D_n\}_{n \in \omega}$ be a sequence of closed symmetric almost neighbourhoods of the diagonal Δ_X of a space X such that $D_0 = X^2$ and $D_{n+1}^{\circ 3} \subseteq D_n$ for $n \in \omega$. A sequence $\mathcal{V} = \{V_n\}_{n \in \omega}$ of elements in \mathfrak{A}_X is adapted to \mathcal{D} if $D_n \subseteq V_n$ and $V_{n+1}^{\circ 3} \subseteq V_n$ for $n \in \omega$.

Lemma 1.10. *Let $\mathcal{D} = \{D_n\}_{n \in \omega}$ be a sequence of closed symmetric almost neighbourhoods of the diagonal of a compact space X with the properties that $D_0 = X^2$, $D_{n+1}^{o3} \subseteq D_n$ for $n \in \omega$ and $\bigcap_{n \in \omega} D_n = \Delta_X$.*

We denote by $\{\mathcal{V}(s)\}_{s \in \mathcal{S}}$ the family of all the sequences of elements in \mathfrak{U}_X which are adapted to \mathcal{D} . For each $s \in \mathcal{S}$ we let $\rho_{\mathcal{V}(s)}$ be the continuous pseudometric on X defined in Theorem 1.5.

The metric $\rho_{\mathcal{D}}^{sup} : X^2 \rightarrow I$ defined by $\rho_{\mathcal{D}}^{sup} = \sup\{\rho_{\mathcal{V}(s)} : s \in \mathcal{S}\}$ is lsc fragmenting on X .

2. FRAGMENTATION, LSC, EVALUATIONS ALONG FINITE PATHS

Lemma 2.1. *a) Let $u : X^2 \rightarrow [0, \infty)$ and $v > 0$. The function $v \cdot u$ is fragmenting iff u is fragmenting and $v \cdot u$ is lsc iff u is lsc.*

b) Suppose $u_1, u_2 : X^2 \rightarrow [0, \infty)$. If u_1 and u_2 are lsc, then so are $\max(u_1, u_2)$, $\min(u_1, u_2)$, $u_1 \cdot u_2$ and $u_1 + u_2$. If u_1 and u_2 are fragmenting, then $\max(u_1, u_2)$, $\min(u_1, u_2)$, $u_1 \cdot u_2$ and $u_1 + u_2$ are fragmenting.

c) If $\{u_i\}_{i \in \mathcal{I}}$ is a pointwise bounded above family of lsc functions defined on a space X^2 , then the function $\sup\{u_i : i \in \mathcal{I}\}$ is lsc. Given $u, u' : X^2 \rightarrow [0, \infty)$ such that u is fragmenting and $u' \leq u$, then u' is also fragmenting.

Proof. a) is immediate from the definition of fragmentation and of lsc.

b) Lsc is well known inherited property for $u_1 \cdot u_2$, $u_1 + u_2$, $\max(u_1, u_2)$ and $\min(u_1, u_2)$. Fragmentation is also easy to observe: If C is a subset of X and ε a positive value, we let $\delta := \min(\frac{\varepsilon}{2}, \sqrt{\varepsilon})$. Then, as u_1 is fragmenting, there is an open set O_1 in X such that $C \cap O_1 \neq \emptyset$ and $u_1\text{-diam}(C \cap O_1) \leq \delta$.

As u_2 is fragmenting, there is an open set O_2 in X such that $(C \cap O_1) \cap O_2 \neq \emptyset$ and $u_2\text{-diam}((C \cap O_1) \cap O_2) \leq \delta$.

Then $C \cap (O_1 \cap O_2) \neq \emptyset$ and its diameter measured in terms of $u_1 \cdot u_2$, $u_1 + u_2$, $\max(u_1, u_2)$ or $\min(u_1, u_2)$ is not greater than ε .

c) It is known that lsc is closed under taking supremum and the property that u' is fragmenting is immediate from the definition. □

Notation 2.2. Let u_A and u_B be functions defined on disjoint subsets A and B of a space X .

We denote by $u_A \wedge u_B$ the function defined on $A \cup B$ by

$$u_A \wedge u_B(x) = \begin{cases} u_A(x) & \text{if } x \in A \\ u_B(x) & \text{if } x \in B. \end{cases}$$

Proposition 2.3. a) Any lsc function $u_S : S \rightarrow I$ on a subset S of a space admits a lsc extension to the closure of S , $u_{\bar{S}} : \bar{S} \rightarrow I$.

Let $u_X : X \rightarrow \mathbb{R}$ be a lsc function on a space X .

b1) If $u_K : K \rightarrow \mathbb{R}$ is a lsc function on a closed subset K of X such that $u_K \leq u_X|_K$, then $u = u_K \wedge u_X|_{X \setminus K}$ is lsc. In particular, any bounded above lsc function defined on a closed subset of X admits a lsc extension to X .

b2) If $u_G : G \rightarrow \mathbb{R}$ is a lsc function on an open subset G of X such that $u_G \geq u_X|_G$, then $v = u_G \wedge u_X|_{X \setminus G}$ is lsc. In particular, any bounded below lsc function defined on an open subset of X admits a lsc extension to X .

Proof. a) We let $u_{\bar{S}} : \bar{S} \rightarrow \mathbb{R}$ be the function given by

$$u_{\bar{S}}(x) := \sup \left\{ \inf \{ u_S(s) : s \in O_x \cap S \} : O_x \text{ open neighbourhood of } x \right\}.$$

We observe that $u_{\bar{S}}$ is a lsc function:

If $u_{\bar{S}}(x) > r$ for $x \in \bar{S}$ and $r \in \mathbb{R}$, then by the definition there is an open set O_x around x such that $\inf \{ u_S(s) : s \in O_x \cap S \} > r$ and so $u_{\bar{S}}(s) > r$ for any $s \in O_x \cap \bar{S}$, open set in \bar{S} .

To show that $u_{\bar{S}}$ is also extending u_S let s_0 be a point in S .

By the lsc of u_S , for any $n \in \mathbb{N}$ there is an open set $O_{s_0}^n$ around s_0 such that $\inf \{ u_S(s) : s \in O_{s_0}^n \cap S \} \geq u_S(s_0) - \frac{1}{n}$ and therefore $u_{\bar{S}}(s_0) \geq u_S(s_0)$.

As $s_0 \in O \cap S$ for any open set O around s_0 , we also have $u_{\bar{S}}(s_0) \leq u_S(s_0)$ and so $u_{\bar{S}}(s_0) = u_S(s_0)$.

b1) $u^{-1}((-\infty, v]) = u_K^{-1}((-\infty, v]) \cup (u_X^{-1}((-\infty, v]) \setminus K)$ for $v \in \mathbb{R}$. As $u_K \leq u_X|_K$, then $u_X^{-1}((-\infty, v]) \cap K \subseteq u_K^{-1}((-\infty, v])$ and so $u^{-1}((-\infty, v])$ is a closed subset of X .

If $u_K : K \rightarrow (-\infty, v_0]$ is lsc on K closed subset of X , then $u_K \wedge v_0|_{X \setminus K}$ is lsc on X , where v_0 is the function mapping all the points of X to v_0 .

b2) can be shown similarly to b1). □

Proposition 2.4. *Let $\{K, U\}$ be a partition of a space X into a closed and respectively open set. If $u': K^2 \rightarrow [0, \infty)$ is separating fragmenting and $u'': U^2 \rightarrow [0, \infty)$ is separating fragmenting we obtain a separating fragmenting function $u: X^2 \rightarrow [0, \infty)$ by defining*

$$u(x_1, x_2) := \begin{cases} u'(x_1, x_2) & \text{if } (x_1, x_2) \in K^2 \\ u''(x_1, x_2) & \text{if } (x_1, x_2) \in U^2 \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let C be nonempty subset of X and ε positive. If $U \cap C \neq \emptyset$, then there is an open subset O of U such that $O \cap (U \cap C) \neq \emptyset$ and $u''\text{-diam}(O \cap (U \cap C)) = u\text{-diam}((O \cap U) \cap C) \leq \varepsilon$. If $C \subseteq K$, then there is an open set O in K such that $O \cap C \neq \emptyset$ and $u'\text{-diam}(O \cap C) \leq \varepsilon$. Then $O \cup U$ is an open set in X and $u'\text{-diam}(O \cap C) = u\text{-diam}((O \cup U) \cap C) \leq \varepsilon$. The definition of u makes it separating when u' and u'' are separating. □

Proposition 2.5. *Let $f: X \rightarrow Y$ be a continuous mapping onto Y . Given $u: Y^2 \rightarrow [0, \infty)$, we define $h: X^2 \rightarrow [0, \infty)$ by*

$$h(x, x') := u(f(x), f(x')) \text{ for } (x, x') \in X^2.$$

- a1) *If u is lsc, then h is lsc.*
- a2) *If u is fragmenting, then h is fragmenting.*

Let $f: X \rightarrow Y$ be a continuous mapping onto Y . Given $u: Y^2 \rightarrow [0, \infty)$, we define $h: Y^2 \rightarrow [0, \infty)$ by

$$h(y, y') := u\text{-dist}(f^{-1}(y), f^{-1}(y')) \text{ for } (y, y') \in Y^2.$$

- b1) *If f is perfect and u is fragmenting, then h is fragmenting.*
- b2) *If f is perfect and u is lsc, then h is lsc.*
- b3) *If X^2 is countably compact and u is lsc separating, then h separates all the pairs of disjoint closed subsets of Y . In particular u itself separates all the pairs of disjoint closed subsets of X .*

Proof. a1) $h = u \circ f^2$ with u lsc and f^2 continuous.

a2) For a nonempty subset S of X and ε positive we have $f(S)$ nonempty and so there is an open set O in Y such that $O \cap f(S) \neq \emptyset$ and $u\text{-diam}(O \cap f(S)) \leq \varepsilon$. The set $f^{-1}(O)$ is open in X , with $f^{-1}(O) \cap S \neq \emptyset$. As $f(f^{-1}(O) \cap S) \subseteq O \cap f(S)$, we have $h\text{-diam}(f^{-1}(O) \cap S) \leq \varepsilon$.

b1) Let C be a closed subset of Y and ε positive.

As $f|_{f^{-1}(C)}$ is continuous with compact fibers, there is by 3.1.C.a of [3] a restriction of $f|_{f^{-1}(C)}$ to a irreducible mapping onto C , $f|_D : D \rightarrow C$, where D is a closed subset of $f^{-1}(C)$.

Having u fragmenting, there is a nonempty relatively open subset U of D such that $u\text{-diam}(U) \leq \varepsilon$.

By the irreducibility of $f|_D$, $f|_D(D \setminus U) \neq C$, so $O := C \setminus f|_D(D \setminus U) \neq \emptyset$. O is relatively open in C as f is closed and also $f|_D^{-1}(O) \subseteq U$. Then $h\text{-diam}(O) = \sup \{u\text{-dist}(f^{-1}(y), f^{-1}(y')) : y, y' \in O\} \leq \sup \{u\text{-dist}((f|_D)^{-1}(y), (f|_D)^{-1}(y')) : y, y' \in O\} \leq u\text{-diam}(U) \leq \varepsilon$.

b2) Let $y, y' \in Y$ be such that $h(y, y') = v > \varepsilon > 0$.

Then $u(x, x') > \frac{v+\varepsilon}{2}$ for any $x \in f^{-1}(y)$, $x' \in f^{-1}(y')$.

By the lsc of u , for x, x' fixed we can find neighbourhoods of x and x' , O_x and $O_{x'}$ such that $u(a, b) > \frac{v+\varepsilon}{2}$ for any $a \in O_x$, $b \in O_{x'}$.

Using the compactness of $f^{-1}(y)$ and of $f^{-1}(y')$, we find open sets $V \supseteq f^{-1}(y)$, $V' \supseteq f^{-1}(y')$ such that $u(a, b) > \frac{v+\varepsilon}{2}$ for $a \in V$, $b \in V'$.

As f is a closed mapping, we can find open sets W around y and W' around y' with the property that $f^{-1}(W) \subseteq V$ and $f^{-1}(W') \subseteq V'$. Then $u(w, w') \geq \frac{v+\varepsilon}{2} > \varepsilon$ for any $w \in W$, $w' \in W'$.

b3) The proof is by contradiction. So assume that K and K' are two closed subsets of Y such that $h\text{-dist}(K, K') = 0$.

We can find sequences $\{k_n\}_{n \in \mathbb{N}}$ and $\{k'_n\}_{n \in \mathbb{N}}$ of points from K and respectively K' such that $h(k_n, k'_n) \leq \frac{1}{n+1}$. This implies that for $n \in \mathbb{N}$ we have that $f^{-1}(k_n)$ and $f^{-1}(k'_n)$ contain points x_n and respectively x'_n with $u(x_n, x'_n) \leq \frac{1}{n}$.

By countable compactness, the set $A = \{(x_n, x'_n) : n \in \mathbb{N}\}$ has an accumulation point, $(x, x') \in f^{-1}(K) \times f^{-1}(K')$.

If $u(x, x') > \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$, then by the lsc of u there is an open set O around (x, x') such that $u(a, b) > \frac{1}{n_0}$ for any $(a, b) \in O$.

As only finitely many of the pairs of points of A are at u -distance greater than $\frac{1}{n_0}$, we conclude that $u(x, x')$ cannot be positive.

As u is separating, $u(x, x') = 0$ implies $x = x'$, so $K \cap K' \neq \emptyset$.

For the last part of b3) it suffices to consider f the identity of X . \square

Proposition 2.6. *Let Y be a space and $g: Y^2 \rightarrow [0, \infty)$ lsc function with the property that $g^\Delta = g^{\Delta_{n_0}}$ for some $n_0 \in \mathbb{N}$.*

a) *If Y^{n_0} is countably compact and g is separating, then g^Δ is a metric.*

b) *If Y is compact, then g^Δ is a lsc pseudometric.*

Proof. a) If $g^\Delta(y_1, y_2) = 0$, then for every $n \in \mathbb{N}$ there are paths $p^n = (y_1, z_1^n, z_2^n, \dots, z_{n_0}^n, y_2)$ between y_1 and y_2 such that $\ell_g(p^n) \leq \frac{1}{n}$. In Y^{n_0} the sequence $\{(z_1^n, z_2^n, \dots, z_{n_0}^n)\}_{n \in \mathbb{N}}$ has an accumulation point, $(z_1, z_2, \dots, z_{n_0})$.

By the lsc of g we then conclude that $g(y_1, z_1) = g(z_1, z_2) = \dots = g(z_{n_0}, y_2) = 0$ and g separating implies further that $y_1 = y_2$.

b) Let $\{y_1^\sigma\}_{\sigma \in \Sigma}$ and $\{y_2^\sigma\}_{\sigma \in \Sigma}$ be nets in Y converging to y_1 and respectively y_2 and ε positive such that $g^\Delta(y_1^\sigma, y_2^\sigma) \leq \varepsilon$ for $\sigma \in \Sigma$, but $g^\Delta(y_1, y_2) = v > \varepsilon$. For $\sigma \in \Sigma$ we can find paths of points in Y , $p^\sigma = (y_1^\sigma, z_1^\sigma, z_2^\sigma, \dots, z_{n_0}^\sigma, y_2^\sigma)$, such that $\ell_g(p^\sigma) \leq \varepsilon + \frac{v-\varepsilon}{3}$.

We may assume that $\{z_i^\sigma\}_{\sigma \in \Sigma}$ converges to $z_i \in Y$. By the lsc of g we have that the path $p = (y_1, z_1, \dots, z_{n_0}, y_2)$ has $\ell_g(p) \leq \varepsilon + \frac{v-\varepsilon}{3} < v = g^\Delta(y_1, y_2)$, which is impossible. \square

3. APPLICATIONS

3.1. REDUCING THE SCATTERED PART OF A SPACE

One reduction that we can apply when looking at the continuous images of RN compacts is through the scattered part of the space.

Proposition 3.1. *a) For a space X the following are equivalent:*

- i) *The 0-1 metric on X is fragmenting;*
- ii) *X is scattered;*
- iii) *The diagonal Δ_X is an almost neighbourhood of itself.*

b) If $\{S, D\}$ partitions a compact space X into the scattered subspace S and the dense in itself subspace D , then X is RN compact iff D is RN compact.

Proof. a) is immediate from definitions.

b) RN compactness is hereditary with respect to closed subsets. If D is RN compact, we let $\rho_D: D^2 \rightarrow I$ be a lsc fragmenting metric and define $\rho_X: X^2 \rightarrow I$ by

$$\rho_X(x, x') := \begin{cases} 0 & \text{if } x = x' \\ \rho_D(x, x') & \text{if } (x, x') \in D^2 \\ 1 & \text{otherwise.} \end{cases}$$

As ρ_D is fragmenting the closed set D and the 0-1 metric is fragmenting the scattered set S , we conclude by Proposition 2.4 that ρ_X is fragmenting. The lsc of ρ_X follows from b1) of Proposition 2.3. \square

Proposition 3.2. a) *If the dense in itself part D_X of a compact space X is metrizable, then X is RN compact.*

b) *If $f: X \rightarrow Y$ is continuous from X space as in a) onto Y , then Y has the dense in itself part also metrizable and therefore is RN compact.*

Proof. a) is immediate from b) of Proposition 3.1.

b) Taking a irreducible restriction, $f|_E: E \rightarrow D_Y$ of $f|_{f^{-1}(D_Y)}$, we observe that E does not contain isolated points from X , and therefore E is a closed subset of the metrizable subspace D_X of X . Then D_Y is metrizable and therefore Y is RN compact. \square

In particular we can argue that continuous images of the Alexandroff double circle are RN compacts.

3.2. CHARACTERIZING RN COMPACTNESS OF IMAGES OF RN COMPACTS THROUGH THE CLOSURE OF THE POINTS WITH NONTRIVIAL FIBERS

Theorem 3.3. *Let $f: X \rightarrow Y$ be a continuous mapping from an RN compact X onto Y . Then Y is RN compact iff there exist a closed subspace K of Y with $\{y \in Y : |f^{-1}(y)| > 1\} \subseteq K$ and lsc fragmenting metrics $\rho_X: X^2 \rightarrow I$ and $\rho_K: K^2 \rightarrow I$ with the property that*

$$\rho_K(k, k') \leq \rho_X\text{-dist}(f^{-1}(k), f^{-1}(k')) \text{ for any } (k, k') \in K^2 \quad (\star).$$

Proof. Assume first that Y is RN compact and let ρ and ρ_Y be lsc fragmenting metrics on X , respectively Y .

The pseudometric $\rho_Y \circ f^2$ is lsc fragmenting by a) of Proposition 2.5.

Let $\rho_X := \max(\rho, \rho_Y \circ f^2)$. As ρ is separating, ρ_X is a metric which, by b) of Lemma 2.1, is also lsc and fragmenting.

We take $K = Y$, $\rho_K = \rho_Y$ and then (\star) is satisfied as $\rho_K(y, y') = \rho_Y \circ f^2\text{-dist}(f^{-1}(y), f^{-1}(y')) \leq \rho_X\text{-dist}(f^{-1}(y), f^{-1}(y'))$ for $y, y' \in Y$.

Assume now that we have a closed subset K of Y containing the points with nontrivial fibers and that ρ_X and ρ_K are metrics satisfying (\star) .

The function which maps $(y, y') \in Y$ to $\rho_X\text{-dist}(f^{-1}(y), f^{-1}(y'))$ is lsc fragmenting separating by b) of Proposition 2.5.

By defining $g: Y^2 \rightarrow I$ as

$$g(y, y') := \begin{cases} \rho_K(y, y') & \text{if } (y, y') \in K^2 \\ \rho_X\text{-dist}(f^{-1}(y), f^{-1}(y')) & \text{if } (y, y') \in Y^2 \setminus K^2, \end{cases}$$

we obtain a lsc function by (\star) and b1) of Proposition 2.3.

We further consider the pseudometric $g^\Delta : Y^2 \rightarrow I$. As $g^\Delta \leq g$, g^Δ also fragments X by c) of Lemma 2.1.

We shall see that g^Δ is also a lsc metric.

For this purpose we first observe that the formula for calculating g^Δ reduces to an infimum over paths with at most two intermediate points:

We let k and y represent points of K and $Y \setminus K$ respectively.

As $g_{\uparrow K^2}$ and $g_{\uparrow (Y \setminus K)^2}$ are metrics, we have the following reduction rules:

- (1) a path of the type (k, k, k) reduces to that of (k, k) ;
- (2) a path of the type (y, y, y) reduces to that of (y, y) .

We can further reduce measurings along paths by the following

- Remark 3.4.** i) $g(y', y) + g(y, k) \geq g(y', k)$ for $y, y' \in Y \setminus K$, $k \in K$;
 ii) $g(k, y) + g(y, k') \geq g(k, k')$ for $k, k' \in K$, $y \in Y \setminus K$.

Proof. i) $g(y', y) + g(y, k) = \rho_X(f^{-1}(y'), f^{-1}(y)) + \inf\{\rho_X(f^{-1}(y), x) : x \in f^{-1}(k)\} = \inf\{\rho_X(f^{-1}(y'), f^{-1}(y)) + \rho_X(f^{-1}(y), x) : x \in f^{-1}(k)\} \geq \inf\{\rho_X(f^{-1}(y'), x) : x \in f^{-1}(k)\} = \rho_X\text{-dist}(f^{-1}(y'), f^{-1}(k)) = g(y', k)$.

ii) $g(k, y) + g(y, k') = \rho_X\text{-dist}(f^{-1}(k), f^{-1}(y)) + \rho_X\text{-dist}(f^{-1}(y), f^{-1}(k'))$. As ρ_X is lsc metric on X compact, the distance between pairs of closed sets is attained. We let a and b be two points in $f^{-1}(k)$ and respectively $f^{-1}(k')$ such that $\rho_X\text{-dist}(f^{-1}(k), f^{-1}(y)) = \rho_X(a, f^{-1}(y))$ and $\rho_X\text{-dist}(f^{-1}(y), f^{-1}(k')) = \rho_X(f^{-1}(y), b)$.

By the triangle inequality and (\star) we have $g(k, y) + g(y, k') \geq \rho_X(a, b) \geq \rho_X\text{-dist}(f^{-1}(k), f^{-1}(k')) \geq g(k, k')$. \square

The remark above adds the reduction rules:

- (3) a path of type (y, y, k) reduces to that of (y, k) ;
- (4) a path of the type (k, y, y) reduces to that of (k, y) ;
- (5) a path of the type (k, y, k) reduces to that of (k, k) .

By an easy combinatorial argument, from the reduction rules (1) to (5), we see readily that there are only 8 types of irreducible paths: (y, y) , (k, k) , (y, k) , (k, y) , (y, k, k) , (k, k, y) , (y, k, y) and (y, k, k, y) . Therefore $g^\Delta = g^{\Delta^2}$.

As g satisfies the conditions in Proposition 2.6, g^Δ is also lsc separating on Y , finally witnessing that Y is RN compact. \square

Remark 3.5. We observe that the metric g^Δ constructed above is calculated as

- i) $g^\Delta(y, y') = \min(\rho_X(y, y'), \inf\{\rho_X\text{-dist}(f^{-1}(y), f^{-1}(z_1)) + \rho_K(z_1, z_2) + \rho_X\text{-dist}(f^{-1}(z_2), f^{-1}(y')) : z_1, z_2 \in K\})$ if $(y, y') \in (Y \setminus K)^2$;
- ii) $g^\Delta(y, k) = \min(\rho_X\text{-dist}(f^{-1}(y), f^{-1}(k)), \inf\{\rho_X\text{-dist}(f^{-1}(y), f^{-1}(z)) + \rho_K(z, k) : z \in K\})$ if $(y, k) \in (Y \setminus K) \times K$;
- iii) $g^\Delta(y, y') = \rho_K(y, y')$ if $(y, y') \in K^2$.

3.3. THE CHARACTERIZATION THEOREM WITH 0-DIMENSIONALITY

It is easy to see that an extremally disconnected image of an RN compact is RN compact: arguing through irreducible mappings and the Gleason absolute, such an image is homeomorphic to a closed subspace of its preimage.

Arvanitakis [2] and Fabian, Heisler and Matouskova [4] proved the RN compactness of 0-dimensional continuous images of RN compacts. We present a proof of the result through a construction that we further apply.

Proposition 3.6. *If $f : X \rightarrow Y$ is a continuous mapping from a RN compact X onto a 0-dimensional Y , then Y is RN compact.*

Proof. For distinct points y and y' of Y we let

$$\mathcal{P}_{y, y'} = \{(A, B) : \{A, B\} \text{ clopen partition of } Y \text{ with } y \in A, y' \in B\}.$$

Given a lsc fragmenting metric $\rho_X : X^2 \rightarrow I$ we define $\rho_Y : Y^2 \rightarrow I$ by

$$\rho_Y(y, y') = \begin{cases} \sup\{\rho_X\text{-dist}(f^{-1}(A), f^{-1}(B)) : (A, B) \in \mathcal{P}_{y, y'}\} & \text{if } y \neq y' \\ 0 & \text{if } y = y'. \end{cases}$$

This defines a pseudometric in general. In particular ρ_Y is an metric due to the fact that ρ_X has positive distance between closed disjoint sets by b3) of Proposition 2.5.

For the lsc of ρ_Y let $\rho_Y(y, y') > \varepsilon$ for $y, y' \in Y$ and $\varepsilon > 0$.

There is a clopen partition $\{A_0, B_0\}$ of Y separating y and y' such that $\rho_X\text{-dist}(f^{-1}(A_0), f^{-1}(B_0)) > \varepsilon$, hence $\rho_Y(a, b) > \varepsilon$ for $(a, b) \in A_0 \times B_0$.

The function mapping $(y, y') \in Y^2$ to $\rho_X\text{-dist}(f^{-1}(y), f^{-1}(y'))$ is fragmenting as seen in b) of Proposition 2.5.

As $\rho_Y(y, y') \leq \rho_X\text{-dist}(f^{-1}(y), f^{-1}(y'))$ for $y, y' \in Y$, ρ_Y is also fragmenting. □

Proposition 3.7. *a) If $f : X \rightarrow Y$ is a continuous mapping from an RN compact space X onto Y such that $\{y \in Y : |f^{-1}(y)| > 1\}$ is included in a 0-dimensional closed subspace of Y , then Y is RN compact.*

b) Quotients collapsing finitely many closed subsets of RN compacts are RN compacts.

Proof. a) Let K be a closed 0-dimensional subspace containing the the set of points with nontrivial fibers, $\{y \in Y : |f^{-1}(y)| > 1\}$.

By Theorem 3.6 we observe that having a lsc fragmenting metric ρ_X on X we can produce a lsc fragmenting metric ρ_K on K which also has the property that $\rho_K(k, k') \leq \rho_X\text{-dist}(f^{-1}(k), f^{-1}(k'))$ for any $(k, k') \in K$. Condition (\star) is satisfied and so we can invoke Theorem 3.3 to obtain the RN compactness of Y .

b) $\{y \in Y : |f^{-1}(y)| > 1\}$ is finite and therefore closed 0-dimensional. □

3.4. THE CHARACTERIZATION THEOREM WITH METRIZABILITY

For the analogue of Proposition 3.7a in which we replace 0-dimensionality by metrizability we first prove the following:

Proposition 3.8. *Let $\rho_K : K^2 \rightarrow I$ be a continuous pseudometric on a closed subset K of a compact space X and let E_X be a closed idempotent superset of Δ_X such that $E_X \cap K^2 = \rho_K^{-1}(\{0\})$.*

There exists a continuous pseudometric ρ_ν on X such that $E_X \subseteq \rho_\nu^{-1}(\{0\})$ and $\rho_\nu^{-1}(\{0\}) \cap K^2 = \rho_K^{-1}(\{0\})$.

Proof. By the continuity of ρ_K , $\rho_K^{-1}(\{0\})$ is a closed set which is G_δ relatively to K^2 , $\rho_K^{-1}(\{0\}) = \bigcap_{n \in \omega} G_n$, where $G_0 := K^2$ and $G_n := \rho_K^{-1}([0, \frac{1}{n}))$ for $n \in \mathbb{N}$.

As E_X is closed and idempotent, for any $n \in \mathbb{N}$, for any $U \in \mathfrak{U}_X$ such that $E_X \subseteq U$ there exists a $V \in \mathfrak{U}_X$ such that $E_X \subseteq V$ and $V^{\circ n} \subseteq U$ by Lemma 1.6 .

We construct a sequence $\mathcal{V} = \{V_n\}_{n \in \mathbb{N}}$ of open symmetric neighbourhoods of Δ_X with the properties that

- a) $E_X \subseteq V_n$ for $n \in \omega$;
- b) $V_n \cap K^2 \subseteq G_n$ for $n \in \omega$;
- c) $V_{n+1}^{\circ 3} \subseteq V_n$ for $n \in \omega$.

Let $V_0 = X^2$ and assume that we defined $V_i \in \mathfrak{U}_X$ for $i < n$ such that $E_X \subseteq V_i$ and $V_i \cap K^2 \subseteq G_i$ for $i < n$ and that $V_{i+1}^{\circ 3} \subseteq V_i$ for $i < n - 1$.

As $E_X^{\circ 3} \subseteq V_{n-1}$ and $V_{n-1} \in \mathfrak{U}_X$, there is by Lemma 1.6, $W_n \in \mathfrak{U}_X$ such that $E_X \subseteq W_n$ and $W_n^{\circ 3} \subseteq V_{n-1}$.

We now let V_n be the symmetric open neighbourhood of Δ_X given by

$$V_n = (W_n \setminus K^2) \cup (W_n \cap G_n).$$

V_n has the properties that $E_X \subseteq V_n$, $V_n \cap K^2 \subseteq G_n$ and $V_n^{\circ 3} \subseteq V_{n-1}$. By Theorem 1.5, we construct a continuous pseudometric ρ_ν on X . As $E_X \subseteq V_n \subseteq \{(x_1, x_2) : \rho_\nu(x, y) \leq \frac{1}{2^n}\}$ for $n \in \mathbb{N}$, the pseudometric ρ_ν has the property that $E_X \subseteq \rho_\nu^{-1}(\{0\})$.

We also have by b) that $\rho_\nu^{-1}(\{0\}) \cap K^2 \subseteq \bigcap_{n \in \omega} G_n = \rho_K^{-1}(\{0\})$.

Therefore $\rho_\nu^{-1}(\{0\}) \cap K^2 = E_X \cap K^2 = \rho_K^{-1}(\{0\})$. □

Corollary 3.9. a) For any continuous pseudometric ρ_K defined on a closed subset K of a compact space X there exists a continuous pseudometric ρ_ν on X with the property that the pairs of points from K separated by ρ_ν are exactly the ones separated by ρ_K .

b) If X is compact with a closed metrizable subspace K , then there is a continuous metric on K which extends to a continuous pseudometric on X .

c) Let $f: X \rightarrow Y$ be a continuous mapping from the compact space X onto Y and let K be closed metrizable subspace of Y .

There exists a continuous pseudometric $\rho_Y: X^2 \rightarrow I$ such that $(f^2)^{-1}(\Delta_Y) \subseteq \rho_Y^{-1}(\{0\})$ and $\rho_Y^{-1}(\{0\}) \cap (f^2)^{-1}(K^2) = (f^2)^{-1}(\Delta_K)$. The pseudometric ρ_Y given by $\rho_Y(y_1, y_2) = \rho_Y(f^{-1}(y_1), f^{-1}(y_2))$ for $(y_1, y_2) \in Y^2$ is continuous on Y^2 and separating on K .

Proof. a) Given a continuous pseudometric ρ_K on K , we have $\rho_K^{-1}(\{0\})$ closed idempotent subset of K^2 . Then $E_X = \rho_K^{-1}(\{0\}) \cup \Delta_X$ is a closed idempotent superset of Δ_X for which we can apply Proposition 3.8.

b) If ρ_K is a continuous metric on K , then $E_X = \Delta_X$ is closed idempotent with $E_X \cap K^2 = \Delta_K = \rho_K^{-1}(\{0\})$. By Proposition 3.8 we obtain a continuous pseudometric on X whose restriction to K is a metric.

c) If $\rho_K: K^2 \rightarrow I$ is a continuous metric on K , then the pseudometric $\rho = \rho_K \circ f^2_{|(f^{-1}(K))^2}$ is continuous as composition of continuous functions, with $\rho^{-1}(\{0\}) = (f^2)^{-1}(\Delta_K)$.

For the construction of ρ_Y we make use again of Proposition 3.8. We let the idempotent set E_X be $(f^2)^{-1}(\Delta_Y)$ and then $E_X \cap (f^{-1}(K))^2 = \rho^{-1}(\{0\})$. Therefore there exists a continuous pseudometric ρ_Y on X with

$$(f^2)^{-1}(\Delta_Y) \subseteq \rho_Y^{-1}(\{0\}) \text{ and } \rho_Y^{-1}(\{0\}) \cap (f^2)^{-1}(K^2) = (f^2)^{-1}(\Delta_K).$$

As ρ_Y does not distinguish distances between points on the same fiber, $\rho_Y(y_1, y_2) = \rho_Y(f^{-1}(y_1), f^{-1}(y_2))$ for $y_1, y_2 \in Y$ inherits the triangle inequality. For k_1 and k_2 distinct points of K we have $(f^{-1}(k_1) \times f^{-1}(k_2)) \cap (f^2)^{-1}(\Delta_K) = \emptyset$ and so $\rho_Y(k_1, k_2) > 0$ by the property of ρ_Y to distinguish all the (equal) distances between points on distinct fibers in $f^{-1}(K)$.

Let y_1 and y_2 be points of Y with $\rho_Y(y_1, y_2) = v \in (\varepsilon_1, \varepsilon_2)$. For $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ we have $\rho_Y(x_1, x_2) = v \in (\varepsilon_1, \varepsilon_2)$ and using the continuity of ρ_Y and finite covers for compact fibers we find O_1 and O_2 open sets containing $f^{-1}(y_1)$ and $f^{-1}(y_2)$ such that $\rho_Y(a_1, a_2) \in (\varepsilon_1, \varepsilon_2)$ for any $a_1 \in O_1, a_2 \in O_2$.

As f is closed, we can choose open sets V_1 and V_2 around y_1 and respectively y_2 such that $f^{-1}(V_1) \subseteq O_1$ and $f^{-1}(V_2) \subseteq O_2$. By the definition of ρ_Y we obtain $\rho_Y(v_1, v_2) \in (\varepsilon_1, \varepsilon_2)$ for any $v_1 \in V_1, v_2 \in V_2$. □

Lemma 3.10. *Let $\mathcal{C} = \{C_n\}_{n \in \omega}$ be a sequence of closed symmetric almost neighbourhoods of the diagonal of a space X such that $C_0 = X^2$, $C_{n+1}^{\circ 3} \subseteq C_n$ for $n \in \omega$ and $\bigcap_{n \in \mathbb{N}} C_n = \Delta_X$.*

Let $\mathcal{V} = \{V_n\}_{n \in \omega}$ be a sequence of elements in \mathfrak{A}_X such that $V_0 = X^2$, $V_{n+1}^{\circ 3} \subseteq V_n$ for $n \in \omega$.

Then \mathcal{C} has a subsequence to which \mathcal{V} is adapted.

Proof. As $\bigcap_{n \in \omega} C_n = \Delta_X$, for every $m \in \omega$ there exists an $n(m) \in \omega$ such that $C_{n(m)} \subseteq V_m$. We can easily choose $n(0) = 0$ and $n(m)$ such that $n(m) > n(m - 1)$ for $n \in \mathbb{N}$.

The sequence \mathcal{V} is adapted to $\mathcal{C}' := \{C_{n(m)}\}_{m \in \omega}$. □

Proposition 3.11. *If $f: X \rightarrow Y$ is a continuous mapping from X RN compact space onto Y such that $\{y \in Y : |f^{-1}(y)| > 1\}$ is included in a closed metrizable subspace of Y , then Y is RN compact. In particular, if X is an RN compact space and K is a closed metrizable subspace of it, then X admits a lsc fragmenting metric whose restriction to K is a continuous metric.*

Proof. Let K be closed metrizable with $K \supseteq \{y \in Y : |f^{-1}(y)| > 1\}$. We intend to produce metrics ρ_X and ρ_K on X and respectively K which satisfy condition (\star) of Theorem 3.3.

We first use the metrizability of K to produce the continuous pseudometrics ρ_X on X and ρ_Y on Y with the properties from Corollary 3.9c.

We define $\rho_K := \rho_{Y|K^2}$, continuous metric with the property that $\rho_K(k_1, k_2) = \rho_Y(f^{-1}(k_1), f^{-1}(k_2))$ for $k_1, k_2 \in K$.

Let ρ_{RN} be lsc fragmenting metric on X and define $\rho_X := \max(\rho_Y, \rho_{RN})$. Since ρ_Y is continuous, ρ_X is lsc fragmenting metric. As $\rho_K(k_1, k_2) = \rho_Y(f^{-1}(k_1), f^{-1}(k_2)) \leq \rho_X(f^{-1}(k_1), f^{-1}(k_2))$ for $k_1, k_2 \in K$, condition (\star) is satisfied.

The lsc fragmenting metric g^Δ constructed as in Theorem 3.3 from $g: Y^2 \rightarrow I$ given by

$$g(y, y') := \begin{cases} \rho_K(y, y') & \text{if } (y, y') \in K^2, \\ \rho_X\text{-dist}(f^{-1}(y), f^{-1}(y')) & \text{if } (y, y') \in Y^2 \setminus K^2 \end{cases}$$

coincides with ρ_K on the pairs of points from K by Remark 3.5iii. The special case when f is the identity of X gives the lsc fragmenting metric g^Δ equal to the continuous metric ρ_K on the pairs of points from K . □

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