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**MINIMAL PRIME INITIALS AND  
ZERO-DIMENSIONAL SPACES**

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*Dedicated to Mel Henriksen on his 75th birthday.*

**ABSTRACT.** For commutative semigroups, prime ideals minimal among all primes containing a fixed ideal were characterized in [6], where it was shown that families of such minimal primes form zero-dimensional spaces when provided with the hull-kernel topology. Here, we consider the analogous condition for initial sets in preordered sets: it is sufficient for minimality of a prime initial set among all prime initial sets containing a fixed initial, but it is not necessary. Several examples are presented.

Let  $(X, \leq)$  be a preordered set (i.e.,  $\leq$  is reflexive and transitive but anti-symmetry may fail:  $a \leq b$ ,  $b \leq a$ , and  $a \neq b$  can happen). A subset  $I$  of  $X$  is an **initial set** if  $a \leq b \in I \implies a \in I$ . Note that the empty set is initial and that the intersection of any family of initial sets is initial. The initial set determined by  $a$  is  $\downarrow a = \{x \in X : x \leq a\}$ , the **final set** determined by  $a$  is  $\uparrow a = \{x \in X : x \geq a\}$ . In [3], we defined a **prime** initial set as a proper initial set  $P$  which satisfied the condition:  $x, y \notin P \implies (\downarrow x) \cap (\downarrow y) \not\subseteq P$ ; equivalently,  $I, J$  initial with  $I \cap J \subseteq P \implies I \subseteq P$  or  $J \subseteq P$ . For a given initial set  $I$ , an initial set  $P$  is said to **belong** to  $I$  if  $I \subseteq P$ .

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Initial and final sets have been used by several authors under several different names; see, for example, [1], [2], and [7].

Minimal prime ideals belonging to an ideal  $I$  in a commutative semigroup  $X$  were characterized in [6]; in case  $X$  is a meet semilattice, the notions “ideal” and “initial set” coincide, and that characterization reads:

**Theorem 1.** *Suppose  $I$  is an initial set in  $X$ . A prime  $(x, y \notin P \implies x \wedge y \notin P)$  initial set  $P$  belonging to  $I$  is a minimal prime initial belonging to  $I$  (i.e., is minimal in the family of prime initials containing  $I$ ) if and only if*

$$x \in P \implies \exists y \notin P \text{ with } x \wedge y \in I.$$

In terms of the order, this condition reads:

$$(0.1) \quad x \in P \implies \exists y \notin P \text{ with } (\downarrow x) \cap (\downarrow y) \subseteq I.$$

We ask if this result holds in preordered sets, finding that condition 0.1 implies that  $P$  is a minimal prime initial belonging to  $I$ , but that the converse fails. This positive result is easily proved.

**Proposition 1.** *Suppose  $I$  is an initial set in the preordered set  $X$  and that  $P$  is a prime initial belonging to  $I$ . Then  $P$  is a minimal prime initial belonging to  $I$  if it satisfies condition 0.1.*

*Proof.* Suppose  $x \in P \setminus I$ . Then there is  $y \notin P$  with  $(\downarrow x) \cap (\downarrow y) \subseteq I$ , so no initial between  $I$  and  $P$  that misses  $x$  can be prime.  $\square$

Examples displaying the variety of modes of failure of the converse are presented in the last two sections.

When a collection of minimal prime ideals in a commutative semigroup is provided with the hull-kernel topology, the result is a zero-dimensional space. More, the usual base for this topology consists of clopen sets. We show that this phenomenon depends upon condition 0.1, and fails for prime initial sets that do not satisfy this condition. Again, examples illustrating this failure are presented in the last two sections.

**Note:** We will express all results in terms of preordered sets and families of initial sets belonging to a given initial. To be sure, each such setting can be reduced to initial sets belonging to  $\{0\}$  in an ordered set having smallest element  $0$ , and our results can be proved there. But the resulting increase

in efficiency would be minor, and the resulting generality of outlook is often of use (e.g., for preorderings of semigroups in [4], and in any case where the focus is on a given family  $\mathcal{X}$  of initials, which is then viewed as a family of initials belonging to the initial  $\bigcap \mathcal{X}$ ).

For each  $a \in X$ , set

$$P^a = X \setminus (\uparrow a).$$

$P^a$  is a prime initial not containing  $a$ . It follows that

- Every proper initial set is an intersection of prime initials.
- More:
- If  $I$  is initial, then  $I$  is an intersection of minimal prime initials belonging to  $I$ .

This follows from

**Proposition 2.** *If  $I$  is initial and  $I \subseteq P$ , a prime initial, then there is a minimal prime initial between  $I$  and  $P$ .*

*Proof.* A standard Zorn’s Lemma argument, as on p.6 in [5]. □

### 1. THE MAIN RESULTS

Now suppose  $\mathcal{P}$  is a family of initial sets in the preordered set  $X$ . For  $A \subseteq X$ : the **hull** of  $A$  (in  $\mathcal{P}$ ) is  $\mathcal{P}(A) = \{P \in \mathcal{P} : A \subseteq P\}$ . For  $\mathcal{S} \subseteq \mathcal{P}$ : the **kernel** of  $\mathcal{S}$  is  $k(\mathcal{S}) = \bigcap \mathcal{S}$ .

The map  $2^{\mathcal{P}} \ni \mathcal{S} \rightarrow \mathcal{P}(k(\mathcal{S})) \in 2^{\mathcal{P}}$  is a closure operator on  $\mathcal{P}$  (i.e.,  $\iota$ )  $\mathcal{S} \subseteq cl\mathcal{S}$ ,  $\iota$ )  $cl\mathcal{S} = cl(cl\mathcal{S})$ ,  $\iota$ )  $\mathcal{S} \subseteq \mathcal{T} \implies cl\mathcal{S} \subseteq cl\mathcal{T}$ ). The following result is a consequence of the proof of Theorem 1 in [3].

**Lemma 1.** *This is a topological (i.e., Kuratowski) closure operator when  $\mathcal{P}$  consists of prime initials.*

*Proof.* Kuratowski requires that  $cl\mathcal{S} \cup cl\mathcal{T} = cl(\mathcal{S} \cup \mathcal{T})$  for all  $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}$ . It is always true that the left-hand side, which is  $\mathcal{P}(k\mathcal{S}) \cup \mathcal{P}(k\mathcal{T})$ , is included in the right-hand side,  $\mathcal{P}(k(\mathcal{S} \cup \mathcal{T}))$ . If  $Q \in cl(\mathcal{S} \cup \mathcal{T}) = \mathcal{P}(k(\mathcal{S} \cup \mathcal{T})) = \mathcal{P}(k\mathcal{S} \cap k\mathcal{T})$ , then  $Q \supseteq k\mathcal{S} \cap k\mathcal{T}$ . If  $Q$  is prime, then  $k\mathcal{S} \subseteq Q$  or  $k\mathcal{T} \subseteq Q$ : i.e.,

$$Q \in \mathcal{P}(k\mathcal{S}) \cup \mathcal{P}(k\mathcal{T}) = cl\mathcal{S} \cup cl\mathcal{T}. \quad \square$$

Write  $x \perp_{\mathcal{P}} y$  if  $(\downarrow x) \cap (\downarrow y) \subseteq k(\mathcal{P})$ . (We will suppress the use of the subscript when no confusion can result.) For  $A \subseteq X$ , set  $A^{\perp} = \{y \in X : x \perp y \text{ for each } x \in A\}$ .

**Lemma 2.**  $x \perp y$  iff  $\mathcal{P}(x) \cup \mathcal{P}(y) = \mathcal{P}$ .

*Proof.* Suppose  $x \perp y$  and  $P \in \mathcal{P} \setminus \mathcal{P}(x)$ . Then,  $x \notin P$ ; but  $(\downarrow x) \cap (\downarrow y) \subseteq k(\mathcal{P}) \subseteq P$  and  $P$  is prime, so it follows that  $y \in P$ . Thus,  $P \in \mathcal{P}(y)$ . Conversely, suppose  $\mathcal{P}(x) \cup \mathcal{P}(y) = \mathcal{P}$ . Then  $k(\mathcal{P}) = k(\mathcal{P}(x) \cup \mathcal{P}(y)) = k(\mathcal{P}(x)) \cap k(\mathcal{P}(y)) \supseteq (\downarrow x) \cap (\downarrow y)$ ; i.e.,  $x \perp y$ .  $\square$

**Corollary 1.**  $x^\perp = k(\mathcal{P} \setminus \mathcal{P}(x))$ .

**Theorem 2.** Let  $(X, \leq)$  be a preordered set and  $\mathcal{P}$  a family of prime initial sets in  $X$ .

- (1) For  $x \in X$ , the following are equivalent:
- $\mathcal{P}(x^\perp) = \mathcal{P} \setminus \mathcal{P}(x)$ ;
  - $\mathcal{P} \setminus \mathcal{P}(x)$  is clopen in the hull-kernel topology on  $\mathcal{P}$ .
- (2) For  $P \in \mathcal{P}$ , the following are equivalent:
- $\mathcal{P}(x^\perp) = \mathcal{P} \setminus \mathcal{P}(x)$  for each  $x \in P$ ;
  - $\mathcal{P} \setminus \mathcal{P}(x)$  is clopen in the hull-kernel topology on  $\mathcal{P}$  for each  $x \in X$ .

Each of these conditions implies that  $P$  satisfies condition 0.1, so is a minimal prime initial belonging to  $k(\mathcal{P})$ .

- (3) The following are equivalent:
- Each member of  $\mathcal{P}$  satisfies condition 0.1;
  - $\mathcal{P}(x^\perp) = \mathcal{P} \setminus \mathcal{P}(x)$  for each  $x \in X$ ;
  - $\{\mathcal{P} \setminus \mathcal{P}(x) : x \in X\}$  is a clopen base for the hull-kernel topology on  $\mathcal{P}$ .

Each of these conditions implies that each member of  $\mathcal{P}$  is a minimal prime initial belonging to  $k(\mathcal{P})$ .

*Proof.* 1.  $\mathcal{P} \setminus \mathcal{P}(x)$  is a basic open set in the hull-kernel topology on  $\mathcal{P}$ , so it is clopen if and only if

$$\mathcal{P} \setminus \mathcal{P}(x) = cl_{\mathcal{P}}(\mathcal{P} \setminus \mathcal{P}(x)) = \mathcal{P}(k(\mathcal{P} \setminus \mathcal{P}(x))) = \mathcal{P}(x^\perp),$$

by Corollary 1.  $\square$

2. The equivalence of the two bulleted statements is part 1. If  $x \in P$  and  $\mathcal{P}(x^\perp) = \mathcal{P} \setminus \mathcal{P}(x)$ , then  $P \notin \mathcal{P} \setminus \mathcal{P}(x) = \mathcal{P}(x^\perp)$ :  $x^\perp \not\subseteq P$ . Thus condition 0.1 holds for  $P$ ; apply Proposition 1.

3. By Lemma 2, it is always true that  $\mathcal{P}(x^\perp) \supseteq \mathcal{P} \setminus \mathcal{P}(x)$ . If each member of  $\mathcal{P}$  satisfies condition 0.1, then the reverse inclusion also holds:  $x^\perp \subseteq P \in \mathcal{P} \implies x \notin P$ . The other implications here are merely a slight re-casting of those of part 2.

2. EXAMPLE I

We here present examples showing that Theorem 1, which is valid for meet semilattices, fails for (pre)ordered sets, and that the last of the three equivalent conditions in Theorem 2.3 cannot be weakened to “the hull-kernel topology is zero-dimensional.” The examples are found by choosing various initial sets in a single ordered set  $X$  and considering the spaces of minimal prime initials belonging to these initials; obtaining, among others: copies of the real line with its usual topology, homeomorphs of disjoint unions of unit intervals, discrete spaces of varying sizes, and a zero-dimensional space not all of whose points are prime initial sets satisfying condition 0.1.

Let  $X$  denote the plane,  $\mathbf{R} \times \mathbf{R}$ , provided with the *strict product order*:  $(x_1, y_1) \leq (x_2, y_2)$  if  $(x_1, y_1) = (x_2, y_2)$  or if  $x_1 < x_2$  and  $y_1 < y_2$ .

In  $X$ ,

$$\uparrow(x, y) = \{(u, v) : u > x \text{ and } v > y\} \cup \{(x, y)\},$$

the open “upper right-hand quadrant” determined by the point  $(x, y)$ , together with the point  $(x, y)$ .

Similarly,

$$\downarrow(x, y) = \{(u, v) : u < x \text{ and } v < y\} \cup \{(x, y)\},$$

and

$$P^{(x,y)} = \{(u, v) : u \leq x\} \cup \{(u, v) : v \leq y\} \setminus \{(x, y)\}.$$

For  $(x, y) \in X$ , set

$$Q^{(x,y)} = P^{(x,y)} \cup \{(x, y)\}.$$

It is clear that each  $Q^{(x,y)}$  is a prime initial. Finally, note that

$$P^{(z, -\infty)} = \{(u, v) : u \leq z\}$$

and

$$P^{(-\infty, z)} = \{(u, v) : v \leq z\}$$

are prime initials for each  $z \in \mathbf{R}$ . We show that the foregoing names all of the prime initials in  $X$ . To that end, suppose  $P$  is a prime initial in  $X$  and define  $f : \mathbf{R} \rightarrow \mathbf{R} \cup \{\pm\infty\}$  by

$$f(x) = \begin{cases} -\infty, & \text{if } \{y : (x, y) \in P\} = \emptyset \\ \sup \{y : (x, y) \in P\}, & \text{otherwise.} \end{cases}$$

Note that there is at least one  $x \in \mathbf{R}$  with  $f(x) < +\infty$ , since  $P$  is proper.

**Lemma 3.** *Suppose  $f(x_1) = t \in \mathbf{R}$ . Then*

$$\downarrow(x_1, t) \setminus (x_1, t) \subseteq P$$

and, for  $x > x_1$ :

- (1)  $f(x) = t$ , and
- (2)  $(x, t) \in P$ .

*Proof.* If  $(x_1, t) \in P$ , then certainly  $\downarrow(x_1, t) \setminus (x_1, t) \subseteq P$ . If not, there is a sequence  $\{\delta_n : n \in \mathbf{N}\}$  of real numbers with  $0 < \delta_n < \frac{1}{n}$  and  $(x_1, t - \delta_n) \in P$ . Clearly,

$$\downarrow(x_1, t) \setminus (x_1, t) \subseteq \bigcup \{\downarrow(x_1, t - \delta_n) : n \in \mathbf{N}\} \subseteq P.$$

1. For any  $\varepsilon > 0$ ,  $(x_1, t + \frac{\varepsilon}{2}) \notin P$ , but  $(x_1, t + \frac{\varepsilon}{2}) \in \downarrow(x, t + \varepsilon)$  so that  $(x, t + \varepsilon) \notin P$ , so  $f(x) \leq t$ . Also, we know that  $(x_1, t + \varepsilon) \notin P$ , and

$$(\downarrow(x, t - \varepsilon)) \cap (\downarrow(x_1, t + \varepsilon)) \subseteq \downarrow(x_1, t) \setminus (x_1, t) \subseteq P,$$

so  $(x, t - \varepsilon) \in P$ ; i.e.,  $f(x) \geq t$ .

2. If  $\varepsilon > 0$ , then  $(x_1, t + \varepsilon) \notin P$  while  $(\downarrow(x_1, t + \varepsilon)) \cap (\downarrow(x, t)) \subseteq \downarrow(x_1, t) \setminus (x_1, t) \subseteq P$ , so  $(x, t) \in P$ .  $\square$

**Theorem 3.** *If  $P$  is a prime initial set in  $X$ , then either  $P = P^{(x,y)}$ , for some  $-\infty \leq x, y < +\infty$  (except that  $x = -\infty = y$  is not a possibility) or  $P = Q^{(x,y)}$  for some  $(x, y) \in \mathbf{R} \times \mathbf{R}$ .*

*Proof.* Suppose that  $f(x_1) = t \in \mathbf{R}$  for some  $x_1 \in \mathbf{R}$ . Set

$$s = \inf \{x : f(x) = t\}.$$

By the Lemma, we know that  $x > s \implies f(x) = t$ . Then, either

- (1)  $s = -\infty$ , in which case  $P = P^{(-\infty, t)}$ , or
- (2)  $s \in \mathbf{R}$ . In this case, define

$$g(y) = \sup \{x : (x, y) \in P\}.$$

We know that  $g(y) = +\infty$ , for  $y \leq t$  and  $g(y) < +\infty$  otherwise. Reasoning as above, we conclude that there is  $s_0 \in \mathbf{R}$  with  $g(y) = s_0$  and  $(s_0, y) \in P$  for all  $y > t$ . It must be that  $s_0 = s$ : anything else would be contradictory.

Thus,

$$P \supseteq (\{(x, y) : x \leq s\} \cup \{(x, y) : y \leq t\}) \setminus \{(s, t)\} = P^{(s,t)}$$

and

$$P \subseteq Q^{(s,t)},$$

so  $P = P^{(s,t)}$  or  $P = Q^{(s,t)}$ , depending upon the absence or presence of  $(s, t)$  in  $P$ .

One situation remains: that in which  $f(x) \notin \mathbf{R}$  for any  $x$ . But then, utilization of the function  $g$  defined above leads one to conclude that  $P = P^{(s, -\infty)}$  for some  $s$ .  $\square$

**Example 1.** Let  $I = \{(x, y) \in X : x + y \leq 0\}$ , and let  $\mathcal{M}$  denote the space of minimal prime initials belonging to  $I$  (with the hull-kernel topology).  $\mathcal{M}$  consists of the initials  $Q^{(x, -x)}$  for  $x \in \mathbf{R}$ . Thus,

$$\mathcal{M} \ni Q^{(x, -x)} \longmapsto x \in \mathbf{R}$$

is a bijection. It is, in fact, a homeomorphism (where  $\mathbf{R}$  carries the usual topology).

*Proof.* The basic open sets in  $\mathcal{M}$  have the form

$$\mathcal{M} \setminus \mathcal{M}(a) = \{Q^{(x, -x)} : a \notin Q^{(x, -x)}\}$$

for each  $a \in X \setminus I$ , and each of these maps onto an open interval. E.g., for  $a = (3, -1)$ ,

$$\mathcal{M} \setminus \mathcal{M}(a) = \{Q^{(x, -x)} : 1 < x < 3\} \longrightarrow (1, 3). \quad \square$$

**Remark 1.** A simple geometric sketch convinces that none of the  $Q^{(x, -x)}$  satisfies condition 0.1 for this  $I$  - the failure occurring along the “edges” of  $Q^{(x, -x)}$  (i.e.,  $\{(x, v) : v > -x\}$  and  $\{(u, -x) : u > x\}$ ).

**Example 2.** Let  $S$  be a subset of the (strictly) positive  $y$ -axis ( $S \subseteq \{(0, y) : y > 0\}$ ), and set  $I = Q^{(0,0)} \setminus S$  and let  $\mathcal{M}$  denote the space of minimal prime ideals belonging to  $I$ .

- (1) If  $S = \emptyset$ , then  $I$  is prime, and  $\mathcal{M}$  is the one-point space.
- (2) Otherwise,  $\mathcal{M} = \{Q^{(0,0)}\} \cup \{P^{(0,y)} : (0, y) \in S\}$ . For each  $(0, y) \in S$ ,  $\mathcal{M} \setminus \mathcal{M}((0, y)) = \{P \in \mathcal{M} : (0, y) \notin P\} = \{P^{(0,y)}\}$ : i.e.,  $P^{(0,y)}$  is isolated.



- (3) If  $(0, 0)$  does not lie in the closure of  $S$ , then  $Q^{(0,0)}$  is also isolated, so we can in this way display examples of initial sets whose corresponding spaces of minimal prime initials are discrete and of size any cardinal up to and including that of the continuum.
- (4) If  $(0, 0)$  does lie in the closure of  $S$ , the result is a discrete space plus a single accumulation point.

**Example 3.** Let  $I = \{(x, y) : x + y < 0\} \cup \{(x, -x) : x \in \mathbf{Q}\}$ , where  $\mathbf{Q}$  denotes the rationals. The space of minimal prime initials belonging to  $I$  is

$$\mathcal{M} = \left\{ P^{(x,-x)} : x \in \mathbf{R} \setminus \mathbf{Q} \right\} \cup \left\{ Q^{(x,-x)} : x \in \mathbf{Q} \right\}.$$

In the hull-kernel topology, the members of the first set are isolated:

$$\mathcal{M} \setminus \mathcal{M}(x, -x) = \left\{ P^{(x,-x)} \right\}$$

for each irrational  $x$ . On the other hand, if  $y > -x$ ,

$$\begin{aligned} \mathcal{M} \setminus \mathcal{M}(x, y) &= \left\{ P^{(u,-u)} : -y < u < x, u \in \mathbf{R} \setminus \mathbf{Q} \right\} \\ &\cup \left\{ Q^{(u,-u)} : -y < u < x, u \in \mathbf{Q} \right\}. \end{aligned}$$

This displays the topological structure of  $\mathcal{M}$ : it has an open dense uncountable discrete subspace whose complement is a copy of  $\mathbf{Q}$  in its usual topology. Thus,  $\mathcal{M}$  is zero dimensional, but not all of the members of the usual base for the hull-kernel topology on  $\mathcal{M}$  are clopen (i.e., not all of the members of  $\mathcal{M}$  satisfy condition 0.1 -in fact, none of the  $Q^{(x,-x)}$ , for  $x \in \mathbf{Q}$  do).

**Remark 2.** By combining features and techniques from the preceding examples, other spaces of prime initials minimal among those belonging to a specified initial set can be generated. Two general observations:

- (1) Any line with negative slope could have been used for Example 1.
- (2) Example 2 could as easily have been constructed using a subset of the positive  $x$ -axis.

**The examples:**

- (1) Let  $I = P^{(0,-\infty)} \cup \{(x, y) : x + y \leq 0\}$ . The space of minimal prime ideals belonging to  $I$  is a homeomorph of a half-closed interval.

(2) Let  $I = Q^{(0,0)} \cup \{(x, y) : x + y \leq 1\}$ . Here, the space is a homeomorph of a closed interval.

(3)  $I = P^{(0,-\infty)} \cup \{(x, y) : x + y \leq 1, 0 \leq x \leq 1\}$   
 $\cup \{(x, y) : x + y \leq 0, x > 1\}$

yields the disjoint union of two intervals: in this case, each is half-open. By removing the point  $(1, 0)$  from  $I$ , one adds an isolated point; by inserting a horizontal edge between the slanted edges, the first of the intervals becomes closed and the second open.

(4) It is clear how one goes about creating an example where the space in question is the disjoint union of an infinite sequence of intervals and it is a straightforward exercise to create an example for which the space is a

homeomorph of the subspace  $\bigcup_{n=1}^{\infty} \left\{ \left( \frac{1}{n}, y \right) : 0 \leq y < \frac{1}{n} \right\}$

of the plane or of that subspace with  $(0, 0)$  adjoined.

### 3. EXAMPLE II

We present a procedure for producing topologically specified examples of spaces of prime initials which is probably not new. It is employed to yield a zero-dimensional space of prime initials containing no minimal primes.

Let  $\mathcal{Y}$  be a topological space and let  $\mathfrak{B}$  be a base for the open sets in  $\mathcal{Y}$ . We assume that  $\emptyset \in \mathfrak{B}$ .

For each  $\alpha \in \mathcal{Y}$ , let  $P_\alpha = \{\mathcal{C} \in \mathfrak{B} : \alpha \notin \mathcal{C}\}$ .

**Lemma 4.**

- (1)  $P_\alpha$  is a prime initial set in the ordered set  $(\mathfrak{B}, \subseteq)$ .
- (2) If each member of  $\mathfrak{B}$  is also closed, then each  $P_\alpha$  is a minimal prime initial in  $\mathfrak{B}$ .

*Proof.* That each  $P_\alpha$  is initial is clear. If  $\mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{B} \setminus P_\alpha$ , then  $\alpha \in \mathcal{C}_1 \cap \mathcal{C}_2$ , which is an open set containing  $\alpha$  so there is a member  $\mathcal{C}$  of  $\mathfrak{B}$  with  $\alpha \in \mathcal{C} \subseteq \mathcal{C}_1 \cap \mathcal{C}_2$ :  $P_\alpha$  is prime.

Now suppose that  $\mathcal{C} \in P_\alpha$  and that  $\mathcal{C}$  is closed. Then  $\mathcal{Y} \setminus \mathcal{C}$  is an open set containing  $\alpha$ , so there is  $\mathcal{D} \in \mathfrak{B}$  with  $\alpha \in \mathcal{D} \subseteq \mathcal{Y} \setminus \mathcal{C}$ . Thus,  $\mathcal{C} \cap \mathcal{D} = \emptyset$ : no initial set contained in  $P_\alpha$  and missing  $\mathcal{C}$  can be prime.  $\square$

**Theorem 4.**

- (1) If  $\mathcal{Y}$  is a  $T_0$ -space, then there is an ordered set  $X$  with smallest element  $0$  and a collection  $\mathcal{P}$  of prime initial sets in  $X$  having  $k(\mathcal{P}) = \{0\}$  such that when  $\mathcal{P}$  is provided with the hull-kernel topology it is homeomorphic to  $\mathcal{Y}$ .
- (2) If, in addition,  $\mathcal{Y}$  is zero-dimensional, then  $X$  and  $\mathcal{P}$  can be chosen so that  $\mathcal{P}$  consists of minimal prime initials.

*Proof.* If  $\mathfrak{B}$  is a base for the topology on  $\mathcal{Y}$ , let  $X$  be the ordered set  $(\mathfrak{B}, \subseteq)$  and set

$$\mathcal{P} = \{P_\alpha : \alpha \in \mathcal{Y}\}.$$

The map  $P : \mathcal{Y} \ni \alpha \mapsto P_\alpha \in \mathcal{P}$  is one-to-one and onto and, for  $\mathcal{C} \in \mathfrak{B}$ ,

$$P_{\mathcal{C}} = \{P_\alpha : \alpha \in \mathcal{C}\} = \{P \in \mathcal{P} : \mathcal{C} \notin P\} = \mathcal{P} \setminus \mathcal{P}(\mathcal{C}).$$

I.e.,  $P$  maps the base  $\mathfrak{B}$  for the topology on  $\mathcal{Y}$  onto the base  $\{\mathcal{P} \setminus \mathcal{P}(\mathcal{C}) : \mathcal{C} \in \mathfrak{B}\}$  for the open sets in the hull-kernel topology on  $\mathcal{P}$ .  $\square$

**Remark 3.** In general, the ordered sets obtained in this way from a given space using different bases can be quite different.

Consider the space  $\mathbf{Q}$  of rational numbers with the usual topology, and bases

$$\begin{aligned} \mathfrak{B}_1 &= \{(a, b) : a, b \text{ irrational, } a < b\}, \\ \mathfrak{B}_2 &= \{(r - \varepsilon_n, r + \varepsilon_n) : r \in \mathbf{Q}\}, \end{aligned}$$

where  $\{\varepsilon_n\}_{n \in \mathbf{N}}$  is a fixed decreasing sequence of irrationals with  $\lim \varepsilon_n = 0$ . Each is a clopen basis for the open sets in  $\mathbf{Q}$ . These yield ordered sets  $X_1, X_2$ ; the corresponding spaces of (minimal) prime ideals are each homeomorphic to  $\mathbf{Q}$ , even though  $X_2$  is countable while  $X_1$  is not.

**Remark 4.** In general, this process does not yield all minimal prime initials.

In  $\mathbf{Q}$ , with  $\mathfrak{B}_1$  as above, let

$$P = \{\mathcal{C} \in \mathfrak{B}_1 : \delta \notin \mathcal{C}\},$$

where  $\delta$  is some fixed irrational. Then:

- (1)  $P$  is initial.

- (2)  $P_\alpha \not\subseteq P$  for any  $\alpha \in \mathbf{Q}$ .
- (3)  $P$  is prime.

*Proof.* For, suppose  $\mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{B} \setminus P$ . Then  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are open intervals with irrational endpoints containing  $\delta$ , so their intersection is another such: it is in  $\mathfrak{B}_1$  - and it contains  $\delta$ . Hence,  $\mathcal{C}_1 \cap \mathcal{C}_2 \in \mathfrak{B} \setminus P$ . □

- (4)  $P$  contains minimal prime initials, none of which are in  $\mathcal{P}$ .

**Remark 5.** A zero dimensional space of prime initials need not consist of minimal primes.

Consider  $\mathbf{Q}$  with base

$$\mathfrak{B}_3 = \{(a, b) : a, b \text{ rational, } a < b\}.$$

This provides a zero dimensional space of prime initials in which none of the standard hull-kernel basic open sets is closed. (That the prime initials in question are not minimal primes is readily seen: for any  $a \in \mathbf{Q}$ , the prime initial it defines is

$$P_a = \{(c, d) \in \mathfrak{B}_3 : a \notin (c, d)\}.$$

Then

$$P'_a = \{(c, d) \in \mathfrak{B}_3 : a \notin (c, d]\}$$

defines a smaller initial that is prime:

$$(c, d), (e, f) \notin P'_a \iff a \in (c, d] \cap (e, f],$$

and it follows that  $(c, d) \cap (e, f) \in \mathfrak{B}_3 \setminus P'_a$ .

[Note that each  $P'_a$  is minimal. For, if  $(x, y) \in P'_a$ , then  $a \notin (x, y]$ , so either

- $(a - \varepsilon, a + \varepsilon) \cap (x, y) = \emptyset$  for some  $\varepsilon > 0$  (Note that  $(a - \varepsilon, a + \varepsilon) \notin P'_a$ ), or
- $x = a$ , in which case  $(a - 1, a) \cap (x, y) = \emptyset$  (Here,  $(a - 1, a) \notin P'_a$ ).

In either case, this shows that no initial contained in  $P'_a$  and missing  $(x, y)$  can be prime.]

[Note, also, that when one uses the map

$$\mathcal{P}' = \{P'_a; a \in \mathbf{Q}\} \ni P'_a \longmapsto a \in \mathbf{Q}$$

to transfer topologies, one gets a refinement of the usual topology on  $\mathbf{Q}$  with clopen base consisting of the intervals  $(a, b]$ , where  $a, b \in \mathbf{Q}$  with  $a < b$ .]

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