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**THE CONDITION WEAK-P(γ, α) AND ITS
IMPORTANCE TO COUNT THE NUMBER OF
DENSE ω -BOUNDED SUBGROUPS**

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ABSTRACT. Let α and γ be infinite cardinals. We say that condition $\text{weak-P}(\gamma, \alpha)$ holds if there is a Hausdorff, zero-dimensional *weak-P-space* X of size γ and weight at most α . We say that condition $\text{P}(\gamma, \alpha)$ holds if there is a Tychonoff, *P-space* X of size γ and weight at most α . Comfort and van Mill [Topology and Its Applications 77 (1977), 105-113] introduced those conditions and asked the following question: Do conditions $\text{P}(\gamma, \alpha)$ and $\text{weak-P}(\gamma, \alpha)$ hold for all cardinals $\alpha > \omega$? For all cardinals $\alpha > \mathfrak{c}$? For all cardinals α such that $\text{cf}(\alpha) > \omega$? In the present paper, a partial answer to this question is given. For an ordinal ξ , it is proved that (i) condition $\text{P}(2^{\aleph_\xi}, \aleph_\xi)$ holds provided ξ is either a successor ordinal or a limit ordinal with uncountable cofinality; (ii) condition $\text{P}(2^{\aleph_\xi}, \aleph_\xi)$ fails provided ξ is a limit ordinal with countable cofinality. Indeed, if κ is a strong limit cardinal with countable cofinality then condition $\text{P}(\kappa^+, \kappa)$ fails. Also it is shown that if α is an infinite cardinal of uncountable cofinality then condition $\text{weak-P}(\alpha^+, \alpha)$ holds. Using this fact, it is proved that the number of dense ω -bounded subgroups of a compact group G of $w(G) = \alpha$ with $\text{cf}(\alpha) > \omega$ which in addition is either Abelian or connected is at least $2^{(\alpha^+)}$. If in addition $2^\alpha < 2^{(\alpha^+)}$ then the number of such subgroups is at least $|G|^{(\alpha^+)}$. This also gives a partial answer to a question of Itzkowitz and Shakhmatov [Math. Japonica 45 (1997), 497-501].

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1. NOTATION, DEFINITIONS, AND PRELIMINARIES

The symbols α and γ denote infinite cardinals, and η and ξ denote ordinals. Following [5], I define $\beth_0 := \omega$. If ξ is a successor ordinal, say $\xi = \eta + 1$, then $\beth_\xi := 2^{\beth_\eta}$. If ξ is a limit ordinal, then $\beth_\xi := \cup_{\eta < \xi} \beth_\eta$. The German letter \mathfrak{c} will be reserved for the power of the continuum.

Given a set X and a cardinal α , I denote by $[X]^\alpha$ the set $\{A \subseteq X : |A| = \alpha\}$ and by $[X]^{\leq \alpha}$ the set $\{A \subseteq X : |A| \leq \alpha\}$. Following [8], I denote by $\Omega(G)$ the set of dense, ω -bounded subgroups of a topological group G .

Definition 1.1. Let α and γ be infinite cardinals. We say that:

- (a) Condition $\text{weak-P}(\gamma, \alpha)$ holds if there is a Hausdorff, zero-dimensional *weak-P-space* X of size γ and weight at most α , and
- (b) Condition $\text{P}(\gamma, \alpha)$ holds if there is a Tychonoff, *P-space* X of size γ and weight at most α .

Remark 1.2. For infinite cardinals α and γ , we have that condition $\text{P}(\gamma, \alpha)$ implies condition $\text{weak-P}(\gamma, \alpha)$.

Proof. It follows from the fact that in a Tychonoff *P-space* X the family of *cozero* sets is a base consisting of open-and-closed sets. \square

Part (c) of the following Lemma of Comfort and van Mill will be crucial for my application of the condition $\text{weak-P}(\gamma, \alpha)$ to topological groups.

Lemma 1.3. [Comfort and van Mill] *For α and γ any two infinite cardinals, the following conditions are equivalent:*

- (a) *Condition $w\text{-P}(\gamma, \alpha)$ holds;*
- (b) *The compact space $\{0, 1\}^\alpha$ contains a subspace X such that $|X| = \gamma$ and X is a weak-P-space; and*
- (c) *The set $\{0, 1\}^\alpha$ contains a subset X with these properties: $|X| = \gamma$, and if $C \in [X]^\omega$ and $p \in X \setminus C$ then there exists $\xi < \alpha$ such that $\pi_\xi(p) = p_\xi = 1$ and $\pi_\xi \upharpoonright C \equiv 0$.*

2. NEW RESULTS REGARDING CONDITIONS $w\text{-P}(\gamma, \alpha)$ AND $\text{P}(\gamma, \alpha)$

In [8], these authors prove the following Theorem. Also, they prove that the least cardinal $\alpha > \mathfrak{c}$ where condition $w\text{-P}(2^\alpha, \alpha)$ fails satisfies $cf(\alpha) = \omega$. In this section, I extend a little bit of our knowledge of conditions $w\text{-P}(2^\alpha, \alpha)$ and $\text{P}(2^\alpha, \alpha)$.

Theorem 2.1. [Comfort and van Mill] *If $1 < \alpha = \alpha^\omega$, then condition $P(2^\alpha, \alpha)$ holds.*

For an infinite cardinal α , we defined condition $w\text{-}P(2^\alpha, \alpha)$. For my results in the present section, it is important to have the notion of condition $w\text{-}P(2^\xi, \xi)$ when ξ is an infinite ordinal. So, we have the following definition.

Definition 2.2. Let ξ be an infinite ordinal. We say that condition $w\text{-}P(2^\xi, \xi)$ holds if condition $w\text{-}P(2^{|\xi|}, |\xi|)$ holds.

The following is a new result and it improves a Theorem due to Comfort and van Mill in [8]. The main idea in the argument presented here closely parallels that in [8, 2.7]. We also deduce [8, Theorem 2.7] as a Corollary to our Theorem 2.3.

Theorem 2.3. *Let α and κ be two infinite cardinals such that $cf(\alpha) > \omega$ and $\kappa < \alpha$. If condition $w\text{-}P(2^\xi, \xi)$ holds for each $\kappa \leq \xi < \alpha$, then condition $w\text{-}P(2^\alpha, \alpha)$ holds.*

Proof. Given $\kappa \leq \xi < \alpha$, we can define a Hausdorff, zero-dimensional *weak-P-space* topology, say τ_ξ , on the set $\{0, 1\}^\xi$ such that $w(\{0, 1\}^\xi) \leq |\xi| < \alpha$. This is possible since $w\text{-}P(2^\xi, \xi)$ holds and Lemma 1.3-(b) applies. For each $\kappa \leq \xi < \alpha$, we choose a τ_ξ -clopen base, say \mathcal{S}_ξ , of $\{0, 1\}^\xi$ with $|\mathcal{S}_\xi| \leq |\xi|$ and consider the “natural” projection π_ξ from $\{0, 1\}^\alpha$ onto $\{0, 1\}^\xi$. We define $\mathcal{S} = \bigcup_{\kappa \leq \xi < \alpha} \{\pi_\xi^{-1}[S] : S \in \mathcal{S}_\xi\}$ and notice that $|\mathcal{S}| \leq \sum_{\kappa \leq \xi < \alpha} |\{\pi_\xi^{-1}[S] : S \in \mathcal{S}_\xi\}| \leq \sum_{\kappa \leq \xi < \alpha} |\mathcal{S}_\xi| \leq \sum_{\kappa \leq \xi < \alpha} |\xi| \leq \alpha \cdot \alpha = \alpha$. We set $\mathcal{B} = \mathcal{S} \cup \mathcal{S}'$, where $\mathcal{S}' = \{\{0, 1\}^\alpha \setminus B : B \in \mathcal{S}\}$. Since $|\mathcal{S}| \leq \alpha$, it is clear that \mathcal{B} is a subbase for a Hausdorff, zero-dimensional topology, say τ , on $\{0, 1\}^\alpha$ such that $w(\{0, 1\}^\alpha) \leq \alpha$. We claim that $(\{0, 1\}^\alpha, \tau)$ is a *weak-P-space*. For this, we take $C \in [\{0, 1\}^\alpha]^\omega$ and $p \in (\{0, 1\}^\alpha \setminus C)$. We then show that $p \notin \overline{C}^{\{0, 1\}^\alpha}$. We write $C = \{c_n : n \in \omega\}$. Since $p \neq c_n$, there exists $\xi_n < \alpha$ such that $p(\xi_n) \neq c_n(\xi_n)$, for each $n \in \omega$. Now, we can find $\kappa \leq \xi < \alpha$ such that $\xi_n < \xi$ for each $n \in \omega$, since $cf(\alpha) > \omega$. Then $\pi_\xi(p) = p \upharpoonright \xi \neq c_n \upharpoonright \xi$ for each $n \in \omega$. Since $\{0, 1\}^\xi$ is a *weak-P-space*, and $\{c_n \upharpoonright \xi : n \in \omega\} \in [\{0, 1\}^\xi]^\omega$, and $p \upharpoonright \xi \notin \{c_n \upharpoonright \xi : n \in \omega\}$, there exists a basic open set S in \mathcal{S}_ξ such that $p \upharpoonright \xi \in S \subseteq (\{0, 1\}^\xi \setminus \{c_n \upharpoonright \xi : n \in \omega\})$. Then $p \in \pi_\xi^{-1}[S] \subseteq (\{0, 1\}^\alpha \setminus C)$.

Since $\pi_\xi^{-1}[S]$ is a subbasic open set in $\{0, 1\}^\alpha$ and $p \in (\{0, 1\}^\alpha \setminus C)$ was chosen arbitrarily, we have that $\{0, 1\}^\alpha \setminus C$ is a τ_ξ -open set, hence C is a closed set. Therefore $\{0, 1\}^\alpha$ is a *weak-P-space* as claimed, since C was an arbitrary countable set in $\{0, 1\}^\alpha$. Thus condition $w\text{-P}(2^\alpha, \alpha)$ holds. \square

Corollary 2.4. *Let κ be an infinite cardinal such that condition $w\text{-P}(2^\kappa, \kappa)$ holds. Then the least cardinal $\alpha > \kappa$ for which condition $w\text{-P}(2^\alpha, \alpha)$ fails has uncountable cofinality.*

Proof. It follows from Theorem 2.3. To see this, we take $\alpha > \kappa$ such that condition $w\text{-P}(2^\alpha, \alpha)$ fails with α the least cardinal with respect to this property. Hence, for each $\kappa \leq \xi < \alpha$, condition $w\text{-P}(2^\xi, \xi)$ holds. If $cf(\alpha)$ were uncountable, then we would have that condition $w\text{-P}(2^\alpha, \alpha)$ holds, due to Theorem 2.3 which is a contradiction. Thus $cf(\alpha) = \omega$. \square

The following Corollary is Theorem 2.7 in [8].

Corollary 2.5. [Comfort and van Mill] *Let α be the least cardinal such that $\alpha > \mathfrak{c}$ and condition $w\text{-P}(2^\alpha, \alpha)$ fails. Then $cf(\alpha) = \omega$.*

Proof. In view of Corollary 2.4, it suffices to show that $w\text{-P}(2^\mathfrak{c}, \mathfrak{c})$ holds. Since $\mathfrak{c}^\omega = \mathfrak{c}$ and Theorem 2.1 applies, we have that condition $w\text{-P}(2^\mathfrak{c}, \mathfrak{c})$ holds. Therefore $cf(\alpha) = \omega$. \square

Corollary 2.6. *Let κ be an infinite cardinal such that condition $w\text{-P}(2^\kappa, \kappa)$ holds. Then condition $w\text{-P}(2^{(\kappa^+)}, \kappa^+)$ holds.*

Proof. Since condition $w\text{-P}(2^\xi, \xi)$ holds for each $\kappa \leq \xi < \kappa^+$ and $cf(\kappa^+) > \omega$, Theorem 2.3 applies. Therefore condition $w\text{-P}(2^{(\kappa^+)}, \kappa^+)$ holds. \square

Lemma 2.7. *Let X be the disjoint union of the family $\{X_i : i \in I\}$. Then X is a P -space (weak- P -space) if and only if each X_i is a P -space (weak- P -space).*

Proof. It is routine. \square

Theorem 2.8. *Let κ and α be two cardinals such that $cf(\kappa) \leq \alpha$. Assume that condition $P(\mu, \alpha)$ ($w\text{-P}(\mu, \alpha)$) holds for each $\mu < \kappa$. Then condition $P(\kappa, \alpha)$ ($w\text{-P}(\kappa, \alpha)$) also holds.*

Proof. Case 1. κ is a successor, say $\kappa = \mu^+$. Let $Y = Y(\mu, \alpha)$ witness condition $P(\mu, \alpha)$ ($w\text{-}P(\mu, \alpha)$) and define X as the topological union $\dot{\cup}_\kappa Y$. Then $|X| = \kappa \cdot |Y| = \kappa \cdot \mu = \kappa$, and $wX = \kappa \cdot (wY) \leq \kappa \cdot \alpha = cf(\kappa) \cdot \alpha = \alpha$, and X is a P -space (*weak-P-space*), due to Lemma 2.7.

Case 2. κ is a limit cardinal and take $\kappa_\xi \uparrow \kappa$ ($\xi < cf(\kappa)$). Let $Y(\kappa_\xi, \alpha)$ witness condition $P(\kappa_\xi, \alpha)$ ($w\text{-}P(\kappa_\xi, \alpha)$) and define $X = \dot{\cup}_{\xi < cf(\kappa)} Y(\kappa_\xi, \alpha)$. Then $|X| = \sum_{\xi < cf(\kappa)} |Y(\kappa_\xi, \alpha)| = \sum_{\xi < cf(\kappa)} \kappa_\xi = \kappa$, and $wX \leq \sum_{\xi < cf(\kappa)} \alpha = cf(\kappa) \cdot \alpha = \alpha$ and X is a P -space (*weak-P-space*), by Lemma 2.7. \square

The following Theorem answers partially a question posed by Comfort and van Mill in [8].

Theorem 2.9. *Let ξ be an ordinal.*

(a) *If ξ is a successor ordinal or a limit ordinal with uncountable cofinality, then condition $P(2^{\beth_\xi}, \beth_\xi)$ holds.*

(b) *If ξ is a limit ordinal with countable cofinality, then condition $P(2^{\beth_\xi}, \beth_\xi)$ fails.*

Proof. (a) Let ξ be a successor ordinal. Since $(\beth_\xi)^\omega = \beth_\xi$, Theorem 2.1 applies. Therefore condition $P(2^{\beth_\xi}, \beth_\xi)$ holds.

Let ξ be a limit ordinal with $cf(\xi) > \omega$. In order to prove that $\beth_\xi = (\beth_\xi)^\omega$, it suffices to show that $(\sum_{\eta < \xi} \beth_\eta)^\omega \leq \sum_{\eta < \xi} (\beth_\eta)^\omega$. For this, we take $f \in (\sum_{\eta < \xi} \beth_\eta)^\omega$. Then for each $n < \omega$ there exists $\eta < \xi$ such that $f(n) \in \beth_{\eta_n}$. Since $cf(\xi) > \omega$, we can find $\eta' < \xi$ such that $f \in (\beth_{\eta'})^\omega$. Now Theorem 2.1 applies. Therefore condition $P(2^{\beth_\xi}, \beth_\xi)$ holds.

(b) Let ξ be a limit ordinal with countable cofinality. It is well known that \beth_ξ is a strong limit with $cf(\beth_\xi) = \omega$; see [5]. Suppose on the contrary that condition $P(2^{\beth_\xi}, \beth_\xi)$ holds. Let X witness condition $P(2^{\beth_\xi}, \beth_\xi)$. Since $wX \leq \beth_\xi$ and $d(X) \leq wX$, we can take D a dense subset of X such that $|D| = \beth_\xi$. Write D as a countable union of subsets with cardinality less than \beth_ξ , say $D = \cup_{n \in \omega} D_n$ and $|D_n| < \beth_\xi$ for each $n \in \omega$. This is possible because $cf(\beth_\xi) = \omega$. Since X is a P -space, and in such spaces the countable union of closed sets is a closed set, we have that $\overline{\cup_{n \in \omega} D_n}^X = \cup_{n \in \omega} \overline{D_n}^X$. Hence $2^{\beth_\xi} = |X| = |\overline{D}^X| = |\cup_{n \in \omega} \overline{D_n}^X| = |\cup_{n \in \omega} \overline{D_n}^X|$. It is well known that for each regular T_1 space Y , we have $|Y| \leq 2^{2^{d(Y)}}$.

Hence $|\overline{D_n^X}| \leq 2^{2^{|D_n|}}$ for each $n \in \omega$. Since $|D_n| < \beth_\xi$ and \beth_ξ is a strong limit cardinal, we have $2^{2^{|D_n|}} < \beth_\xi$ for each n . So $2^{\beth_\xi} = |\cup_{n \in \omega} \overline{D_n^X}| \leq \sum_{n \in \omega} |\overline{D_n^X}| \leq \sum_{n \in \omega} 2^{2^{|D_n|}} \leq \beth_\xi < 2^{\beth_\xi}$ which is impossible. Therefore condition $P(2^{\beth_\xi}, \beth_\xi)$ fails. This completes the proof. \square

Remark 2.10. The proof of part (b) shows even more. That is, if κ is a strong limit cardinal with countable cofinality, then condition $P(\kappa^+, \kappa)$ fails.

Proof. It is known that each strong limit cardinal, say κ , is of the form \beth_ξ for some limit ordinal ξ and that $cf(\beth_\xi) = cf(\xi)$; see [5]. The Remark follows from the proof of part (b) in the previous Theorem applied to $\kappa = \beth_\xi$. \square

The following Theorem 2.11 appears in [5].

Theorem 2.11. *Let α be an infinite cardinal and consider the following four conditions:*

- (i) α is a measurable cardinal.
- (ii) α is a strongly inaccessible cardinal (that is, it is a regular and strong limit cardinal).
- (iii) $\alpha = 2^{<\alpha}$ and α is a regular cardinal.
- (iv) $\alpha = \alpha^{<\alpha}$.

These are related as follows: (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).

Remark 2.12. The consistency of $\mathbf{MA} + \neg \mathbf{CH}$ is well known; see [15]. In such a model, we can find cardinals α satisfying (iv) but not (i). For example, consider $\alpha = \mathfrak{c}$. It is well known that \mathfrak{c} is not measurable. To see that $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$, recall from [15] that under \mathbf{MA} , we have $2^\lambda = \mathfrak{c}$ for each $\omega \leq \lambda < \mathfrak{c}$. We then have $\mathfrak{c}^{<\mathfrak{c}} = \sum_{\lambda < \mathfrak{c}} \mathfrak{c}^\lambda = \sum_{\omega \leq \lambda < \mathfrak{c}} \mathfrak{c}^\lambda = \sum_{\omega \leq \lambda < \mathfrak{c}} (2^\omega)^\lambda = \sum_{\omega \leq \lambda < \mathfrak{c}} (2^\lambda)^\omega = \sum_{\omega \leq \lambda < \mathfrak{c}} \mathfrak{c} \leq \mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$, as required.

Corollary 2.13. *Let α be an uncountable measurable cardinal. Then condition $P(2^\alpha, \alpha)$ holds.*

Proof. In view of Theorem 2.1, it suffices to show that $\alpha^\omega = \alpha$. Since α is a measurable cardinal, Theorem 2.11 implies that α is a regular cardinal and $\alpha = 2^{<\alpha}$. We know that $(2^{<\alpha})^\beta = 2^{<\alpha}$ for each $\beta < cf(\alpha)$; see [5]. Hence $(2^{<\alpha})^\omega = 2^{<\alpha}$, since α is an uncountable regular cardinal. Therefore $\alpha^\omega = (2^{<\alpha})^\omega = 2^{<\alpha} = \alpha$. Thus condition $P(2^\alpha, \alpha)$ holds. \square

The following Lemma is well-known; see [5, 12.19].

Lemma 2.14. *Let α be an infinite cardinal. Then there exists $\mathcal{F} \subseteq \alpha^\alpha$ such that $|\mathcal{F}| = \alpha^+$ and $|\{\xi \in \alpha : f(\xi) = g(\xi)\}| < \alpha$ whenever $f \neq g$, and $f, g \in \mathcal{F}$.*

Theorem 2.15. *Let α be a cardinal number with uncountable cofinality. Then condition $w\text{-}P(\alpha^+, \alpha)$ holds.*

Proof. It is clear that α^α is a Hausdorff, zero-dimensional space and that $w(\alpha^\alpha) = \alpha$, when α has the discrete topology and α^α the Tychonoff product topology. We take X as in Lemma 2.14 and claim that X with the topology inherited from α^α witnesses condition $w\text{-}P(\alpha^+, \alpha)$. Since X as a subspace of α^α is a Hausdorff, zero-dimensional space with $w(X) \leq w(\alpha^\alpha) = \alpha$ and $|X| = \alpha^+$, it suffices to prove that X is a *weak-P-space*. For this, we take an arbitrary countable set $\{f_n : n \in \omega\}$ in X , and an arbitrary point $f \in (X \setminus \{f_n : n \in \omega\})$. For each $n \in \omega$, we set $A_n = \{\xi \in \alpha : f(\xi) = f_n(\xi)\}$. Then $|A_n| < \alpha$ for each $n \in \omega$, since $\{f_n : n \in \omega\} \cup \{f\} \subseteq X$ and $f \neq f_n$ for each $n \in \omega$. Since $cf(\alpha) > \omega$, we can find $\xi \in \alpha$ such that ξ is not in $\cup_{n \in \omega} A_n$. Hence $f_n(\xi) \notin \{f(\xi)\}$ for each $n \in \omega$. So, we have $\{f_n : n \in \omega\} \cap \pi_\xi^{-1}[\{f(\xi)\}] = \emptyset$. Now, it is clear that $\pi_\xi^{-1}[\{f(\xi)\}] \cap X$ is an open set in X containing f and missing $\{f_n : n \in \omega\}$. Hence $f \notin \overline{\{f_n : n \in \omega\}}^X$. Therefore f is a *weak-P-point* of X for each $f \in X$. Thus X is a *weak-P-space*. \square

Remark 2.16. We notice that the hypothesis $cf(\alpha) > \omega$ in Theorem 2.15 cannot be omitted. For example, condition $w\text{-}P(\omega^+, \omega)$ fails (and hence condition $w\text{-}P(\gamma, \omega)$ fails for all $\gamma \geq \omega^+$). To see this, suppose there is a Hausdorff *weak-P-space* X such that $|X| = \omega^+$ and $w(X) \leq \omega$. Since $d(X) \leq w(X) \leq \omega$, we can find a countable dense set in X . But each countable set in X is closed, because X is a *weak-P-space*. Therefore X has a countable dense closed set, so $|X| = \omega$ which is a contradiction. Thus condition $w\text{-}P(\omega^+, \omega)$ fails.

3. AN APPLICATION TO COMPACT GROUPS

Definition 3.1. Let X be a topological space. We say that X is ω -bounded if each countable subset of X has compact closure (in X).

Lemma 3.2. *Let X be a Hausdorff topological space. Then X is an ω -bounded space if, and only if, for each $A \in [X]^\omega$ there exists a compact set K such that $A \subseteq K \subseteq X$.*

Proof. Trivial. □

The following Lemma 3.3 is taken from [8]. It gives to us a method to generate ω -bounded subgroups.

Lemma 3.3. [Comfort-van Mill] *Let X be an ω -bounded space. For any subset A of X , let $\omega(A)$ be the set $\bigcup\{\overline{D}^X : D \in [A]^\omega\}$. Then:*

(a) $\omega(A)$ is the smallest ω -bounded subset of X containing A ; and

(b) if X is a topological group and A is a subgroup, then $\omega(A)$ is a subgroup of X .

The following Theorem improves the lower bound for the number of dense ω -bounded subgroups given by Itzkowitz and Shakhmatov in [14]. Its proof closely parallels an argument presented by Comfort and van Mill in [8].

Theorem 3.4. *Let G be a compact group which is either Abelian or connected and whose weight $wG = \alpha$ has uncountable cofinality. Then:*

(a) $|\Omega(G)| \geq 2^{(\alpha^+)}$ and;

(b) $|\Omega(G)| \geq |G|^+$ provided $2^\alpha < 2^{(\alpha^+)}$.

Proof. It is well known that a compact group G of weight $w(G) = \alpha$ which in addition is either Abelian or connected can be mapped onto a product $\prod_{\xi < \alpha} G_\xi$ with each G_ξ a non-trivial compact group via a continuous epimorphism (see [7]). Indeed if $cf(\alpha) > \omega$ then we can take $G_\xi = F$ for all $\xi < \alpha$ with F a non-trivial compact group. Therefore we can find a continuous epimorphism $\varphi: G \rightarrow F^\alpha$. Since α has uncountable cofinality, Theorem 2.15 applies. Then condition $w\text{-P}(\alpha^+, \alpha)$ holds. Using Lemma 1.3-(c), we can find a set X in the set $\{0, 1\}^\alpha \subseteq F^\alpha$ such that $|X| = \alpha^+$, and if $C \in [X]^\omega$ and $p \in (X \setminus C)$, then there exists an index $\xi \in \alpha$ such that $\pi_\xi(p) = p_\xi = 1$ and $\pi_\xi \upharpoonright C = 0$. Given $A \subseteq X$, we know from Lemma 3.3-(b) that $\omega(\langle A \rangle) = \bigcup\{\overline{D}^{F^\alpha} : D \in [\langle A \rangle]^\omega\}$ is an ω -bounded subgroup of F^α containing $\langle A \rangle$. We claim that $\omega(\langle A \rangle) \neq \omega(\langle B \rangle)$ whenever A and B are subsets of X and $A \neq B$.

For this, it suffices to show that $\omega(\langle A \rangle) \cap X = A$ for each $A \subseteq X$. Now, given $A \subseteq X$, it is clear that $A \subseteq \omega(\langle A \rangle) \cap X$. Conversely, take $p \in \omega(\langle A \rangle) \cap X$. Then $p \in \overline{D}^{F^\alpha}$ for some $D \in [\langle A \rangle]^\omega$. Given $x \in D$, there exists a finite set C_x in A such that $x \in \langle C_x \rangle$. So $D \subseteq \cup_{x \in D} \langle C_x \rangle \subseteq \langle \cup_{x \in D} C_x \rangle$. Since $p \in \overline{D}^{F^\alpha}$ and $\overline{D}^{F^\alpha} \subseteq \overline{\langle \cup_{x \in D} C_x \rangle}^{F^\alpha}$, then $p \in \overline{\langle \cup_{x \in D} C_x \rangle}^{F^\alpha}$. We set $C = \cup_{x \in D} C_x \subseteq A$. If p were not in C , then $p \in (X \setminus C)$. Since X has property (c) in Lemma 1.3 and $|C| \leq \omega$, there exists an index $\xi \in \alpha$ such that $\pi_\xi(p) = 1$ and $\pi_\xi(x) = 0$ for each $x \in C$. Since π_ξ is a homomorphism between groups such that $\pi_\xi(x) = 0$ for each $x \in C$, we have that $\pi_\xi \upharpoonright \langle C \rangle \equiv 0$. Since π_ξ is a continuous mapping, we have $\pi_\xi[\overline{\langle C \rangle}^{F^\alpha}] \subseteq \overline{\pi_\xi \langle C \rangle}^F = \overline{\{0\}}^F = \{0\}$. But, this is impossible because $p \in \overline{D}^{F^\alpha} \subseteq \overline{\langle C \rangle}^{F^\alpha}$ would imply $1 = \pi_\xi(p) = 0$ which is a contradiction. So $p \in C \subseteq A$. Hence $p \in \omega(\langle A \rangle) \cap X$. Then $\omega(\langle A \rangle) \cap X = A$. So $\omega(\langle A \rangle) \neq \omega(\langle B \rangle)$ whenever $A, B \subseteq X$ and $A \neq B$, as claimed. Therefore $\{\omega(\langle A \rangle) : A \subseteq X\}$ is a family consisting of distinct ω -bounded subgroups of F^α . Thus F^α has at least $2^{(\alpha^+)}$ -many ω -bounded subgroups.

Moreover, the ω -bounded subgroups in F^α can be taken dense, due to the following argument: Set $K = F^\alpha$. We consider the diagonal mapping from K into K^α . That is to say, for each point $p \in K$, we define a point φ_p in K^α as follows: For each $\xi \in \alpha$, we define $\varphi_p(\xi) = p$. Now set $\Delta = \{\varphi_p \in K^\alpha : p \in K\}$. It is clear that $\Delta \approx K$ in the sense that Δ and K are isomorphic as groups and homeomorphic as topological spaces. Since $\Delta \approx K = F^\alpha$ and F^α has $2^{(\alpha^+)}$ -many ω -bounded subgroups, we can find $2^{(\alpha^+)}$ -many distinct ω -bounded subgroups, say $\{\Delta(\eta) : \eta < 2^{(\alpha^+)}\}$, in Δ . Since K^α is a compact space, we have that $\Sigma(0)$ is a dense ω -bounded subspace. Here, we recall that $\Sigma(0)$ is the subset of K^α defined as $\{\varphi \in K^\alpha : |\{\xi \in \alpha : \varphi(\xi) \neq 0\}| \leq \omega\}$. Now, since $\Delta(\eta)$ and $\Sigma(0)$ are ω -bounded subspaces of K^α , we have that $\Delta(\eta) \times \Sigma(0)$ is an ω -bounded subspace of $K^\alpha \times K^\alpha$ for each $\eta < 2^{(\alpha^+)}$. Since K^α is a topological group, we have that $+: K^\alpha \times K^\alpha \rightarrow K^\alpha$ is a continuous function from $K^\alpha \times K^\alpha$ onto K^α . Since every continuous onto function takes ω -bounded subspaces to ω -bounded subspaces, we have that $\Delta(\eta) + \Sigma(0)$ is an ω -bounded subspace of K^α for each $\eta < 2^{(\alpha^+)}$. It is clear that $\Sigma(0)$ is a normal subgroup of K^α .

Then $\Delta(\eta) + \Sigma(0)$ is a subgroup of K^α for each $\eta < 2^{(\alpha^+)}$. Since $\Sigma(0)$ is a dense subspace of K^α , and $\Sigma(0) \subseteq \Delta(\eta) + \Sigma(0)$ for each $\eta < 2^{(\alpha^+)}$, we have that $\Delta(\eta) + \Sigma(0)$ is a dense subspace of K^α for each $\eta < 2^{(\alpha^+)}$. Therefore, the family given by the sets $\Delta(\eta) + \Sigma(0)$ for each $\eta < 2^{(\alpha^+)}$ consists of dense ω -bounded subgroups of K^α . It remains to prove that $|\{\Delta(\eta) + \Sigma(0) : \eta < 2^{(\alpha^+)}\}| = 2^{(\alpha^+)}$. For this, given $\eta, \bar{\eta} < 2^{(\alpha^+)}$ with $\eta \neq \bar{\eta}$, we claim that $\Delta(\eta) + \Sigma(0) \neq \Delta(\bar{\eta}) + \Sigma(0)$. We will argue by contradiction. Indeed, suppose that $\Delta(\eta) + \Sigma(0) = \Delta(\bar{\eta}) + \Sigma(0)$ for some $\eta, \bar{\eta} < 2^{(\alpha^+)}$ and $\eta \neq \bar{\eta}$. We know that $\Delta(\eta) \neq \Delta(\bar{\eta})$. So, we can find a point φ_p such that either $\varphi_p \in (\Delta(\eta) \setminus \Delta(\bar{\eta}))$ or $\varphi_p \in (\Delta(\bar{\eta}) \setminus \Delta(\eta))$ according to whether $\Delta(\eta) \not\subseteq \Delta(\bar{\eta})$ or $\Delta(\bar{\eta}) \not\subseteq \Delta(\eta)$. Without loss of generality we can assume that $\varphi_p \in (\Delta(\eta) \setminus \Delta(\bar{\eta}))$. We then have $\varphi_p = \varphi_p + 0 \in \Delta(\eta) + \Sigma(0) = \Delta(\bar{\eta}) + \Sigma(0)$ would imply that $\varphi_p - \varphi_q \in \Sigma(0)$ for some $\varphi_q \in \Delta(\bar{\eta})$. Since $|\text{supp}(\varphi_p - \varphi_q)| \leq \omega$ and $\alpha > \omega$, there exists an index $\xi \in \alpha$ such that $(\varphi_p - \varphi_q)(\xi) = 0$. Hence $\varphi_p = \varphi_q$ which is a contradiction. So $\{\Delta(\eta) + \Sigma(0) : \eta < 2^{(\alpha^+)}\}$ is faithfully indexed, therefore it has cardinality $2^{(\alpha^+)}$. Since $K \approx K^\alpha$ (isomorphic as groups and homeomorphic as topological spaces) and K^α has $2^{(\alpha^+)}$ -many distinct dense ω -bounded subgroups, then $K = F^\alpha$ has $2^{(\alpha^+)}$ -many distinct dense ω -bounded subgroups.

Now, we recall that $\varphi: G \twoheadrightarrow F^\alpha$ is a continuous epimorphism. Since G is compact we have that φ is a closed mapping with compact fibers. Now, it is easy to show that inverse images under φ of ω -bounded subgroups of F^α are ω -bounded subgroups of G . Hence, each ω -bounded subgroup of F^α will produce an ω -bounded subgroup of G . Since φ is an open mapping due to [12, 5.29], we have that dense subgroups of F^α will produce dense subgroups of G . Hence, the inverse image under φ of each dense ω -bounded subgroup of F^α will be a dense ω -bounded subgroup of G . So $|\Omega(G)| \geq |\Omega(F^\alpha)|$. This proves that $|\Omega(G)| \geq 2^{(\alpha^+)}$ and hence (a). For (b) it is enough to note that if $2^{(\alpha^+)} > 2^\alpha$ then $|\Omega(G)| \geq 2^{(\alpha^+)} \geq (2^\alpha)^+ = |G|^+$. \square

We observe that if **GCH** holds, then $|G|^+ = 2^{|G|}$. Therefore we conclude from the previous Theorem that $|\Omega(G)| = 2^{|G|}$.

Remark 3.5. The previous theorem depends heavily on the fact that G is Abelian or connected and that its weight has uncountable cofinality. The relevance of the next Theorem is that we do not require such hypothesis. The main argument used here is similar to that in [14, 2.1] presented by Itzkowitz and Shakhmatov. Its proof was suggested to me by Comfort.

Theorem 3.6. *Let F be a non-trivial compact topological group and α an uncountable cardinal. Then $|\Omega(F^\alpha)| \geq 2^\alpha$. Hence assuming $|F| \leq 2^\alpha$, it follows that $|\Omega(F^\alpha)| \geq |F|^\alpha$.*

Proof. Since $\alpha \geq \aleph_1$, we can find a family \mathcal{F} in $\wp(\alpha)$ such that $|\mathcal{F}| = 2^\alpha$, and such that either $|A \setminus B| \geq \aleph_1$ or $|B \setminus A| \geq \aleph_1$ for each $A, B \in \mathcal{F}$. For $A \in \mathcal{F}$, the set $\Sigma_A(0) := \{x \in F^A : |\{i \in A : x(i) \neq 0\}| \leq \omega\}$ is a dense ω -bounded subgroup of F^A . Since ω -boundedness is preserved under products, we have that $\Sigma_A(0) \times F^{\alpha \setminus A}$ is a dense ω -bounded subgroup of F^α . Now we claim that if $A, B \in \mathcal{F}$ with $A \neq B$, then $\Sigma_A(0) \times F^{\alpha \setminus A} \neq \Sigma_B(0) \times F^{\alpha \setminus B}$. To see this, suppose without loss of generality that $|A \setminus B| \geq \aleph_1$, and define $x = (x_i) \in F^\alpha$ as follows: For $i \in (A \setminus B)$, we define $x_i = 0$, and $x_i \neq 0$ otherwise. It is easy to see that $x \in [\Sigma_A(0) \times F^{\alpha \setminus A}] \setminus [\Sigma_B(0) \times F^{\alpha \setminus B}]$, as required. Thus $|\Omega(F^\alpha)| \geq |\mathcal{F}| = 2^\alpha$, and if $|F| \leq 2^\alpha$ then $|\Omega(F^\alpha)| \geq 2^\alpha = |F|^\alpha$. \square

If κ is a strong limit cardinal with countable cofinality, then condition $P(\kappa^+, \kappa)$ fails due to Remark 2.10. Also condition $w\text{-}P(\omega^+, \omega)$ fails due to Remark 2.16. The author of this paper has not been able to answer the following questions.

Question 3.7. *Does condition $w\text{-}P(\kappa^+, \kappa)$ hold for all uncountable strong limit cardinals κ with countable cofinality?*

Question 3.8. *Does condition $P(\alpha^+, \alpha)$ hold for all infinite cardinals α of uncountable cofinality?*

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