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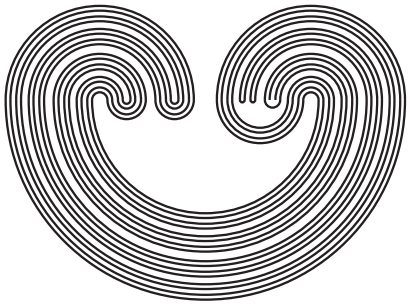

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
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Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 26, 2001–2002
Pages 763–778

DYNAMICS OF POLYNOMIAL MAPS 2×2 REAL MATRICES

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ABSTRACT. We investigate the dynamics of iterated polynomial maps of the form $p_C(Z) = a_n Z^n + \cdots + a_1 Z + C$, where the a_k are real numbers, and the variable Z and constant C are 2×2 real matrices. Under the action of these maps, every orbit is confined to an invariant 3-dimensional subspace. For any map p_C, at most three of these invariant 3-spaces are dynamically distinct, with the exact number determined by the eigenvalues of C. It is well known that for polynomial maps of the complex plane of degree at least two, every attracting cycle must attract the orbit of a critical point. We demonstrate that in the case of polynomial matrix maps, it is possible for attracting cycles to exist that do not attract the orbit of any critical point. In light of this fact, we discuss the potential dynamical significance of a quadratic matrix Mandelbrot set.

1. INTRODUCTION

Iterated maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ exhibit so many different behaviors that few general conclusions can be made about the dynamics of this class of maps. It is helpful to restrict to a narrower class of maps that has enough structure to yield general results, but is still broad enough to admit an interesting range of dynamical behaviors. For example, the rigid structure of the algebra of complex numbers yields many powerful and beautiful results about the dynamics of polynomial maps of the complex plane that do not hold for more general maps.

2000 *Mathematics Subject Classification.* 39B12.
Key words and phrases. Matrix dynamics, polynomial maps, Mandelbrot set, attracting cycles.

Iterated maps from $\mathbb{R}^4$ to $\mathbb{R}^4$ exhibit an even wider range of behaviors than those in $\mathbb{R}^2$. We consider the more restricted class of maps on $\mathbb{R}^4$ defined by

$$p_C(Z) = a_n Z^n + \ldots + a_1 Z + C,$$

where the a_k are real numbers and the variable Z and constant C are elements of $\mathcal{M}$, the set of 2×2 real matrices.

In this article, we investigate whether the algebraic properties of matrix operations provide enough structure to make p_C a tractable and dynamically interesting class of maps. We will show that the structure of these maps is indeed quite rigid, making it possible for us to establish the following results:

- When C is a multiple of the identity matrix I, we can completely describe the dynamics of p_C in terms of the dynamics of polynomial maps on the complex plane and the real line.
- When C is not a multiple of the identity, there is a unique invariant plane containing all multiples of the identity.
- Any 3-space containing this plane will be invariant, so every orbit is confined to a 3-dimensional subspace of $\mathcal{M}$.
- For C not a multiple of the identity, there are one, two, or three dynamically distinct invariant 3-spaces for any map p_C depending on whether the eigenvalues of C are complex, real and identical, or real and distinct, respectively.
- For polynomial maps of the complex plane of degree at least 2, every attracting cycle must attract the orbit of a critical point. However, in the case of polynomial matrix maps it is possible for attracting cycles to exist that do not attract a critical orbit.
- The quadratic matrix Mandelbrot set for the map $q_C(Z) = Z^2 + C$ is defined as the set of parameter values C such that the orbit of the zero matrix $Z_0 = 0$ remains bounded under iteration in q_C. Points C in the complement of this set correspond to maps q_C that admit no attracting cycles in any invariant plane containing all multiples of the identity.

2. Reducing the Dimension of the Problem

The Cayley-Hamilton theorem describes an important property of matrix operations that will enable us to describe the dynamics of p_C in the 4-dimensional matrix space in terms of the dynamics in a

collection of 3-dimensional subspaces. The Cayley-Hamilton theorem states that a matrix satisfies its own characteristic polynomial. For $Z \in \mathcal{M}$, this implies that

$$Z^2 = \operatorname{tr}(Z)Z - \det(Z)I.$$

Thus, Z^2 lies in the plane spanned by Z and the identity matrix I. This fact together with the distributive property for matrix multiplication tells us that for all positive integers k, Z^k lies in the span of Z and I.

Planes of matrices containing the line spanned by the identity will play an important role in our discussion of the dynamics of p_C. For any matrix Z, let $\mathcal{P}(Z)$ denote the subspace spanned by Z and I. This subspace is a plane unless Z is a multiple of the identity.

Proposition 1. *If C is a multiple of the identity, $\mathcal{P}(Z)$ is invariant for all Z.*

Proof. Maps p_C are composed of two transformations: a polynomial in Z and a translation by a constant matrix C. The polynomial transformation always leaves the image of Z in $\mathcal{P}(Z)$. If C is a multiple of the identity, translation by C will not remove the image of Z from this subspace. □

We will see later that we can completely describe the dynamics in any invariant subspace $\mathcal{P}(Z)$ in terms of the dynamics of polynomial maps of the complex plane and the real line. Therefore, unless otherwise stated, we assume that C is not a multiple of the identity.

Proposition 2. *If C is not a multiple of the identity, $\mathcal{P}(C)$ is the unique invariant plane containing the line spanned by I.*

Proof. We can see from the previous discussion that $\mathcal{P}(C)$ will be invariant. If a matrix Z is not a linear combination of C and I, then translation by C will necessarily remove the image $p_C(Z)$ out of the plane spanned by Z and I. Thus, no other plane containing all multiples of I can be invariant. □

By these arguments, it is clear that the image $p_C(Z)$ of any matrix Z lies in the span of I, C, and Z, and so we have the following proposition.

Proposition 3. *Any 3-space containing $\mathcal{P}(C)$ is invariant.*

This proposition points out an important property of maps p_C; no orbit explores more than one 3-dimensional subspace of the 4-dimensional matrix space. The dynamics of p_C in $\mathcal{M}$ always reduces to the dynamics of p_C in a collection of 3-dimensional subspaces. We will see later that there are no more than three dynamically distinct invariant 3-spaces for any map p_C.

3. Dynamics in Invariant Planes

The Jordan canonical form theorem reveals another important property of matrix operations that will enable us to further simplify our task. This theorem allows us to describe the dynamics within any invariant plane in terms of the dynamics in one of three planes of matrices in Jordan canonical form.

3.1. Jordan Canonical Planes

Let x, y, h, u, and v be real numbers. Every matrix $A \in \mathcal{M}$ that is not a multiple of the identity is similar to a unique matrix having one of three Jordan canonical forms:

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix}, \qquad \begin{bmatrix} x & h \\ 0 & x \end{bmatrix}, \qquad \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix},$$

depending on whether the eigenvalues of A are complex, real and identical, or real and distinct, respectively [5]. Each of these three Jordan canonical forms defines a plane in matrix space containing all multiples of the identity. We will refer to them as *Jordan canonical planes.*

The following proposition is easily verified.

Proposition 4. *For S an invertible 2×2 matrix, let $S_* : \mathcal{M} \to \mathcal{M}$ be the associated similarity transformation: $S_*(A) = S^{-1}AS$. S_* is an algebra automorphism for $\mathcal{M}$, i.e. it preserves I and commutes with addition, multiplication, and multiplication by scalars. In particular, S_* maps $\mathcal{P}(A)$ to $\mathcal{P}(S_*(A))$. Furthermore, it maps the dynamical system p_C to $p_{S_*(C)}$, that is,*

$$S^{-1}[p_C(Z)]S = p_{S^{-1}CS}(S^{-1}ZS).$$

Corollary 5. *Without loss of generality, we can assume that C lies in one of the three Jordan canonical planes.*

3.2. Dynamics in Invariant Jordan Canonical Planes

In this section we explain the dynamics in each of the three potential invariant Jordan canonical planes. In the Jordan canonical plane with complex eigenvalues, the dynamics of p_C is equivalent to the dynamics of a related polynomial map in the complex plane. In the Jordan canonical plane with distinct real eigenvalues, the dynamics of p_C is equivalent to the topological product of the dynamics of related polynomial maps on two copies of the real line. In the Jordan canonical plane with identical real eigenvalues, the dynamics of p_C is dominated by the dynamics of a related polynomial map on the real line.

3.2.1. The Jordan Canonical Plane with Complex Eigenvalues

Consider the plane of matrices in Jordan form with complex eigenvalues. Let ϕ be the canonical isomorphism between matrices Z in this plane and points $\phi(Z)$ in the complex plane given by

$$\phi\left(\begin{bmatrix} x & y \\ -y & x \end{bmatrix}\right) = x + iy.$$

This isomorphism respects the matrix operations of addition and multiplication so that $\phi(p_C(Z)) = p_{\phi(C)}(\phi(Z))$. This means that the dynamics of p_C in any invariant plane of matrices with complex eigenvalues is equivalent to the dynamics of $p_{\phi(C)}$ in the complex plane.

3.2.2. The Jordan Canonical Plane with Distinct Real Eigenvalues

Consider the plane of matrices in Jordan form with distinct real eigenvalues. Suppose C has diagonal entries c_u and c_v. For any matrix Z in this plane with diagonal entries u and v,

$$p_C(Z) = \begin{bmatrix} p_{c_u}(u) & 0 \\ 0 & p_{c_v}(v) \end{bmatrix}.$$

The dynamics in each coordinate is independent of the other. Therefore the dynamics in any invariant plane with distinct real eigenvalues is equivalent to the product space of the dynamics of p_{c_u} and p_{c_v} on their respective real lines.

3.2.3. The Jordan Canonical Plane with Identical Real Eigenvalues

Consider the plane of matrices in Jordan form with identical real eigenvalues. Let Z_i have diagonal entries x_i and off-diagonal entry h_i. The dynamics of the x coordinates of an orbit in this plane is independent of the values of the h coordinates. Moreover, the behavior of the x coordinates of an orbit dominates the behavior of the h coordinates in a sense made precise by the following proposition.

Proposition 6. *Let Z_i have diagonal entries x_i and off-diagonal entry h_i, and let C have diagonal entries c_x and off-diagonal entry c_h. If the map p_{c_x} induces an attracting (repelling) n-cycle on the real line, then the map p_C induces an n-cycle that attracts (repels) nearby orbits in the Jordan canonical plane with identical real eigenvalues. If the x coordinates of an orbit in this plane grow without bound, the h coordinates of the orbit will also be unbounded.*

Proof. The polynomial map p_C sends a point Z_i in this Jordan canonical plane to the point

$$Z_{i+1} = p_C(Z_i) = \begin{bmatrix} x_{i+1} & h_{i+1} \\ 0 & x_{i+1} \end{bmatrix} = \begin{bmatrix} p_{c_x}(x_i) & p'_{c_x}(x_i)h_i + c_h \\ 0 & p_{c_x}(x_i) \end{bmatrix},$$

where p' is the derivative with respect to x. This coordinate form of the map can be established by inductively deriving the form of Z^k.

Suppose that the x coordinates of a particular orbit are on an attracting n-cycle. The definition of an attracting cycle and the associated theorems presented in [2] apply to this situation, so

$$|(p^n_{c_x})'(x_1)| \quad = \quad |p'_{c_x}(x_1) \cdot p'_{c_x}(x_2) \cdots p'_{c_x}(x_n)| < 1.$$

The expression for each iterate h_{i+1} is linear in h_i with slope given by $p'_{c_x}(x_i)$. Composing this expression n times will give us a linear function with slope equal to the product of the slopes of each linear function around the n-cycle. Since this product is less than 1, the composed linear function has an attracting fixed point corresponding to the unique attracting n-cycle which lies over the x coordinate cycle.

Similarly, a repelling cycle in the x coordinates of an orbit will induce a repelling cycle in the h coordinates.

If the x coordinates of an orbit grow without bound, the polynomial $p'_{c_x}(x_i)$ yields values with arbitrarily large magnitude. Since the h coordinate of a point on the orbit is multiplied by this quantity at each iteration, the h coordinates of the orbit also grow without bound. □

3.2.4. References Discussing Dynamics in Isomorphic Planes

There are many sources describing details of the dynamics of polynomial maps of the complex plane and the real line. A good starting reference for this information is [2]. All three of the Jordan canonical planes have isomorphic incarnations as "cycle planes" and as graphic representations of "binary number systems". References discussing the dynamics of polynomial maps in these related systems include [1, 3, 4, 6, 7].

4. Dynamically Distinct Invariant 3-Spaces

In this section, we use similarity transformations to demonstrate the following proposition.

Proposition 7. *There are at most three dynamically distinct invariant 3-dimensional subspaces for any map p_C.*

Proof. There is a standard inner product on $\mathcal{M}$ that establishes an isometry between $\mathcal{M}$ and $\mathbb{R}^4$ with the Euclidean inner product. If A and B are matrices in $\mathcal{M}$, $\frac{1}{2}\text{tr}(AB^T)$ is the expression for the square of this inner product [5]. (The scalar factor of $\frac{1}{2}$ is chosen so that the identity matrix I has unit norm.)

We define the matrices H, G, and D as follows:

$$H = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Together with the identity matrix I, these matrices form an orthonormal basis for $\mathcal{M}$ with respect to the inner product defined above. The matrices $\pm H$ are unique in that they are the only two normal[1] matrices in $\mathcal{M}$ with eigenvalues $\pm i$. A calculation shows that a matrix with unit norm is orthogonal to both H and I if and only if it is a normal matrix with eigenvalues ± 1. Consequently, any 3-dimensional subspace containing all multiples of I must contain a normal matrix with eigenvalues ± 1.

[1] A matrix Z is normal if $Z^T Z = ZZ^T$

Let W be an invariant 3-dimensional subspace for a map p_C. We first show that we may assume without loss of generality that W contains the basis elements I and D.

Normal matrices with distinct real eigenvalues are orthogonally diagonalizable. Therefore, there exists some orthogonal similarity transformation Q_* taking any given normal matrix with eigenvalues ± 1 to the diagonal matrix D. Since every 3-dimensional subspace containing all multiples of I contains a normal matrix with eigenvalues ± 1, there is a map Q_* that takes a 3-dimensional subspace invariant under the action of p_C to an isometric and dynamically equivalent 3-space containing I and D that is invariant under the action of $p_{Q^{-1}CQ}$.

Now, given I and D as two basis elements for an invariant 3-space W, the third orthonormal basis element B must be some linear combination of G and H, having the form

$$B = \begin{bmatrix} 0 & y+z \\ -y+z & 0 \end{bmatrix}.$$

The eigenvalues of this basis element will either be complex, real and identical, or real and distinct depending on whether y is greater than z, the two variables are equal, or z is greater than y, respectively.

Let S be an invertible diagonal matrix, so that it has the form

$$S = \begin{bmatrix} x+w & 0 \\ 0 & x-w \end{bmatrix},$$

where $x \neq \pm w$. The associated similarity transformation S_* preserves the plane of matrices spanned by I and D, possibly transforming the basis element B.

- If B has complex eigenvalues, it is always possible to find a transformation S_* of this form that takes B to some multiple of H.
- If B has distinct real eigenvalues, it is possible to find a transformation S_* taking B to some multiple of G.
- If B has identical real eigenvalues, then B has a single nonzero entry in one of the two off-diagonal positions. The two possible 3-spaces occurring in this case are isometric and dynamically equivalent via conjugation by the matrix H.

Since similarity transformations preserve the dynamics of polynomial maps, we conclude that are at most three dynamically distinct invariant 3-spaces for any map p_C. We can describe the dynamics in any invariant 3-space in terms of the dynamics of an appropriate map acting on either the 3-space spanned by I, D, and H, the 3-space spanned by I, D, and G, or the 3-space of upper triangular matrices. □

If C has complex eigenvalues, only one of these 3-spaces contains the invariant plane $\mathcal{P}(C)$, and so there is only one dynamically distinct invariant 3-space. If C has identical real eigenvalues, there will be exactly two dynamically distinct invariant 3-spaces. If C has distinct real eigenvalues, all three types of dynamically distinct 3-spaces occur as invariant sets.

5. Critical Orbits and Attracting Cycles

Let us consider whether polynomial matrix maps p_C have one of the most well-known properties of polynomial maps of the Riemann sphere. For polynomial maps of the Riemann sphere of degree at least 2, every attracting cycle must attract the orbit of a critical point [2]. This limits the number of attracting cycles that can exist to the number of critical points of p_C. To determine whether an analogous result holds for polynomial matrix maps we must first find a dynamically appropriate definition of *critical point*.

We define a critical point of p_C to be any matrix A at which all partial derivatives within the plane $\mathcal{P}(Z)$ vanish, for some matrix Z not a multiple of the identity. Note that the set of critical points of p_C does not depend on the value of the constant C since it represents a translation in matrix space. Our definition guarantees that the restriction of the map p_C to any potential invariant plane will be equipped with the same set of critical points as the appropriate corresponding 2-dimensional dynamical system. Requiring instead that all partials in the 4-dimensional matrix space vanish can exclude points that are in fact critical to the dynamics.

5.1. Renegade Attracting Cycles Induced by Quadratic Matrix Maps

Let us examine whether every attracting cycle attracts the orbit of a critical point in the case of the family of quadratic maps

$q_C(Z) = Z^2 + C$. For this family, $Z = 0$ is the only critical point coming from planes with complex or distinct real eigenvalues. In planes with identical real eigenvalues, the set of critical points consists of all trace zero matrices. Note that the square of any of the critical points is the zero matrix. Thus, we may simply consider whether the orbit of $Z_0 = 0$ finds all attracting cycles induced by q_C. We further note that the orbit of $Z_0 = 0$ must remain in the invariant plane $\mathcal{P}(C)$. Therefore, if an attracting cycle exists outside of the plane $\mathcal{P}(C)$, the critical orbit could not possibly find it.

We test our conjecture that every attracting cycle attracts the orbit of $Z_0 = 0$ experimentally. We choose values of C from the Jordan canonical plane with complex eigenvalues. As we know from our previous discussion, the dynamics in the invariant plane $\mathcal{P}(C)$ is equivalent to the dynamics of q_C in the complex plane. For C values in this plane, there is only one dynamically distinct invariant 3-space. Let D be the diagonal matrix with entries 1 and -1 as in section 4. Without loss of generality, we may restrict our attention to the dynamics of q_C in the invariant 3-space containing the Jordan canonical plane with complex eigenvalues and the matrix D. For C not a multiple of the identity, the plane $\mathcal{P}(D)$ is not invariant under the action of p_C, and so it will be interesting to see how orbits originating in this plane behave.

We will color each point in the plane $\mathcal{P}(D)$ to indicate the ultimate fate of the orbit beginning at that point under the repeated action of p_C. Black indicates that some matrix in the first 500 iterations has a determinant[2] exceeding 4. White indicates that the orbit comes extremely close to $\mathcal{P}(C)$ after 500 iterations and suggests that the orbit may approach the attracting cycle in the invariant plane. Grey indicates that the orbit does neither of these, suggesting that it may approach a renegade attracting cycle outside of the invariant plane. Thus, the appearance of grey in any of the images suggests that our conjecture is likely to be false.

[2]Note that the implied escape criterion does not actually work for all orbits, since it is possible for a matrix with determinant larger than 4 to have an image under q_C with determinant less than 4. We therefore accept the possibility that some points colored black should perhaps be assigned another color instead, though this difficulty does not seem to occur in the images shown.

The Mandelbrot set for the map of the complex plane $q_C(Z) = Z^2 + C$ gives a convenient visual summary of the periods of attracting cycles corresponding to various values of the parameter C (see figure 1). For example, C values in the main cardioid correspond to maps q_C with an attracting fixed point, points in the large circle to the left of this region correspond to maps with an attracting 2-cycle, points in the next circle to the left yield an attracting 4-cycle. This period doubling continues infinitely, limiting on the Feigenbaum value of C on the real axis before giving way to regions for attracting cycles of all other periods. The complement of the Mandelbrot set (all of the points not colored black in figure 1) consists of the set of C values for which there is no attracting cycle in the complex plane under the action of q_C [2].

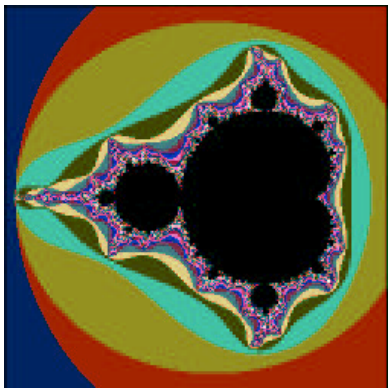
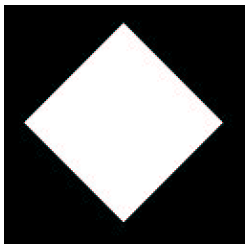
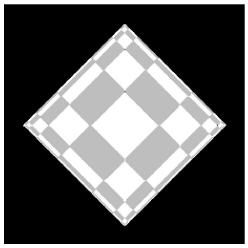


FIGURE 1. The complex quadratic Mandelbrot set.

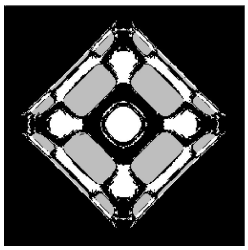
When we choose C values corresponding to the main cardioid of the complex Mandelbrot set for our experiment, the computer algorithm colors no points grey (see figure 2). However, as we choose C values from successive circles along the real axis, a period-doubling, hyperbolic checkerboard pattern with grey and white regions emerges. The checkerboard effect is more pronounced for values of C near (but not on) the real axis and degenerates for values of C that are farther from this axis.



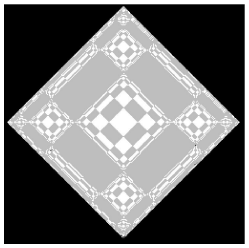
$C = -.7 + .01i$
$\mathcal{P}(C)$ has attracting fixed point



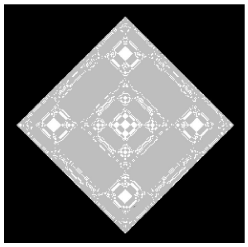
$C = -.77 + .01i$
$\mathcal{P}(C)$ has attracting 2-cycle



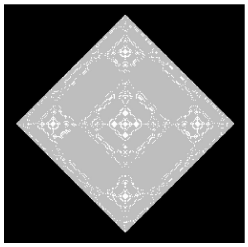
$C = -1 + .22i$
$\mathcal{P}(C)$ has attracting 2-cycle



$C = -1.26 + .001i$
$\mathcal{P}(C)$ has attracting 4-cycle



$C = -1.371 + .0001i$
$\mathcal{P}(C)$ has attracting 8-cycle



$C = -1.399 + .0001i$
$\mathcal{P}(C)$ has attracting 16-cycle

FIGURE 2. Hyperbolic checkerboard patterns suggest the existence of renegade attracting cycles.

5.2. Renegade Attracting Cycles

We can understand the curious hyperbolic checkerboard patterns described in the previous section by first considering what happens when C is a multiple of I and every plane $\mathcal{P}(Z)$ is invariant. Suppose that $C = cI$ is chosen so that q_c induces an attracting 2-cycle on the real line. The dynamics in planes of matrices with distinct real eigenvalues is equivalent to the topological product of the dynamics of q_c on two copies of the real line. Suppose that each of the two independent variables is on the 2-cycle. The two variables may execute the 2-cycle in phase, resulting in a 2-cycle on the axis

spanned by the identity, or they may be out of phase, resulting in a 2-cycle off this line (see figure 3).

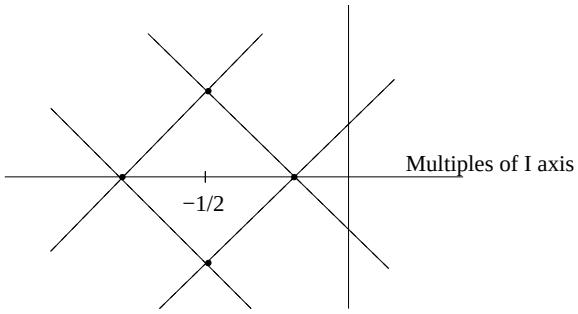


FIGURE 3. The origin of the renegade attracting 2-cycle

The white regions in the hyperbolic checkerboard consist of those initial conditions for which both variables approach their attracting cycles in phase. The grey regions in the hyperbolic checkerboard consist of those initial conditions for which the two variables approach their attracting cycles with different phases. The boundaries of the regions are the topological product of the set of repelling periodic points and all their pre-images under q_c on two copies of the real line.

When C has complex eigenvalues and is no longer a multiple of the identity, exactly one renegade 2-cycle survives (see figure 4). When C is not a multiple of the identity, a calculation shows that the renegade 2-cycle is unique if it exists. While C lies inside the circle to the left of the cardioid in the complex Mandelbrot set (producing an attracting 2-cycle in the invariant plane), the renegade 2-cycle exists and is attracting. The correspondence does not go the other way, however; there are C values that produce an attracting renegade 2-cycle that do not produce any attracting cycles in the invariant plane.

By an argument similar to the one given above for 2-cycles, we know that if $C = cI$ and q_c induces an attracting n-cycle on the real axis, then every plane $\mathcal{P}(Z)$ with distinct, real eigenvalues contains n n-cycles that attract nearby orbits within the plane.

Conjecture 8. *Let H be defined as in the proof of Proposition 7, let $C = cI + \epsilon H$, and suppose that q_c induces an attracting n-cycle on the real axis. For ϵ sufficiently small, q_C admits precisely n*

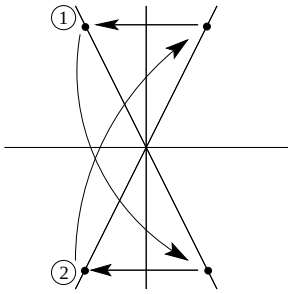


FIGURE 4. The only remaining renegade attracting 2-cycle. In this figure, the axis consisting of multiples of I extends orthogonally out of the page through the center of the diagram. The horizontal line is the invariant plane $\mathcal{P}(C)$ and the vertical line is the plane $\mathcal{P}(D)$.

attracting n-cycles — one in the invariant plane and $n-1$ renegade attracting cycles.

One piece of evidence for this conjecture is that computer-generated hyperbolic checkerboard patterns persist even under significant increase in the number of iterations.

5.3. The Quadratic Matrix Mandelbrot Set

Let us define the quadratic matrix Mandelbrot set for the map $q_C(Z) = Z^2 + C$ to be the set of parameter values C such that the orbit of $Z_0 = 0$ remains bounded under iteration in q_C. This orbit must remain in the invariant subspace $\mathcal{P}(C)$. We recall that for the quadratic map q_C acting on the complex plane or on the real line that when the orbit of $Z_0 = 0$ is unbounded, no attracting cycle can exist. This fact together with our earlier discussion about the dynamics in each of the 3 types of invariant planes tells us that parameters C in the complement of the quadratic matrix Mandelbrot set admit no attracting cycle in any invariant plane of the form $\mathcal{P}(Z)$.

6. Conclusion

The structure provided by 2×2 matrix operations turns out to be quite rigid. We have seen that for any polynomial matrix map p_C we can explain the dynamics of the entire 4-dimensional

space in terms of the dynamics of p_C in at most three 3-dimensional subspaces. The interaction between the easily understood dynamics of the invariant plane $\mathcal{P}(C)$ and the rest of a given invariant 3-space is intricate, and the examples we have shown reveal intriguing relationships that merit additional study.

The quadratic matrix Mandelbrot set does not give us nearly as much information about the dynamics of q_C as the complex Mandelbrot set does, even though it does allow us to determine when no attracting cycle exists in the invariant plane. We have seen that it is possible for renegade attracting cycles to exist when there is no attracting cycle in the invariant plane, and so there is no direct correspondence between attracting cycles in the invariant plane and renegade attracting cycles.

It is possible that there is nevertheless some relationship between attracting cycles within the invariant plane and renegade attracting cycles. We saw that the existence of an attracting 2-cycle implies the existence of a renegade attracting 2-cycle even though the converse does not hold. If some relationship between renegade cycles and invariant plane cycles does exist, the information encoded in the quadratic matrix Mandelbrot set may turn out to have broader dynamical significance after all. Without any such relationship between the attracting cycles, another method would be required to determine which parameters C admit attracting cycles under the action of q_C.

This discussion leaves a number of interesting open questions about the dynamics of polynomial matrix maps.

- Are all renegade cycles produced out of planes with distinct real eigenvalues in the manner of the 2-cycles presented in section 5.2 or can renegade cycles emerge spontaneously when C is nowhere near the real axis?
- Are renegade cycles possible in the 3-space that includes the Jordan canonical plane with complex eigenvalues if C has identical real eigenvalues or distinct real eigenvalues?
- For the family of quadratic maps q_C, do all C values with complex eigenvalues that admit an attracting n-cycle in the invariant plane also admit $n-1$ renegade attracting n-cycles, as we know to be the case for 2-cycles?

- Is there a convenient way of characterizing those C values that permit at least one attracting cycle to exist when there is no attracting cycle in the invariant plane?
- How do renegade cycles behave in the other 2 potential invariant 3-spaces? (Both of these 3-spaces contain planes with distinct, real eigenvalues. We know that whenever $C = cI$ and q_c induces an n-cycle on the real line, these planes will contain n n-cycles. Do some of these persist as renegade cycles? How does the stability of these cycles compare with the stability of cycles in the invariant plane?)

7. Acknowledgements

I would like to thank Michael Kinyon for suggesting this problem as an undergraduate research project and for his continuing guidance and encouragement. I am grateful to Robert Benedetto, David Fried, and Tasso Kaper for helpful comments and discussions. Many thanks also to the referee (unknown to me) who made many excellent suggestions for improving this article.

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