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# INDEPENDENCE

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ABSTRACT. We survey the very few known results on independent sets of topologies on topological spaces. We introduce the notion of *multiplicatively independent* ring topologies on a field.

The purpose of the first section of this note is to present a topological notion which

- has been stated in the literature in general topological terms over twenty years ago,
- has an easy intuitive interpretation,
- has an application in algebra dating back to ancient times,

and yet fails to attract the attention of general topologists. We attempt to generate interest by "surveying" the very few known results and asking some natural questions. In the second section, we consider a related notion for fields with a ring topology.

## 1. INDEPENDENCE IN TOPOLOGICAL SPACES

A collection $\mathbf{T}$ of topologies on a set $X$ is *independent* if, for each nonempty finite subset $\mathbf{F}$ of $\mathbf{T}$ and each choice of nonempty sets $U_{\mathcal{T}} \in \mathcal{T}$, for $\mathcal{T} \in \mathbf{F}$, $\cap\{U_{\mathcal{T}} : \mathcal{T} \in \mathbf{F}\} \neq \emptyset$. One readily checks that this is equivalent to the *Approximation Theorem*: Given $x_i$, $1 \leq i \leq n$, in $X$ and $\mathcal{T}_i$-neighborhoods $V_i$ of $x_i$, where $\mathcal{T}_i$, $1 \leq i \leq n$,

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are distinct topologies in $\mathbf{T}$, there exists $x \in X$ such that $x \in V_i$ for all $i$. Another condition clearly equivalent to independence of $\mathbf{T}$ is that the diagonal is dense in the product $\prod_{\mathcal{T}\in\mathbf{T}}(X,\mathcal{T})$.

The natural interpretation is that a pair of topologies form an independent set if they are "very different" in the sense that every neighborhood of any point in one topology is dense in the other topology: all points collected into a small region in one topology are spread throughout the space in the other topology.

The algebraic application mentioned is the Chinese Remainder Theorem: Given integers $a_1, \ldots, a_n$ and pairwise relatively prime integers $m_1, \ldots, m_n$, there exists an integer $a$ such that $a \equiv a_i \quad (m_i)$. We will rephrase this statement to show that it is a special case of the Approximation Theorem. Observe that, by factoring $m_i = \prod p_{ij}^{k_{ij}}$ into powers of primes and replacing the congruence $a \equiv a_i$ $(m_i)$ by the congruences $a \equiv a_i \quad (p_{ij}^{k_{ij}})$ for all $j$, we may assume in the statement of the Remainder Theorem that $m_i = p_i^{k_i}$, with the $p_i$ all distinct.

For each prime $p$, a metric is defined on $\mathbf{Z}$ by the conditions

$$d_p(x,y) = |x-y|_p, \qquad \text{where} \qquad |p^n a|_p = p^{-n}, \ \text{ when } p \nmid a.$$

This may be extended to a metric, with associated topology $\mathcal{T}_p$ called the $p$-adic topology, on $\mathbf{Q}$ by letting $|p^n(a/b)|_p = p^{-n}$, when $p \nmid a$ and $p \nmid b$. We let $V_i = \{a \in \mathbf{Z} : |a - a_i|_{p_i} \le p_i^{-k_i}\}$. For a fixed $i$, as $k_i$ varies over the positive integers, the sets $V_i$ vary over a base at $a_i$ for $\mathcal{T}_{p_i}$. In this notation, the Remainder Theorem becomes exactly the Approximation Theorem for the set of all $p$-adic topologies.

By a valuation we mean a Krull valuation to an ordered group whose operation we write multiplicatively. By an absolute value we mean a rank one valuation or a power of the usual absolute value on a subfield of the complex numbers.

The generalization of this Approximation Theorem form of the Remainder Theorem to any set of distinct topologies induced by absolute values on a field is due to Artin and Whaples [1]. The term independence is due to Shanks who first considered the notion as applying just to ring topologies on a field. An unpublished result of Shanks, quoted as Theorem 3 in [11], is the first theorem about independence stated in terms of topologies: A topology on a field induced by a valuation and a field topology not finer than the valuation induced topology are independent. Weber [9] extended Shanks' result to say that a finite number of valuation topologies and a field topology not finer than any of them form an independent set.

The only discussion in the literature of independence for topological spaces without an algebraic structure is in the form of brief introductory comments (which precede a deeper analysis of independent ring topologies) by Weber in [9] and [10]. Statements of some of these basic properties of independence also appear in [4, Theorems 3.4.1 and 3.4.2]. Some elementary results which include those made by Weber are described below.

We let $\mathbf{0}$ and $\mathbf{1}$ denote the trivial and discrete topologies, respectively, on any set. All spaces are assumed to have more than one point. For a set $\mathbf{T}$ of topologies on a set $X$, $\vee\mathbf{T}$ denotes the topology generated by $\cup\mathbf{T}$. The cardinality of a set $A$ is denoted by $\#A$.

When will a collection be independent?

Clearly, for any topology $\mathcal{T}$, $\{\mathbf{0}, \mathcal{T}\}$ is independent and, for $\mathcal{T} \neq \mathbf{0}$, $\{\mathbf{1}, \mathcal{T}\}$ is not independent. If $\mathbf{T}$ is an independent collection and for each $\mathcal{T} \in \mathbf{T}$, $\mathcal{S}_{\mathcal{T}}$ is a topology weaker than $\mathcal{T}$, then $\{\mathcal{S}_{\mathcal{T}} : \mathcal{T} \in \mathbf{T}\}$ is an independent collection. If $\mathbf{T}_i$ is an independent collection of topologies on a set $X$ for each $i \in I$ and the collections $\mathbf{T}_i$ are disjoint, then $\cup_{i\in I}\mathbf{T}_i$ is independent if and only if $\{\vee\mathbf{T}_i : i \in I\}$ is independent.

Let $\mathbf{T}$ be a collection of topologies on $X$; let $Y$ be a subset of $X$; and let $\mathbf{T}_Y = \{\mathcal{T}|_Y : \mathcal{T} \in \mathbf{T}\}$: (1) If $\mathbf{T}$ is independent and $Y \in \mathcal{T}\backslash\{\emptyset\}$ for some $\mathcal{T} \in \mathbf{T}$, then $\mathbf{T}_Y$ is independent. (2) If $\mathbf{T}_Y$ is independent and $Y$ is dense with respect to each $\mathcal{T} \in \mathbf{T}$, then $\mathbf{T}$ is independent. Consequently, if $Y$ is cofinite and each $\mathcal{T} \in \mathbf{T}$ satisfies the $\mathrm{T}_1$ separation axiom and has no isolated points, then $\mathbf{T}$ is independent if and only if $\mathbf{T}_Y$ is independent.

What can be asserted about an independent collection $\mathbf{T}$?

If $\mathcal{S} \in \mathbf{T}$ is $\mathrm{T}_1$, then, for $\mathcal{T} \in \mathbf{T}\backslash\{\mathcal{S}\}$, $\mathcal{T}$ has no isolated points. (Proof: $(\complement\{x\}) \cap \{x\} = \emptyset$ would contradict independence.)

If either (1) some topology in $\mathbf{T}$ is $\mathrm{T}_1$ and has no isolated points or (2) two distinct topologies in $\mathbf{T}$ are $\mathrm{T}_1$, then $\vee\mathbf{T} \neq \mathbf{1}$. (Proof: From the paragraph above, (2) implies (1). If $\mathcal{T}_1 \in \mathbf{T}$ is $\mathrm{T}_1$ and has no isolated points and $\cap_{i=1}^{n} U_i = \{x\}$, where $U_i \in \mathcal{T}_i \in \mathbf{T}$, then

$\complement\{x\} \cap (\cap_{i=1}^n U_i) = \emptyset$, which may be written as

$$(\complement\{x\} \cap U_1) \cap (\cap_{i=2}^n U_i) = \emptyset.$$

Independence implies $\complement\{x\} \cap U_1 = \emptyset$, so $U_1 = \{x\}$, contradicting the hypothesis.)

If $\mathcal{T}_1, \mathcal{T}_2 \in \mathbf{T}$ and $\mathcal{T} \subset \mathcal{T}_1 \cap \mathcal{T}_2$, then there are no pairs of disjoint nonempty $\mathcal{T}$-open sets. Hausdorff spaces and nontrivial regular (not necessarily $\mathrm{T}_1$) spaces have pairs of disjoint nonempty open sets. Example: $\mathcal{T}$ in the last statement need not be $\mathbf{0}$. Let $X$ be a set of cardinality greater than an infinite cardinal $\alpha$; let $\mathcal{T}_\alpha$ be the topology whose nonempty open sets are the complements of sets of cardinality at most $\alpha$; and let $\mathbf{T}$ consist of all topologies weaker than or equal to $\mathcal{T}_\alpha$. The cofinite topology $\mathcal{T}$ is in $\mathbf{T}$, and $\mathcal{T} \subset \mathcal{T} \cap \mathcal{T}_\alpha$.

A useful consequence of independence is the following: If each topology $\mathcal{T} \in \mathbf{T}$ arises from a uniformity $\mathcal{U}_\mathcal{T}$, then the completion of $X$ with respect to the least uniformity finer than each $\mathcal{U}_\mathcal{T}$ (which has $\vee\mathbf{T}$ as its associated topology) is the product of the completions of $X$ with respect to each $\mathcal{U}_\mathcal{T}$. (Proof: The natural embedding into the diagonal takes $X$ onto a dense subset of the product of the completions.)

**Question 1.** The $p$-adic topologies on $\mathbf{Z}$ constitute an infinite independent collection of topologies with a rich structure: they are homogeneous totally bounded metric spaces. It is also known ([3]; see also [8]) that on an algebraically closed field $K$ such that $(\#K)^{\aleph_0} = \#K$ (for example, the complex field) there is an independent collection consisting of $2^{\#K}$ topologies induced by complete absolute values. Thus, given a cardinal $\alpha$, there are independent collections of cardinality greater than $\alpha$ consisting of homogeneous complete metric spaces.

*Do there exist independent collections of arbitrary cardinality consisting of compact Hausdorff spaces?*

**Question 2.** An independent collection of topologies on a set $X$ will be called maximal if it is maximal (under set containment) among independent collections of proper (i.e., nontrivial, nondiscrete) topologies on $X$. By Zorn's Lemma, each independent collection of proper topologies is contained in a maximal independent collection. Let $I(\alpha)$ denote the least upper bound of the cardinalities of independent collections of proper topologies on a set of cardinality $\alpha$.

$I(2) = 1$ and there are two maximal independent collections, each having a single member; the topologies involved are homeomorphic. On the set $X = \{a, b, c\}$, a direct computation yields $I(3) = 6$. Among the maximal independent collections are the following, listed (omitting $\emptyset$ and $X$ in each topology) with commas separating sets in a topology and slashes separating topologies:

$$\{a/ab/ac/a, ab/a, ac/a, ab, ac\} \qquad \{a, bc/ab\}.$$

*For which $\alpha$ do all maximal collections of independent topologies on a set of cardinality $\alpha$ have cardinality $I(\alpha)$? For a given $\alpha$ what is $I(\alpha)$?*

**Question 3.** Let $\mathbf{S}$ be a set of topologies on $X$, for example, all nontrivial topologies or all Hausdorff topologies on $X$; let $\mathcal{T}$ be a topology on $X$; and let $\mathbf{T}(\mathbf{S}, \mathcal{T})$ be the set of topologies in $\mathbf{S}$ such that $\{\mathcal{S}, \mathcal{T}\}$ is independent.

*For $\mathbf{S}$ being the set of all Hausdorff topologies, when is $\mathbf{T}(\mathbf{S}, \mathcal{T})$ empty?* Equivalently: when does the collection consisting of the empty set and all $\mathcal{T}$-dense subsets not contain a Hausdorff topology? Restricting consideration to Hausdorff topologies eliminates the possibility of including very weak topologies which do not constitute a substantive solution to the problem.

**Example 1.** We construct a Hausdorff topology on $[0, 1]$ which is independent of the usual topology. To begin, let $(X, \mathcal{T})$ be an arbitrary topological space, and let $\mathcal{D}$ be the collection consisting of the empty set and all $\mathcal{T}$-dense subsets of $X$. Let $\mathcal{E}$ be a pairwise disjoint subcollection of $\mathcal{D}\backslash\{\emptyset\}$. If, for each $E \in \mathcal{E}$, $\mathcal{T}_E$ is a topology on $E$ such that $\mathcal{T}_E \subset \mathcal{D}$, then

$$\mathcal{S} = \{X\} \cup (\{\cup_{E\in\mathcal{E}} U_E : U_E \in \mathcal{T}_E \text{ for all } E \in \mathcal{E}\})$$

is a topology and $\mathcal{S} \subset \mathcal{D}$. Now let $X = [0, 1]$ with the usual topology, and let $\mathcal{E}$ consist of the equivalence classes of the relation $x \equiv y$ if $x - y \in \mathbf{Q}$. The equivalence class $\hat{x}$ of $x$ is $(x + \mathbf{Q}) \cap [0, 1]$. Let $A \subset [0, 1]$ consist of a single representative from each equivalence class, with 0 being the representative of $\mathbf{Q} \cap [0, 1]$. From Section 1, we see that, for any prime $p$, $\mathcal{T}_p|_{\mathbf{Q}\cap(0,1)}$, the $p$-adic topology restricted to the rationals in the open interval $(0, 1)$, is independent of the usual topology on $\mathbf{Q} \cap (0,1)$. Since $(0,1)$ is dense in $[0,1)$ and $[0,1]$, $\mathcal{T}_p|_{\mathbf{Q}\cap[0,1)} \subset \mathcal{D}$ and $\mathcal{T}_p|_{\mathbf{Q}\cap[0,1]} \subset \mathcal{D}$. For $x \in A\backslash\{0\}$ we let $\varphi_x : [0,1) \longrightarrow [0,1)$ be defined by $\varphi_x(t) = t + x$ if $t < 1 - x$; $\varphi_x(t) = t + x - 1$ if $1 - x \le t < 1$. Then $\varphi_x|_{Q\cap[0,1)}$ is a bijection onto

$$(x + \mathbf{Q}) \cap [0,1) = (x + \mathbf{Q}) \cap [0,1] = \hat{x},$$

and the restriction of $\varphi_x$ carries sets in $\mathcal{D}$ into $\mathcal{D}$. We let $\mathcal{T}_{\hat{0}} = \mathcal{T}_p|\mathbf{Q} \cap [0,1]$ and, for $x \in A\backslash\{0\}$, we let $\mathcal{T}_{\hat{x}} = \{\varphi_x(U) : U \in \mathcal{T}_p|_{\mathbf{Q}\cap[0,1)}\}$. We still obtain a topology independent of the usual topology if we choose a prime $p(x)$ for each $x \in A$ instead of fixing $p$ at the outset.

This example is not entirely satisfying, in the sense that it relies heavily on algebra, while our question is whether or not independence is (pardon the pun and vague language) independent of algebra.

**Question 4.** Call two maximal independent collections on a set $X$ *equivalent* if there is a bijection $\Phi$ between them such that $(X, \mathcal{T})$ and $(X, \Phi(\mathcal{T}))$ are always homeomorphic. Call two maximal independent families on a set $X$ *strongly equivalent* if there is a bijection $f$ of $X$ onto itself and a bijection $\Phi$ between the two independent collections such that $f : (X, \mathcal{T}) \longrightarrow (X, \Phi(\mathcal{T}))$ is always a homeomorphism.

For any point $x$ in any set $X$, the collection $\mathbf{T}_x$ consisting of all nontrivial topologies $\mathcal{T}$ on $X$ for which $x$ belongs to every nonempty set in $\mathcal{T}$ is readily seen to be a maximal independent collection. The families $\mathbf{T}_x$ for various choices of $x$ are strongly equivalent.

*How many inequivalent (strongly inequivalent) maximal independent collections are there on a set of a given cardinality?*

## 2. Independence In Fields With a Ring Topology

Let $E^*$ denote the nonzero elements of a subset $E$ of a field. A topology on a ring is called a ring topology if subtraction and multiplication are continuous. A ring topology on a field (or, in Theorem 1, a division ring) $K$ is called a field topology if inversion on $K^*$ is continuous. For a ring topology $\mathcal{T}$, let $\mathcal{B}(\mathcal{T})$ denote the neighborhood filter at zero.

On a field, the only ring topology which is not Hausdorff is $\mathbf{0}$. We denote by $\mathcal{S}\wedge\mathcal{T}$ ($\mathcal{S}+\mathcal{T}$) the finest ring (additive group) topology on a field which is weaker than ring topologies $\mathcal{S}$ and $\mathcal{T}$. It is possible that $\mathcal{S}\wedge\mathcal{T}$, $\mathcal{S}+\mathcal{T}$ and $\mathcal{S}\cap\mathcal{T}$ all be distinct (see [7]). The set $\{U+V : U\in\mathcal{B}(\mathcal{S}), V\in\mathcal{B}(\mathcal{T})\}$ is a neighborhood base at zero for $\mathcal{S}+\mathcal{T}$. We interpret $\mathcal{S}\wedge\mathcal{T}=\mathbf{0}$ as meaning that $\mathcal{S}$ and $\mathcal{T}$ are quite different in the sense that $\mathcal{S}\cap\mathcal{T}$ is too small to contain a Hausdorff ring topology. A subset $B$ of a field with a ring topology $\mathcal{T}$ is *bounded* if, for all $U\in\mathcal{B}(\mathcal{T})$, there exists $V\in\mathcal{B}(\mathcal{T})$ such that $VB\subset U$.

In [5] we introduced a notion of two ring topologies $\mathcal{S}$ and $\mathcal{T}$ "being different" which is closely related to the condition $\mathcal{S}\vee\mathcal{T}=\mathbf{1}$. We say that $\mathcal{S}$ is $\mathcal{T}$-*big* if for, for every $\mathcal{T}$-bounded set $B$, there exists an $\mathcal{S}$-neighborhood of zero, $U$, such that $B\cap U^*=\emptyset$. If $\mathcal{S}\vee\mathcal{T}=\mathbf{1}$, then $\mathcal{S}$ is $\mathcal{T}$-big, and the converse is true if $\mathcal{S}$ and $\mathcal{T}$ are first countable or if $\mathcal{T}$ is locally bounded.

In Section 1 we noted three equivalent formulations of independence. For a field with a ring topology we give additional equivalent statements that have elementary proofs. Some of these equivalences and their proofs appear in [9] and [10] (a portion of these were known earlier). For convenience, we give a complete proof.

**Theorem 1.** *Let $\mathbf{T}=\{\mathcal{S},\mathcal{T}\}$ be a set of two nondiscrete, Hausdorff ring topologies on a field (or, more generally, a division ring) $K$. The following are equivalent:*

*(1) $\mathbf{T}$ is independent.*
*(1′) $\mathbf{T}_{K^*}$ (defined as in Section 1) is independent.*
*(2) $1\in U+V$ for all $(U,V)\in\mathcal{B}(\mathcal{S})\times\mathcal{B}(\mathcal{T})$.*
*(3) $U+V=K$ for all $(U,V)\in\mathcal{B}(\mathcal{S})\times\mathcal{B}(\mathcal{T})$ (i.e., $\mathcal{S}+\mathcal{T}=\mathbf{0}$).*
*(4) $U\cap(1+V)\neq\emptyset$ for all $(U,V)\in\mathcal{B}(\mathcal{S})\times\mathcal{B}(\mathcal{T})$.*
*(5) $0\in S+T$ for all $(S,T)\in(\mathcal{S}\backslash\{\emptyset\})\times(\mathcal{T}\backslash\{\emptyset\})$.*
*(6) $S+T=K$ for all $(S,T)\in(\mathcal{S}\backslash\{\emptyset\})\times(\mathcal{T}\backslash\{\emptyset\})$.*

*If at least one of the topologies $\mathcal{S}$ and $\mathcal{T}$ is a field topology, then the above statements are equivalent to each of the following three statements:*

*(3′) $(1+U)^*(1+V)^*=K^*$ for all $(U,V)\in\mathcal{B}(\mathcal{S})\times\mathcal{B}(\mathcal{T})$.*
*(5′) $1\in ST$ for all $(S,T)\in(\mathcal{S}|_{K^*}\backslash\{\emptyset\})\times(\mathcal{T}|_{K^*}\backslash\{\emptyset\})$.*
*(6′) $ST=K^*$ for all $(S,T)\in(\mathcal{S}|_{K^*}\backslash\{\emptyset\})\times(\mathcal{T}|_{K^*}\backslash\{\emptyset\})$.*

*Statements (7), (8) and (8′) below are equivalent to each other and, if at least one of $\mathcal{S}$ and $\mathcal{T}$ is a field topology, are implied by all of the statements above.*

*(7)* $1 \in UV$ *for all* $(U,V) \in \mathcal{B}(\mathcal{S}) \times \mathcal{B}(\mathcal{T})$.

*(8)* $UV = K$ *for all* $(U,V) \in \mathcal{B}(\mathcal{S}) \times \mathcal{B}(\mathcal{T})$.

*(8′)* $U^*V^* = K^*$ *for all* $(U,V) \in \mathcal{B}(\mathcal{S}) \times \mathcal{B}(\mathcal{T})$.

*Proof.* The equivalence of (1) and (1′) was observed in Section 1.

The proof by Weber in [9], repeated below, of the equivalence of (1), (3) and (6) uses only these properties: addition is continuous with respect to both $\mathcal{S}$ and $\mathcal{T}$; negation is continuous with respect to at least one these topologies; $K$ is a (not necessarily commutative) group under addition. Thus, the same proof applies to the multiplicative group $K^*$, under the additional hypothesis that $\mathcal{S}$ or $\mathcal{T}$ be a field topology, and establishes that (1′), (3′) and (6′) are equivalent.

(1) $\Longrightarrow$ (6): If $S \in \mathcal{S}\backslash\{\emptyset\}$, $T \in \mathcal{T}\backslash\{\emptyset\}$ and $x \in K$, then (from the continuity of negation with respect to $\mathcal{T}$) $x - T \in \mathcal{T}\backslash\{\emptyset\}$; $S \cap (x - T) \neq \emptyset$; $s = x - t$ for some $s \in S$ and $t \in T$; and $x = s + t \in S + T$.

(6) $\Longrightarrow$ (3), since every neighborhood contains a nonempty open set in the topology with respect to which it is a neighborhood.

(3) $\Longrightarrow$ (1): Suppose $s \in S \in \mathcal{S}\backslash\{\emptyset\}$ and $t \in T \in \mathcal{T}\backslash\{\emptyset\}$. Then $-s+S \in \mathcal{B}(\mathcal{S})$ and (again using only the continuity of negation with respect to $\mathcal{T}$) $-T+t \in \mathcal{B}(\mathcal{T})$. Therefore, $-s+t \in (-s+S)+(-T+t)$ and $-s + t = -s + s' - t' + t$, for some $s' \in S$ and $t' \in T$, and, consequently, $s' = t' \in S \cap T$.

(2) $\Longrightarrow$ (3): If $U \in \mathcal{B}(\mathcal{S})$ and $V \in \mathcal{B}(\mathcal{T})$ and $x \in K^*$, then $1 \in x^{-1}U + x^{-1}V$, so that $x \in U + V$. Certainly also $0 \in U + V$.

[(1), and thus] (3) $\Longrightarrow$ (4), again because neighborhoods contain nonempty open sets.

(4) $\Longrightarrow$ (2): If $U \in \mathcal{B}(\mathcal{S})$ and $V \in \mathcal{B}(\mathcal{T})$ then $-V \in \mathcal{B}(\mathcal{T})$, so $U \cap (1 - V) \neq \emptyset$. Therefore, $u = 1 - v$, or equivalently $1 = u + v$, for some $u \in U$ and $v \in V$.

(5) $\Longleftrightarrow$ (6): Obviously (6) implies (5). Conversely, given $S \in \mathcal{S}\backslash\{\emptyset\}$, $T \in \mathcal{T}\backslash\{\emptyset\}$ and $x \in K$, then $-x + S \in \mathcal{T}\backslash\{\emptyset\}$. Thus, $0 = -x + s + t$ for some $s \in S$ and $t \in T$.

(5′) $\Longleftrightarrow$ (6′): Certainly (6′) implies (5′). Conversely, if $S \in \mathcal{S}|_{K^*}\backslash\{\emptyset\}$, $T \in \mathcal{T}|_{K^*}\backslash\{\emptyset\}$ and $x \in K^*$, then $1 \in (x^{-1}S)T$, so $x \in ST$.

(8) $\Longleftrightarrow$ (8′) is obvious.
(7) $\Longleftrightarrow$ (8′): The proof is similar to that of (5′) $\Longleftrightarrow$ (6′).
(5′) $\Longrightarrow$ (7) is obvious. □

By replacing $x - T$ by $-T + x$ in the proof that (1) implies (6), we obtain that $T + S = K$. Thus, if $K$ is a division ring, the last statement shows that, also, $T^*S^* = K^*$.

Theorem 1 remains true if, in (5), 0 is replaced by any fixed element $x \in K$; and, in (2), (4) and (5′), 1 can be replaced by any nonzero element.

Topologies satisfying equivalent conditions (7), (8) and (8′) will be called *multiplicatively independent*. Example 2 below shows that in general neither independence nor multiplicative independence imply the other.

Certainly, if multiplicatively independent topologies are weakened, the new topologies will also be multiplicatively independent.

**Theorem 2.** *$\mathcal{S} \wedge \mathcal{T} = \mathbf{0}$ for multiplicatively independent ring topologies $\mathcal{S}$ and $\mathcal{T}$ on a field.*

*Proof.* Suppose $\mathcal{S}$ and $\mathcal{T}$ are ring topologies on $K$. If $K \neq U \in \mathcal{B}(\mathcal{S} \wedge \mathcal{T})$ and $VV \subset U$ for $V \in \mathcal{B}(\mathcal{S} \wedge \mathcal{T})$, then $VV \neq K$ and $V \in \mathcal{B}(\mathcal{S})$ and $V \in \mathcal{B}(\mathcal{T})$. □

**Theorem 3.** *If $\mathcal{U}$ is a topology on the field $K$ induced by a nontrivial valuation and $\mathcal{S}$ and $\mathcal{T}$ are $\mathcal{U}$-big ring topologies on $K$, then $\mathcal{S}$ and $\mathcal{T}$ are not multiplicatively independent.*

*Proof.* Let $\mathcal{U}$ be induced by $\nu$. There exists $U \in \mathcal{B}(\mathcal{S})$ and $V \in \mathcal{B}(\mathcal{T}))$ such that $\nu(u), \nu(v) > 1$ for $u \in U^*$ and $v \in V^*$. Then for such $u$ and $v$, $\nu(uv) > 1$, so $1 \notin UV$. □

Finally, we consider the relation between each pair of the following four topologies on $\mathbf{Q}$, and use the construction of the topology labelled $\mathcal{T}$ to show that independent topologies need not be multiplicatively independent.

$\mathcal{T}_\infty$: the usual topology;
$\mathcal{T}_p$: the $p$-adic topology for any fixed prime $p$;
$\mathcal{T}_Z$: the ring topology on $\mathbf{Q}$ having the nonzero ideals of $\mathbf{Z}$ as a neighborhood base at zero;

$\mathcal{T}$: the ring topology, introduced by Mutylin ([2]) and generalized in [6], which has as a neighborhood base at zero the sequence $U_n$ defined as follows.

Let $\gamma$ be a positive real number less than or equal to 1; let $\{\alpha_{i\,n}\}$, where $i$ and $n$ are positive integers such that $i \geq n$, be real numbers greater than or equal to 1; and let $\{P_i\}$, where $i$ is a positive integer, be positive real numbers satisfying $P_1 \geq 1/\gamma$ and

$$\alpha_{i\,n} \geq \alpha_{i\,n+1}(2\sum_{j=n+1}^{i-1} \alpha_{j\,n+1}P_j + \alpha_{i\,n+1}P_i); \frac{\gamma P_{i+1}}{2 \cdot 2^{3^i}} \geq \sum_{j=1}^{i} \alpha_{j\,1}P_j.$$

Let $\{t_i\}$ be a sequence of integers such that $\gamma P_i \leq |t_i| \leq P_i$, for all $i$.

$$U_n = \{\sum_{i=n}^{k} \frac{a_i}{b_i} t_i : a_i, b_i \in \mathbf{Z}, k \geq n, \left|\frac{a_i}{b_i}\right| \leq \alpha_{i\,n}, 0 < |b_i| \leq 2^{3^{i-n}}\}.$$

**Example 2.** (1) $\mathcal{T}_p$ and $\mathcal{T}_\infty$ are independent field topologies since they are induced by valuations.

(2) $\mathcal{T}_p$ and $\mathcal{T}$ are independent if $p \nmid t_r$ for any $r$: Given a positive integer $n$, choose $r$ such that $r > n$, and $P_r > p^{n+1}$ and solve $kt_r \equiv 1 \ (p^{n+1})$, with $0 < k < p^{n+1}$. The first inequality in the definition of $U_n$ implies that $\alpha_{r\,n} \geq P_r$. Thus,

$$1 = (1 - kt_r) + kt_r \in \{x \in \mathbf{Q} : |x|_p < p^{-n}\} + U_n.$$

Since $\{P_r\}$ increases monotonically to infinity, it is possible to choose the sequence $t_i$ so that $p \nmid t_i$ for any $i$, provided we have chosen $(1-\gamma)P_1 > p+1$. Note that the theorem of Shanks cited in Section 1 does not apply here, since $\mathcal{T}$ is not a field topology (see [6]).

(3) $\mathcal{T}_{\mathbf{Z}}$ and $\mathcal{T}_\infty$ are not independent. However, by writing $\frac{a}{b} \in \mathbf{Q}$, where $a, b \in \mathbf{Z}$, in the form

$$\frac{a}{b} = (k!\,a)(\frac{1}{k!\,b}),$$

we see that $\mathcal{T}_{\mathbf{Z}}$ and $\mathcal{T}_\infty$ are multiplicatively independent.

(4) $\mathcal{T}_\infty$ and $\mathcal{T}$ are multiplicatively independent:

$$\frac{a}{b} = (\frac{a}{b}t_r)(\frac{1}{t_r}).$$

The second inequality preceding the definition of $U_n$ implies that $\pm t_n/2$ are the nonzero elements in $U_n$ with smallest absolute value. Therefore, $\mathcal{T}_\infty \vee \mathcal{T} = \mathbf{1}$, and the topologies are not independent.

(5) $\mathcal{T}$ and $\mathcal{T}_Z$ are not multiplicatively independent: For the topology $\mathcal{U}$ of the usual absolute value, $\mathcal{T}$ and $\mathcal{T}_{\mathbf{Z}}$ are both $\mathcal{U}$-big. Thus, by Theorem 3, $\mathcal{T}$ and $\mathcal{T}_{\mathbf{Z}}$ are not multiplicatively independent.

Numbers $\alpha_{i\,n}$ and $P_i$ satisfying the two inequalities can be determined inductively so that, after $P_1, \ldots, P_n$ have been chosen, there is a real number $a$ such that $P_{n+1}$ may be chosen to be any number larger than $a$. If we choose each $P_n$ to be of the form $k_n! + 1$ and let $t_n = P_n$, then $1 = -k_n! + t_n$ and $\mathcal{T}$ and $\mathcal{T}_{\mathbf{Z}}$ are independent.

(6) $\mathcal{T}_p < \mathcal{T}_{\mathbf{Z}}$.

(7) In the definition of $U_n$ we may choose $\gamma < 1$ and elements $t_i$ such that $\gamma P_i \leq t_i < t_i + 1 \leq P_i$. Let $\mathcal{T}'$ be the topology determined by the sequence $U'_n$, where the sequence $U'_n$ is obtained by substituting $t_i + 1$ for $t_i$ in the definition of $U_n$. Since $1 = (t_n + 1) - t_n \in U'_n + U_n$, $\mathcal{T}$ and $\mathcal{T}'$ are independent. However, as in (5), both topologies are $\mathcal{U}$-big and, hence, not multiplicatively independent.

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