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LOCALLY COMPACT GROUPS WITH FEW ORBITS UNDER AUTOMORPHISMS

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Abstract. The number of orbits under the group of all topological automorphisms was introduced as a new cardinal invariant for topological groups. For locally compact groups, the impact of bounds for this number is investigated.

Compact groups with finitely many orbits under automorphisms are pro-finite groups with an open solvable characteristic subgroup; detailed information is obtained for the case of at most 4 orbits. Connected locally compact groups with countably many orbits are simply connected nilpotent Lie groups. Abelian locally compact groups with finitely many orbits under automorphisms are understood quite well.

Several open problems are formulated.

This paper presents an overview of the results obtained in [18], [19], [20], [21], and [24]. New results are contained in 8.3, 8.4, 4.5, 3.8, and 6.3. We take the opportunity to add a few extensions, and include a list of open problems at the end.

1. Basics

For a topological group G, we denote by $\omega(G)$ the number of orbits under the group $\mathrm{Aut}(G)$ of all (topological) automorphisms of G. More generally, we write $\omega(A)$ for the number of orbits under $\mathrm{Aut}(A)$, for any (topological) algebraic structure A.

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The class of locally compact groups G with $\omega(G) \leq 2$ (so called *homogeneous* groups, see [19]) of course contains the additive groups of topological vector spaces of finite dimension over a local field. Conversely, these groups were characterized by mild (and natural) additional topological hypotheses inside this class, see 2.9 below. However, there also are locally compact abelian homogeneous groups that do not allow any continuous multiplication by scalars from any field: the divisible hulls of powers of the p-adic integers, see 2.6 and 2.7 below.

Notation 1.1. As a rule, mappings will be applied from the right, and written as exponentiation: thus the composition $\alpha\beta$ means $x \mapsto x^{\alpha\beta} = (x^\alpha)^\beta$. Linear maps will also be written as juxtaposition: $v \mapsto v\lambda$, reminding of multiplication of (row) vectors by matrices.

The *cartesian product* of a family $(G_i)_{i\in I}$ of topological groups is denoted by $\mathsf{X}_{i\in I} G_i$; as a rule, this group will be endowed with the product topology. If $G_i = G$ for each $i \in I$, we will write $G^I := \mathsf{X}_{i\in I} G_i$, and $G^{(I)}$ for the set of all elements of finite support in G^I. If two normal subgroups A and B of a topological group G generate G and intersect trivially, there is a continuous bijective group homomorphism $\varphi\colon A \times B \to G\colon (a,b) \mapsto ab$. If this homomorphism is also a homeomorphism, we write $G = A \oplus B$.

The cyclic group (or ring) with n elements will be denoted by $\mathbb{Z}(n) = \mathbb{Z}/n\mathbb{Z}$, and $\mathbb{F}_q$ is the field with q elements.

For each group G (written multiplicatively) and each $g \in G$, we have the *order* $\operatorname{ord}(g) := \inf\left\{n \in \mathbb{N} \mid n > 0, g^n = 1\right\}$. The invariant $\omega(G)$ clearly depends on the size of the sets $\operatorname{Ord}(G) := \left\{\operatorname{ord}(g) \mid g \in G\right\} \subseteq \mathbb{N} \cup \{\infty\}$ of all orders of elements of G, and the set $\operatorname{pord}(G) := \operatorname{Ord}(G) \cap \mathbb{P}$, where $\mathbb{P}$ denotes the set of all prime numbers. If there is a prime p such that $\operatorname{Ord}(G) \subseteq \left\{p^n \mid n \in \mathbb{N}\right\}$ then G is called a *p-group*. We say that G has finite exponent (or that G is *bounded*) if $\operatorname{Ord}(G)$ is bounded; then the *exponent* of G is the least common multiple of $\operatorname{Ord}(G)$. If G has exponent n, then $g^n = 1$ for each $g \in G$.

The subset consisting of all elements of finite order in G is denoted by $\operatorname{Tor}(G)$. A group G is called a *torsion group* if $G = \operatorname{Tor}(G)$.

Let G be a p-group. Given $g \in G$, the greatest integer m for which $g = x^{p^m}$ is solvable with $x \in G$, is called the *height* $\mathrm{ht}(g)$ of g in G. If $g = x^{p^m}$ is solvable whatever m is, we say that g is of *infinite height* in G, and we write $\mathrm{ht}(g) = \infty$. For $a \in \mathbb{Z}(p^k)^{(d)} \smallsetminus \{0\}$ we have $\mathrm{ord}\,(a) = p^{k-\mathrm{ht}(a)}$. The invariants order and height control the existence of homomorphisms between bounded abelian p-groups, see 8.3 below.

A subset of a group G is called *characteristic* if it is invariant under $\mathrm{Aut}(G)$, that is, if it is the union of a collection of orbits under $\mathrm{Aut}(G)$. The following subgroups are characteristic.

For any group G, let $\mathrm{d}(G)$ denote the commutator group. If G is a topological group, put $\mathrm{D}(G) := \overline{\mathrm{d}(G)}$. Then $\mathrm{d}^{n+1}(G) := \mathrm{d}(\mathrm{d}^n(G))$ yields the usual derived series for G, and $\mathrm{D}^{n+1}(G) := \mathrm{D}(\mathrm{D}^n(G))$ gives a sequence of closed characteristic subgroups of G, with solvable quotients $G/\mathrm{D}^n(G)$. Note that $\mathrm{D}^n(G) = \overline{\mathrm{d}^n(G)}$: thus a Hausdorff group is solvable of derived length at most n (that is, $\mathrm{d}^n(G) = \{1\}$) exactly if $\mathrm{D}^n(G) = \{1\}$.

If A is a locally compact abelian Hausdorff group (LCA group for short), we denote its Pontryagin dual by $\widehat{A}$. Additive notation will be used for abelian groups.

Let A be an LCA group. The union $\mathrm{Comp}\,(A)$ of all compact subgroups is a closed characteristic subgroup of A because it is the annihilator of the connected component of $\widehat{A}$, compare [6] 24.17. If A possesses an open compact subgroup (in particular, if A is totally disconnected) then $\mathrm{Comp}\,(A)$ is open. Of course, we have $\mathrm{Tor}\,(A) \le \mathrm{Comp}\,(A)$.

By a "vector subgroup" of A we mean a subgroup isomorphic (as a topological group) to $\mathbb{R}^n$ for some suitable natural number n. It is known (see, for instance, [6] 24.29) that every LCA group A possesses at least one maximal vector subgroup V, and that A splits as a direct product $A = V \oplus B$; here B is the annihilator of a maximal vector subgroup of $\widehat{A}$. Moreover, the connected component A_0 of A splits as $A_0 = V \oplus M$, where $M := A_0 \cap \mathrm{Comp}\,(A) = \mathrm{Comp}\,(A_0)$ is a compact connected group.

Lemma 1.2 ([24] 3.1–3.3)**.** *Let G be a topological group.*

1. *For each characteristic subgroup H of G one has the inequality $\omega(G) \geq \omega(H) + \omega(G/H) - 1$.*
2. *Let $G = H_1 > \cdots > H_n$ be a descending series of characteristic subgroups H_j of G. Then*
$$\omega(G) \geq \omega(H_n) + \sum_{j=1}^{n-1} (\omega(H_j/H_{j+1}) - 1).$$
3. *Assume that there exist characteristic subgroups C and D of G such that $CD = G$ and $C \cap D = \{1\}$. Then $\omega(G) \geq \omega(C) \cdot \omega(D)$.*

In general, one cannot expect equality in 1.2.3, since automorphisms of C or of D need not extend to automorphisms of G. However, one has $\omega(C \oplus D) = \omega(C) \cdot \omega(D)$ if C and D are characteristic.

Corollary 1.3. *Let G be a solvable topological group with $\omega(G) < \infty$.*

1. *The derived length of G is at most $\omega(G) - 1$, and equality implies $\mathrm{d}^n(G) = \mathrm{D}^n(G)$ for each positive integer n.*
2. *If G is nilpotent then its nilpotency class is bounded by $\omega(G) - 1$.*
3. *If $\omega(G) \leq 3$ and the center of G is not trivial then G is nilpotent of class at most 2.*

Of course, we have $\omega(G) \geq |\mathrm{Ord}(G)|$ for every group G. If a divisible group G contains a nontrivial element of finite order, then $\omega(G)$ is infinite [24] 3.9.

Endowed with the discrete topology, the group $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is isomorphic to the direct product of its torsion group $\mathbb{Q}/\mathbb{Z}$ and a vector space V of dimension $2^{\aleph_0}$ over $\mathbb{Q}$. Thus $\omega(\mathbb{Q}/\mathbb{Z} \times V) = \aleph_0$. Without using the topology on $\mathbb{T}$, this yields that $\omega(G)$ is infinite for each group G that allows a nontrivial homomorphism from $\mathbb{T}$ to G.

This bound can be improved considerably if the usual (compact) topology is used on $\mathbb{T}$. The existence of a nontrivial *continuous* homomorphism from $\mathbb{T}$ (or even any compact connected group) into a topological group G has quite strong consequences for $\omega(G)$, see 5.1 below.

Lemma 1.4 ([24] 3.11)**.** *Let $(G_j)_{j\in J}$ be a family of topological groups, and endow $\mathsf{X}_{j\in J} G_j$ with the product topology. Then $\omega(\mathsf{X}_{j\in J} G_j) \leq \prod_{j\in J} \omega(G_j)$.*

The upper bound in 1.4 is attained if the family consists of pairwise non-isomorphic, simple groups: in this case, each of the factors is characteristic in the product. In other cases, however, $\omega\left(\mathsf{X}_{j\in J} G_j\right)$ may be much smaller. For instance, the vector space $\mathbb{Z}(p)^c$ satisfies $\omega\left(\mathbb{Z}(p)^c\right) = 2$ for each cardinal number $c > 0$, see 2.1. Using an isomorphism from the lattice of subsets of I onto the lattice of closed normal subgroups of a cartesian product of non-abelian simple groups, one proves:

Theorem 1.5 ([24] 3.14). *Assume that $(G_i)_{i\in I}$ is a family of* non-abelian *simple groups. Then $\omega\left(\mathsf{X}_{i\in I} G_i\right)$ is at least the cardinality of the set of cardinal numbers $\left\{c \mid\ c \le |I|\right\}$.*

Thus $\omega\left(\mathsf{X}_{i\in I} G_i\right)$ is infinite if $|I|$ is infinite. Under finiteness assumptions, much stronger bounds can be proved, see 1.6 and 1.7 in combination with 4.5, for instance.

Proposition 1.6 ([24] 3.15). *Consider a family $(G_i)_{i\in I}$ of non-abelian simple topological groups, and endow $P := \mathsf{X}_{i\in I} G_i$ with the product topology. Let $\mathcal{R}$ be a set of representatives for the isomorphism types of elements of $\left\{G_i \mid\ i \in I\right\}$. For each $R \in \mathcal{R}$, we put $S_R := \left\{i \in I \mid\ G_i \cong R\right\}$, and embed $C_R := \mathsf{X}_{i\in S_R} G_i$ as a subgroup of P. Then the following hold:*

1. *For each $R \in \mathcal{R}$, the subgroup C_R is closed and characteristic in P. The product topologies on P and on $\mathsf{X}_{R\in\mathcal{R}} C_R$ coincide.*
2. *We have $\mathrm{Aut}(P) = \mathsf{X}_{R\in\mathcal{R}} \mathrm{Aut}(C_R)$.*
3. *For each $R \in \mathcal{R}$, the group $\mathrm{Aut}(C_R)$ is isomorphic to the (complete) wreath product $\mathrm{Aut}(R) \wr \mathrm{Sym}(S_R)$.*
4. *We have $\omega\left(P\right) = \prod_{R\in\mathcal{R}} \omega\left(R^{S_R}\right)$.*

The determination of $\omega\left(P\right)$ is thus reduced to the next observation, due to Martin Schwachhöfer.

Proposition 1.7 ([24] 3.17). *Let G be a non-abelian simple group such that $\omega\left(G\right) = k$, and let n be a natural number. Then $\omega\left(G^n\right) = \binom{n+k-1}{k-1}$.*

2. Examples and Characterizations in LCA

Theorem 2.1 ([19] 4.4). *Let p be a prime, and let c, d be arbitrary cardinals. Endow the group $\mathbb{Z}(p)^c$ with the (compact) product topology, and endow $\mathbb{Z}(p)^{(d)}$ with the discrete topology. The product of these topologies makes $\mathbb{Z}(p)^c \times \mathbb{Z}(p)^{(d)}$ a locally compact group G with $\omega(G) \leq 2$.*

Every LCA group A with $pA = \{0\}$ is of the form $\mathbb{Z}(p)^c \times \mathbb{Z}(p)^{(d)}$. Thus the torsion groups among the homogeneous LCA groups are exactly the vector spaces over finite fields.

Examples 2.2 ([24] 2.7). Let p be a prime, let n be a natural number, and let c, d be arbitrary nonzero cardinals. Then $\omega\left(\mathbb{Z}(p^n)^{(d)}\right) = n + 1 = \omega\left(\mathbb{Z}(p^n)^c\right)$.

The orbits under the automorphism group of a (discrete) abelian group without elements of infinite height are parameterized by the Ulm invariants, see [11], Theorem 24 in Chapter 18. For arbitrary bounded abelian groups, the orbit decomposition under the group of all automorphisms can be determined quite explicitly; see [18], where this is used to prove the following.

Theorem 2.3 ([18] 5.8). *Let A be a discrete abelian group satisfying $p^n A = \{0\} \neq p^{n-1}A$, for some prime p. Then the following hold.*

1. *We have $\omega(A) = \omega\left(\bigoplus_{k \in N} \mathbb{Z}(p^k)\right)$, where $N \subseteq \mathbb{N}$ is chosen so that $A \cong \bigoplus_{k \in N} \mathbb{Z}(p^k)^{(d_k)}$ holds for some family $(d_k)_{k \in N}$ of non-zero cardinals.*
2. *The number $\omega(A)$ only depends on the set $\left\{k \in N \mid d_k \neq 0\right\}$.*
3. *We have $n + 1 \leq \omega(A) \leq 2^n$.*
4. *The value $\omega(A) = n + 1$ is attained if, and only if, $A \cong \mathbb{Z}(p^n)^{(c)}$ for some nonzero cardinal c.*
5. *The inequality $\omega(A) \neq n + 1$ implies $\omega(A) \geq 2n$.*
6. *The value $\omega(A) = 2n$ is attained if, and only if, $A \cong \mathbb{Z}(p^k)^{(c)} \oplus \mathbb{Z}(p^n)^{(d)}$ for nonzero cardinals c, d, and $k \in \{1, n-1\}$.*
7. *The value $\omega(A) = 2^n$ is attained if, and only if, $A \cong \bigoplus_{k=1}^{n} \mathbb{Z}(p^k)^{(c_k)}$ for some family $(c_k)_{k=1}^{n}$ of nonzero cardinals c_k.*

We will transfer this result to the compact case in 8.4 below.

The fact that any abelian torsion group A is the direct sum of its p-components (which, of course, form characteristic subgroups) allows to extend 2.3 to the case of bounded abelian groups:

Corollary 2.4. *Let A be a discrete bounded abelian group, and let A_p be the p-component of A. Then $\omega(A) = \prod_{p \in \mathbb{P}} \omega(A_p)$.*

Examples 2.5 ([24] 2.9)**.** If c is infinite then there are at least $n+1$ different locally compact group topologies on $\mathbb{Z}(p^n)^c$ yielding a topological group A with $\omega(A) = n+1$. Among these, there are the compact product topology, and the discrete one.

Examples 2.6 ([19] 3.5)**.** Let p be a prime, and consider the compact abelian group $\mathbb{Z}_p$ of p-adic integers. For every cardinal number c, the (torsion-free) compact group $(\mathbb{Z}_p)^c$ can be embedded into its divisible hull $\mathrm{D}(p, c)$. There is a unique group topology on $\mathrm{D}(p, c)$ such that $(\mathbb{Z}_p)^c$ is embedded as an open subgroup. With this topology, the group $\mathrm{D}(p, c)$ is locally compact, and satisfies $\omega(\mathrm{D}(p, c)) = 2$.

Remarks 2.7. The group $\mathrm{D}(p, c)$ can be realized as the group of all bounded functions from the set c to the metric space $\mathbb{Q}_p$.

If c is finite then $\mathrm{D}(p, c)$ coincides with $(\mathbb{Q}_p)^c$ (and is, therefore, isomorphic to its Pontryagin dual). However, if c is infinite, then $\mathrm{D}(p, c)$ cannot be made a *topological* vector space (because multiplication by p is not an open map; see [1] or [19] 3.8). The Pontryagin dual of the group $\mathrm{D}(p, c)$ is not even divisible if c is infinite, see [16]. Self-duality for torsion-free LCA groups has been investigated in [28], and in [23].

Theorem 2.8 ([19] 4.5, 5.1, 6.4)**.** *Let A be a homogeneous LCA group. Then either A is the additive group of a discrete vector space over $\mathbb{Q}$, or A is isomorphic to one of the groups $\mathbb{R}^n$, $\mathrm{D}(p, c)$, and $\mathbb{Z}(p)^c \times \mathbb{Z}(p)^{(d)}$, where n is a natural number, p is a prime, and c, d are arbitrary cardinals.*

Corollary 2.9 ([19] 5.2, 6.4)**.**

1. *A locally compact group A forms a topological vector space (of finite dimension) over the reals if, and only if, it is homogeneous and connected.*

2. *A nontrivial LCA group A forms a topological vector space (of finite dimension) over a totally disconnected local field if, and only if, it is homogeneous, totally disconnected, neither compact nor discrete, and has countable weight.*

Note that the topology of a topological vector space over a locally compact (and thus completely valuated) field is unique; cf. [17] I.3.2 or [27] I §2, Th. 3.

3. Examples of Almost Homogeneous Groups

A group G is called *almost homogeneous* if $\omega(G) \leq 3$. There are two important classes of locally compact almost homogeneous groups; see 3.1 and 6.7.

Examples 3.1 ([13] 2.6, 2.7)**.** Let F be a (commutative) field, and assume that the cyclotomic polynomial Φ_q given by $\Phi_q(X) = \frac{X^q-1}{X-1}$ has $q-1$ different roots in F, and that $\mathrm{Aut}(F)$ acts transitively on the set of these roots. For any vector space V over F, define the semidirect product $\Delta(q,V) := \Omega \ltimes V$ where $\Omega = \left\{ f \in F \mid f^q = 1 \right\}$ acts on V by scalar multiplication. Then $\omega(\Delta(q,V)) = 3$.

Finite examples of groups constructed as in 3.1 include the groups $\mathrm{Sym}(3)$, $\mathrm{Alt}(4)$, and all dihedral groups D_p where p is a prime. See 3.2 for the almost connected locally compact examples.

If V is a topological vector space over a topological field F then $\Delta(q,V)$ is a topological group. However, confining ourselves to *topological* automorphisms we may face a substantial increase in the number of orbits. For instance, the field $\mathbb{C}$ of complex numbers has only two continuous automorphisms. Thus $\Delta(5,\mathbb{C}^1)$ splits into 4 orbits under its group of topological automorphisms.

Proposition 3.2 ([24] 2.14)**.** *Let q be a prime, and let n be a positive integer. For $q > 2$ and n even, we consider $\mathbb{R}^n$ as a vector space of dimension $\frac{n}{2}$ over $\mathbb{C}$.*

1. *For $q \in \{2,3\}$, we have $\omega(\Delta(q,\mathbb{R}^n)) = 3$.*
2. *For $q \geq 3$, we have $\omega(\Delta(q,\mathbb{R}^n)) = \frac{q+3}{2}$.*

Our result 2.8 applies to the groups $\Delta(q,V)$: if such a group carries a Hausdorff topology then its commutator subgroup $\mathrm{d}(\Delta) = \{1\} \times V$ is closed because this maximal subgroup is abelian while $\Delta(q,V)$ is not. Since the quotient $\Delta/\mathrm{d}(\Delta)$ is finite, the subgroup

$\mathrm{d}(\Delta)$ is open. Now if (with the given topology!) we have $\omega(\Delta(q,V)) = 3$, then $\omega(V) = 2$, and V is known by 2.8.

Example 3.3. For each non-zero cardinal c, the group $\Delta(2, \mathrm{D}(p,c))$ is a topological group, satisfying $\omega(\Delta(2, \mathrm{D}(p,c))) = 3$. (Note that $\mathrm{D}(p,c)$ is a vector space over $\mathbb{Q}_p$, although not a topological one if c is infinite.)

Definition 3.4. An important class of nilpotent groups is formed by the so-called *Heisenberg groups*, which are defined as follows: Let F be some (commutative) field whose characteristic is different from 2, let V and Z be vector spaces over F, and let $\beta := \langle \cdot, \cdot \rangle \colon V \times V \to Z$ be a bilinear map. Assume that β is symplectic (that is, $\langle v, w\rangle = -\langle w, v\rangle$). Then

$$(v,x) \cdot (w,y) := \big(v + w, x + y + \tfrac{1}{2}\langle v, w\rangle\big)$$

defines a group multiplication on the set $V \times Z$. We denote this group by $\mathrm{GH}(V, Z, \beta)$.

In fact, $[(v,x),(w,y)] := (0, \langle v, w\rangle)$ gives a Lie bracket on the vector space $V \times Z$; this defines a Lie algebra called $\mathfrak{gh}(V, Z, \beta)$. If $F = \mathbb{R}$ and $\mathfrak{gh}(V, Z, \beta)$ has finite dimension then $\mathrm{GH}(V, Z, \beta)$ is the corresponding simply connected group, modeled on $V \times Z$ by Campbell–Hausdorff multiplication. Actually, the Campbell–Hausdorff series on $\mathfrak{gh}(V, Z, \beta)$ makes sense over any field (of characteristic different from 2) because all commutators lie in the center of $\mathfrak{gh}(V, Z, \beta)$.

See [20] 2.1 or [8] for a discussion of automorphisms of Heisenberg groups.

If F is a topological field and both V and Z have finite dimension over F we endow V and Z with the product topologies. Then β is continuous and $\mathrm{GH}(V, Z, \beta)$ is a topological group. If the prime field (the smallest subfield) of F is dense in F then every automorphism of the topological group $\mathrm{GH}(V, Z, \beta)$ is described by F-linear maps. We will only consider this case in the sequel (if all else fails, with the discrete topology on F).

The real Heisenberg groups H with $\omega(H) \leq 3$ have been determined in [20], see 6.7 below. Embeddings between such groups are studied in [21], [9], and in [8].

The space $V \wedge V$ of skew-symmetric tensors provides a kind of "universal" Heisenberg group:

Examples 3.5 ([24] 2.17)**.** Let F be a (commutative) topological field, and let V be a vector space of finite dimension d over F. Then $H := \mathrm{GH}\,(V, V \wedge V, \wedge)$ is a Heisenberg group with $\omega\,(H) \leq 2 + \lfloor d/2 \rfloor$, where $\lfloor d/2 \rfloor$ is the largest integer l with $l \leq d/2$.

Equality holds if the prime field is dense in F.

Remark 3.6. If $H = \mathrm{GH}\,(V, Z, \beta)$ is a Heisenberg group, the commutator subgroup $\mathrm{d}(H)$ equals $\{0\} \times C$, where $C \leq Z$ is additively generated by $(V \times V)^\beta$. For $C = Z$ we obtain H as a quotient of the group $\mathrm{GH}\,(V, V \wedge V, \wedge)$ discussed in 3.5.

In order to get hold of the general situation, we consider the vector subspace $R := \left\{ v \in V \;\middle|\; \forall w \in V \colon (v,w)^\beta = 0 \right\}$. Picking vector space complements S for R in V and K for C in Z, one gets $H = R \times \mathrm{GH}\,(S, C, \beta) \times K$.

See [24] 2.16 for easy examples of Heisenberg groups H with $\omega\,(H) = 4$ or $\omega\,(H) = 5$. A Heisenberg group H with $\omega\,(H) = 2^{\aleph_0}$ is constructed in [8]:

Definitions 3.7. Let $V = (\mathbb{R}^2)^3$ and $Z = \mathbb{R}^2$. Consider the symplectic map

$$\begin{aligned} \beta \colon V \times V \;&\to\; Z \\ \big((v_0, v_1, v_2), (w_0, w_1, w_2)\big) \;&\mapsto\; \left(\det\left(\begin{smallmatrix} v_0 \\ w_0 \end{smallmatrix}\right) + \det\left(\begin{smallmatrix} v_2 \\ w_2 \end{smallmatrix}\right), \det\left(\begin{smallmatrix} v_1 \\ w_1 \end{smallmatrix}\right) + \det\left(\begin{smallmatrix} v_2 \\ w_2 \end{smallmatrix}\right)\right), \end{aligned}$$

and let $J_H := \mathrm{GH}\,(V, Z, \beta)$ be the corresponding Heisenberg group.

$$\text{Let } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } E := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We obtain two linear representations φ and ψ of the symmetric group $\mathrm{Sym}(3)$ by homomorphic extension: we stipulate that the transpositions (01) and (02) are mapped to

$$(01)^\varphi := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } (02)^\varphi := \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\text{or} \quad (01)^\psi := \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \text{ and } (02)^\psi := \begin{pmatrix} 0 & 0 & E \\ 0 & I & 0 \\ E & 0 & 0 \end{pmatrix}, \text{ respectively.}$$

For each permutation $\pi \in \mathrm{Sym}(3)$, every choice of matrices $B_j \in \mathbb{R}^{2\times 2}$ with $\det B_0 = \det B_1 = \det B_2 =: d$, and each $\mathbb{R}$-linear map

$\tau : V \to Z$, we define a map

$$\begin{array}{rcl} \Psi_{\pi,\tau,B_0,B_1,B_2} : J_H & \to & J_H \\ (v,z) = \big((v_0,v_1,v_2),z\big) & \mapsto & \Big((v_0B_0, v_1B_1, v_2B_2)\pi^{\psi}, v\tau + dz\pi^{\varphi}\Big). \end{array}$$

Proposition 3.8 ([8] 3.3)**.** *The automorphisms of the Heisenberg group J_H are exactly the maps of the form $\Psi_{\pi,\tau,B_0,B_1,B_2}$, as considered in 3.7.*

Proof. Multiplicativity of the determinant together with the condition $\det B_0 = \det B_1 = \det B_2 = d$ yields that $\Psi_{\mathrm{id},\tau,B_0,B_1,B_2}$ is an automorphism of J_H. Another simple computation gives that $\Psi_{\pi,0,I,I,I}$ is an automorphism whenever π is one of the transpositions (01) or (02). Combining these generators, we see that each $\Psi_{\pi,\tau,B_0,B_1,B_2}$, as in 3.7, belongs to $\mathrm{Aut}(J_H)$.

The more interesting part is to give a reason why these maps form all of $\mathrm{Aut}(J_H)$. In order to see this, we make a general remark first (cf. [20] 2.1, or use 3.5 and the universal property of $V \wedge V$): if $H := \mathrm{GH}\,(V,Z,\gamma)$ is a Heisenberg group such that the image of $V \times V$ under γ additively generates Z then every automorphism of H is of the form $\alpha_{\sigma,\rho} := \big((v,z) \mapsto (v\sigma, v\tau + z\rho)\big)$, where $\sigma : V \to V$ and $\rho : Z \to Z$ are continuous additive bijections such that $(\sigma \times \sigma)\beta = \beta\rho$, and $\tau : V \to Z$ is an arbitrary continuous additive map. If V and Z are vector groups, these maps are $\mathbb{R}$-linear.

Adapting by multiplication with $\Psi_{\mathrm{id},\tau,I,I,I}$ for a suitable linear map τ, we reduce the determination of $\mathrm{Aut}(J_H)$ to automorphisms $\alpha_{\sigma,\rho}$.

For $v \in V$, we consider $C_v := \big\{w \in V \;\big|\; (v,w)^{\beta} = 0\big\}$. Putting

$$V_0 := \mathbb{R}^2 \times \{0\} \times \{0\}, \quad V_1 := \{0\} \times \mathbb{R}^2 \times \{0\}, \quad \text{and} \quad V_2 := \{0\} \times \{0\} \times \mathbb{R}^2$$

we observe that the elements of the set $D := V_0 \cup V_1 \cup V_2$ have a special property: one checks easily that $D = \big\{v \in V \mid \dim C_v \geq 5\big\}$. Now $\sigma \in \mathrm{GL}\,(V)$ leaves D invariant only if there is $\pi \in \mathrm{Sym}(3)$ such that $\sigma' := \sigma\pi^{\psi}$ leaves each of the subspaces V_j invariant. With $\rho' := \rho\pi^{\varphi}$ we obtain $\alpha_{\sigma',\rho'} = \alpha_{\sigma,\rho}\Psi_{\pi,0,I,I,I} \in \mathrm{Aut}(J_H)$. Let B_j denote the matrix describing the restriction of σ' to V_j. Evaluating $(\sigma' \times \sigma')\beta$ for specially chosen pairs from $C_j \times C_k$, one easily derives the condition $\det B_0 = \det B_1 = \det B_2$, and obtains $z\rho' = \det B_0 \cdot z$. This means that $\alpha_{\sigma,\rho} = \Psi_{\pi^{-1},0,B_0,B_1,B_2}$, and the proof is complete. □

Corollary 3.9. *We have $\omega\,(J_H) = 2^{\aleph_0}$.*

Proof. On the projective line $\mathrm{P}Z := \{\mathbb{R}z \mid z \in Z \setminus \{0\}\}$, the group $\mathrm{Aut}(J_H)$ induces a finite group isomorphic to $\mathrm{Sym}(3)$. Thus $\mathrm{Aut}(J_H)$ acts with $2^{\aleph_0}$ orbits on $\mathrm{P}Z$, and, *a fortiori*, with $2^{\aleph_0}$ orbits on Z and on J_H. □

4. Characterizations of Finite Simple Groups

The alternating group $\mathrm{Alt}(5)$ satisfies $\omega(\mathrm{Alt}(5)) = 4$; in fact, the two conjugacy classes of elements of order 5 are fused under conjugation by the symmetric group $\mathrm{Sym}(5)$. It turns out that the group $\mathrm{Alt}(5)$ is characterized by this property among the compact, non-solvable groups [24] 4.9. The proof of that result consists in a reduction to the following:

Theorem 4.1 ([24] 2.3)**.** *Every finite non-abelian simple group G satisfying $\omega(G) \leq 4$ is isomorphic to* $\mathrm{Alt}(5)$*.*

The isomorphisms $\mathrm{Alt}(5) \cong \mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$ allow to include the smallest simple group in the list of groups $\mathrm{PSL}(2,f)$, where f ranges over the prime powers. For the members of this list, the values of our invariant ω have been determined in [12], by a recursive formula. For each positive integer k and each nonempty set M of divisors of k, define $T_{k,M} = \gcd_{t \in M} \frac{k}{t}$. The set of prime divisors of k is denoted $\pi(k)$. Moreover, we set

$$H(p,k) := \begin{cases} 0 & \text{if } k \text{ is odd,} \\ p^{\frac{k}{2}} - 1 & \text{if } k \text{ is a power of 2,} \\ p^{\frac{k}{2}} + \displaystyle\sum_{\emptyset \neq M \subseteq \pi(k) \setminus \{2\}} (-1)^{|M|} p^{\frac{T_{k,M}}{2}} & \text{otherwise.} \end{cases}$$

Proposition 4.2 ([12] 2.5)**.** *Let p be an odd prime, and let $k > 1$ be an integer.*

1. $\omega(\mathrm{PSL}(2,2)) = \omega(\mathrm{Sym}(3)) = 3.$
2. $\omega(\mathrm{PSL}(2,p)) = \dfrac{p+3}{2}.$
3. $\omega\left(\mathrm{PSL}\left(2,2^k\right)\right) = \omega\left(\mathrm{SL}\left(2,2^k\right)\right) =$
$$= \frac{2^k}{k} + \sum_{\emptyset \neq M \subseteq \pi(k)} (-1)^{|M|} \left(\frac{2^{T_{k,M}}}{k} - \omega\left(\mathrm{SL}\left(2, 2^{T_{k,M}}\right)\right)\right).$$
4. $\omega\left(\mathrm{PSL}\left(2,p^k\right)\right) =$
$$= \frac{p^k + H(p,k)}{2k} + \sum_{\emptyset \neq M \subseteq \pi(k)} (-1)^{|M|} \left(\frac{p^{T_{k,M}}}{2k} - \omega\left(\mathrm{PSL}\left(2, p^{T_{k,M}}\right)\right)\right).$$

In order to prove 4.5 below, we are going to use deep results from the theory of finite simple groups. The proof of the following theorem uses the classification of finite simple groups:

Theorem 4.3 ([29] 3.1)**.** *Let G be a finite simple non-abelian group, and assume that elements of G belong to the same orbit under* $\mathrm{Aut}(G)$ *if they have the same order. Then G is isomorphic to one of the groups* $\mathrm{PSL}\,(2,5)$*,* $\mathrm{PSL}\,(2,7)$*,* $\mathrm{PSL}\,(2,8)$*,* $\mathrm{PSL}\,(2,9)$*,* $\mathrm{PSL}\,(3,4)$*.*

The investigation of groups with few orbits quite naturally leads to CN-groups in the sense of M. Suzuki [25]: that is, groups where each non-trivial element has nilpotent centralizer.

Theorem 4.4 ([25], [26])**.** *Let G be a non-solvable CN-group. Then G is isomorphic to one of the groups* $\mathrm{Sz}\left(2^{2m+1}\right)$*,* $\mathrm{PSL}\,(2,2^n)$*,* $\mathrm{PSL}\,(2,p)$*,* $\mathrm{PSL}\,(2,9)$*, or* $\mathrm{PSL}\,(3,4)$*, where $m \geq 1$, $n \geq 2$, and p is a Fermat or a Mersenne prime.*

Here $\mathrm{Sz}\,(q)$ denotes the Suzuki group satisfying $\omega\,(\mathrm{Sz}\,(q)) = \omega\,(\mathrm{PSL}\,(2,q)) + 2$, cf. [12] 3.4.

Theorem 4.5 ([22])**.** *A finite non-abelian simple group G satisfies $\omega\,(G) \leq 5$ exactly if G is isomorphic to* $\mathrm{PSL}\,(2,f)$ *with* $f \in \{4,5,7,8,9\}$*.*

Proof. After 4.1, we may concentrate on the case $\omega\,(G) = 5$; recall the isomorphisms $\mathrm{PSL}\,(2,4) \cong \mathrm{PSL}\,(2,5) \cong \mathrm{Alt}(5)$.

According to Burnside's $p^\alpha q^\beta$-Theorem (cp. [5] 4.3.3), the order of G has at least 3 prime divisors; say t, p, q, where t is chosen minimally. Picking elements of order $1, t, p$, and q, respectively, we obtain representatives for four of the five orbits. The elements in the remaining orbit all have the same order r.

If r is not contained in $\{t,p,q\}$ then Zhang's result 4.3 applies, giving

$$G \in \{\mathrm{PSL}\,(2,7)\,, \mathrm{PSL}\,(2,8)\,, \mathrm{PSL}\,(2,9)\,, \mathrm{PSL}\,(3,4)\}.$$

Kohl's formula 4.2 yields $\omega\,(\mathrm{PSL}\,(2,7)) = \omega\,(\mathrm{PSL}\,(2,8)) = \omega\,(\mathrm{PSL}\,(2,9)) = 5$. The group $\mathrm{PSL}\,(3,4)$ has order $2^6 \cdot 3^2 \cdot 5 \cdot 7$, and contains elements of order 4. This[1] gives $\omega\,(\mathrm{PSL}\,(3,4)) \geq 6$.

[1] In fact, the GAP [4] command:

```
Length(Orbits(AutomorphismGroup(PSL(3,4)), PSL(3,4));
```

computes $\omega\,(\mathrm{PSL}\,(3,4)) = 6$).

It remains to discuss the cases where $r \in \{t, p, q\}$. Then every element of $G \smallsetminus \{1\}$ has prime order, and G is a CN-group. The group PSL $(3, 4)$ has already been discussed above, and Kohl's formula 4.2 excludes all the remaining groups in 4.4, except for PSL $(2, f)$ with $f \in \{4, 5, 7, 8, 9\}$. □

Remark 4.6. In [22], the result given in 4.5 is proved by arguments that use less deep results. In particular, the classification of finite simple groups is avoided, and $\omega\,(\mathrm{PSL}\,(2, f))$ is directly determined for $f \in \{4, 5, 7, 8, 9\}$.

5. Compact Groups

Theorem 5.1 ([24] 4.2, 4.3). *Let G be a topological group. If G contains a nontrivial compact connected subgroup C then $\omega\,(G) \geq 2^{\aleph_0}$. Consequently, every compact group K with $\omega\,(K) < 2^{\aleph_0}$ is totally disconnected.*

Compact totally disconnected groups are pro-finite: they contain lots of open normal subgroups, and are embedded in products of finite groups.

Combining 1.6 with 1.7, it is possible to determine $\omega\,(P)$ for each cartesian product P of non-abelian simple groups. Such products occur in the context of compact groups:

Proposition 5.2 ([24] 4.7). *Let K be a compact group, and consider the set $\mathcal{S}$ of open maximal normal subgroups of K with non-abelian quotient. Then $K/\bigcap\mathcal{S}$ is isomorphic to the product $\times_{N\in\mathcal{S}} K/N$.*

Theorem 5.3 ([24] 4.6). *Let K be compact with $\omega\,(K) < \aleph_0$.*

1. *The group K has finite exponent.*
2. *There is an open normal subgroup N of K such that K is isomorphic to a subgroup of some suitable power $(K/N)^c$.*
3. *If $|\mathrm{pord}\,(K)| = 1$ then K is nilpotent.*
4. *If $|\mathrm{pord}\,(K)| = 2$ then K is solvable.*

Theorem 5.4 ([24] 4.8). *Let K be a compact group. If $\omega\,(K) < \aleph_0$ then K is the extension of an open solvable totally disconnected characteristic subgroup by a finite quotient.*

As an application of 5.4, we obtain the following.

Theorem 5.5 ([24] 4.9)**.** *Let K be a compact group.*

1. *If $\omega(K) \le 2$ then there are a prime p and a cardinal number c such that K is isomorphic to $\mathbb{Z}(p)^c$ (with the product topology).*

2. *If $\omega(K) = 3$ then K is solvable, and exactly one of the following cases occurs:*

a. *The exponent of K has more than one prime divisor. In this case K is isomorphic to $\Delta(q, F^d)$, where F is the field obtained by adjoining a primitive q-th root of unity to $\mathbb{F}_p$, for a suitable pair (p, q) of primes; see 3.1.*
The commutator group of K is isomorphic to $\mathbb{Z}(p)^c$, for some cardinal number c.

b. *The group K has exponent p^2 for some prime p. Then K is abelian, unless $p = 2$.*
If K is abelian then $K \cong \mathbb{Z}(p^2)^d$ holds for some cardinal number d.

c. *The group K has exponent p, where p is an odd prime. In this case, we have that K is a Heisenberg group* $\mathrm{GH}(V, Z, \beta)$*; see 3.4. There are cardinal numbers c and d such that $Z \cong \mathbb{Z}(p)^c$ and $V \cong \mathbb{Z}(p)^d$, and the symplectic map β is continuous.*

3. *If $\omega(K) = 4$ then either K is solvable or K is isomorphic to* $\mathrm{Alt}(5)$.

Regarding the situation in 5.5.2b, we remark that there do exist non-abelian groups of exponent 4 that are almost homogeneous. Among these, there are the quaternion group (of order 8) and the Suzuki 2-groups, see [13] Section 5. At present, there seems to be little hope for a complete classification of these groups even in the finite case.

6. Connected Groups

The situation for locally compact connected groups turns out to be entirely different from the compact case.

From 5.1 we know that a connected locally compact group H satisfying $\omega(H) < 2^{\aleph_0}$ cannot contain any non-trivial compact subgroup. The solution of Hilbert's Fifth Problem (see [14] Chap. IV) now yields that H is a Lie group. Considering the adjoint representation, one shows that H has non-trivial center. This yields the following:

Theorem 6.1 ([24] 4.4)**.** *Let H be a locally compact group. If $\omega(H) < 2^{\aleph_0}$ then the connected component H_1 of H is a nilpotent, simply connected Lie group.*

Remark 6.2. With arguments similar to those used in the proof of 6.1, S.G. Dani [2] shows: *If G is a locally compact connected group of finite dimension such that $\mathrm{Aut}(G)$ has a dense orbit then G is nilpotent.*

Theorem 6.3. *Let G be a locally compact group, and assume that $\mathrm{Aut}(G)$ is σ-compact. Then $\omega(G) \leq \aleph_0$ implies that $\mathrm{Aut}(G)$ has an open orbit. In particular, this applies whenever G is connected.*

Proof. Pick a set R of representatives for the orbits under $A := \mathrm{Aut}(G)$, and let $(C_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of A such that $A = \bigcup_{n\in\mathbb{N}} C_n$. Then $G = \bigcup_{r\in R} \bigcup_{n\in\mathbb{N}} r^{C_n}$ is a covering of a locally compact space by countably many compact sets. As locally compact spaces are not meager, this yields that one of the sets r^{C_n} has nonempty interior, and we obtain that the orbit r^A is open.

If G is locally compact connected then $\omega(G) \leq \aleph_0$ implies that G is a simply connected Lie group, and its automorphism group is a closed subgroup of the group $\mathrm{GL}(\mathfrak{g})$ of all linear bijections of the vector space underlying the Lie algebra $\mathfrak{g}$ of G. This implies that $\mathrm{Aut}(G)$ is σ-compact. □

Example 6.4. Let D be a discrete countable group. Then $D \times \mathbb{R}$ is a non-discrete, locally compact group with $\omega(D \times \mathbb{R}) \leq \aleph_0$. Putting $D = \mathbb{Q}/\mathbb{Z}$ one sees that equality is possible.

While it is quite simple to construct Heisenberg groups H such that $\omega(H)$ is finite, but arbitrarily large, I do not know any example of a locally compact connected group whose automorphism group has exactly $\aleph_0$ orbits. See Problem 9.2.

Remarks 6.5.

1. The converse of 6.3 is not true, in general: the automorphism group $\mathrm{Aut}(J_H)$ of the nilpotent, simply connected Lie group J_H constructed in 3.7 has an open orbit, namely $(\mathbb{R}^2 \smallsetminus \{0\})^3 \times Z$, cf. 3.8. However, there are uncountably many orbits under $\mathrm{Aut}(J_H)$.

2. There does exist a step 3 nilpotent Lie group whose automorphism group has no open orbit, see [2] 3.1.

From 1.3 one infers that $\omega(H) - 1$ is an upper bound for the nilpotency class of H. As the automorphisms of a nilpotent Lie algebra $\mathfrak{l}$ can be translated to the simply connected Lie group corresponding to $\mathfrak{l}$ via the exponential map, we have the following.

Corollary 6.6. *If H is a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{h}$ then $\omega(H) = \omega(\mathfrak{h})$.*

Every locally compact connected group H with $\omega(H) \le 3$ is a Heisenberg group; these have been studied in [20] and in [21]. We quote the main result from [20]:

Theorem 6.7. *Let H be a locally compact connected group.*

1. *If $\omega(H) \le 2$ then H is isomorphic to $\mathbb{R}^n$, for some natural number n.*
2. *If $\omega(H) = 3$ then H is a Heisenberg group, and the pair of dimensions $(\dim(H/H'), \dim H')$ belongs to the set*

$$\left\{(2n,1) \mid n \ge 1\right\} \cup \left\{(4n,2) \mid n \ge 1\right\} \cup \left\{(4n,3) \mid n \ge 1\right\} \\ \cup \left\{(3,3),(6,6),(7,7),(8,5),(8,6),(8,7)\right\}.$$

3. *If $\omega(H) \le 3$ then the pair $(\dim(H/H'), \dim H')$ determines H, up to isomorphism.*

Proposition 6.8 ([8] 3.9)**.** *Let $H := \mathrm{GH}(V, Z, \beta)$ be a Heisenberg group, endowed with a group topology such that $\omega(H) = 3$. Then $\omega(H^n) = 2n + 1$.*

7. Almost Homogeneous Groups

A group G is called *almost homogeneous* if $\omega(G) \le 3$. Discrete almost homogeneous groups were studied in [13]. Turning to LCA groups, we have to consider the additional characteristic subgroup $\mathrm{Comp}(G)$ obtained as the union of all compact subgroups of G, see [6] 9.10. If G is totally disconnected then $\mathrm{Comp}(G)$ is open, compare [6] 24.18.

Theorem 7.1 ([24] 6.1)**.** *Let G be an LCA group with $\omega(G) = 3$. Then one of the following cases occurs:*

1. *$G \cong \mathbb{R}^n \times D$, where D is a discrete nontrivial vector space over $\mathbb{Q}$.*

2. *$G \cong \mathrm{D}(p,d)\times D$, where p is a prime, d is a nonzero cardinal, and D is a discrete nontrivial vector space over $\mathbb{Q}$.*

3. *G is the additive group of a nontrivial free module over $\mathbb{Z}(p^2)$, with a suitable topology; compare 2.5.*

Note that we have ignored the homogeneous abelian groups in 7.1; these groups are described in 2.8. The groups $\mathrm{D}(p,d)$ are introduced in 2.6.

In the non-abelian case, we need additional topological assumptions. The locally compact *connected* almost homogeneous groups have been described in 6.7: these groups are certain Heisenberg groups over the real numbers.

Totally disconnected examples of almost homogeneous groups are provided by suitable Heisenberg groups over totally disconnected fields, the groups $\Delta(q,V)$ described in 3.1, and certain compact 2-groups (see 5.5). Without additional hypotheses (like compactness or solvability) it seems quite hard to obtain detailed information on the structure of (almost) homogeneous groups in general. Monstrous examples exist even in the case $\omega(G)=2$, as constructions of certain infinite simple groups by G. Higman–B. Neumann–H. Neumann [7] and A.Yu. Ol'shanskiĭ [15] show. While it is quite clear that none of these (countable!) examples carries a nontrivial locally compact group topology, I do not know of any nontrivial group topology at all. See Problems 9.6 and 9.7.

Some more information can be obtained in the intermediate case, where the group is neither connected nor totally disconnected. Then the center Z and the commutator subgroup $C := \mathrm{d}(G)$ both belong to the set $\{G, G_1, \{1\}\}$, see [24] 6.2. It remains to study the different possibilities for $Z, C \in \{G, G_1, \{1\}\}$.

Theorem 7.2 ([24] 6.6)**.** *Let G be a locally compact group with $\omega(G)=3$.*

1. *If G is connected then G is one of the almost homogeneous Heisenberg groups, as determined in* [20]*, compare 6.7.*

2. *If the connected component G_1 is a proper nontrivial subgroup of G then one of the following, mutually exclusive cases occurs.*

$C=\{1\}$**:** *Then $G=Z$ is abelian, and there exist a positive integer n and a discrete vector space D over $\mathbb{Q}$ such that $G \cong \mathbb{R}^n \times D$.*

$(C,Z) = (G_1, \{1\})$**:** *Then $G \cong \Delta(q, \mathbb{F}^n)$, where n is a positive integer, and the pair $(q, \mathbb{F})$ belongs to $\{(2,\mathbb{R}), (3,\mathbb{C})\}$. These groups are not nilpotent.*

$(C,Z) = (G_1, G_1)$**:** *Then G/G_1 is a discrete vector space over $\mathbb{Q}$, and G is nilpotent.*

$(C,Z) = (G, G_1)$**:** *In this case, the group G/G_1 is torsion-free.*

Examples of type $(C,Z) = (G_1, G_1)$ are provided by almost homogeneous Heisenberg groups (see 6.7). One has to replace the ordinary (connected) topology by the group topology that is defined by requiring that the center is embedded (with its original topology) as an open subgroup. See Problem 9.5.

8. Abelian Groups

Theorem 8.1 ([24] 7.8)**.** *Let A be an LCA group with $\omega(A) < \aleph_0$. Then there are a nonnegative integer n and a cardinal d such that A is isomorphic to $\mathbb{R}^n \times \mathbb{Q}^{(d)} \times \mathrm{Comp}\,(A)$. More precisely, the following hold.*

1. *The connected component A_0 is a vector group.*
2. *There is a closed subgroup B such that $A = A_0 \oplus B$.*
3. *The torsion subgroup $\mathrm{Tor}\,(A)$ is a bounded closed subgroup of B.*
4. *We have $\mathrm{Tor}\,(A) = \mathrm{Tor}\,(B) \le \mathrm{Comp}\,(B) = \mathrm{Comp}\,(A) \le B$.*
5. *The quotient $A/(A_0 + \mathrm{Comp}\,(A)) \cong B/\mathrm{Comp}\,(A)$ is a discrete vector space over $\mathbb{Q}$, and the extension $B/\mathrm{Comp}\,(A)$ splits.*
6. *There are a finite set F of primes and a family $(e_p)_{p\in F}$ of cardinal numbers such that $\mathrm{Comp}\,(A)/\mathrm{Tor}\,(A) \cong \bigtimes_{p\in F} \mathrm{D}(p, e_p)$. If F contains no prime divisor of the exponent of $\mathrm{Tor}\,(A)$ then $\mathrm{Tor}\,(A)$ has a closed complement in $\mathrm{Comp}\,(A)$.*
7. *If A possesses a maximal compact subgroup M then $M = \mathrm{Comp}\,(A) = \mathrm{Tor}\,(A)$.*
8. *If A is compactly generated then $\mathrm{Comp}\,(A) = \mathrm{Tor}\,(A)$ is a maximal compact subgroup, and $A = A_0 \oplus \mathrm{Tor}\,(A)$.*

Proposition 8.2 ([24] 8.3)**.** *Let $(A, \mathcal{T})$ be a topological abelian group such that $\omega(A)$ is finite. Put $m := \omega(A) - 1$, and assume that A contains an element of order p^m, for some integer $p > 1$. Then the following hold.*

1. *The integer p is a prime.*

2. *The set* $\{p^j A \mid 0 \le j \le m\}$ *equals the set of all characteristic subgroups of* $(A, \mathcal{T})$. *In particular, we have*
$$p^{m-j}A = \overline{p^{m-j}A} = \{x \in A \mid p^j x = 0\}$$
for each integer $j \in \{0, \dots, \omega(A)\}$.
3. *The group* A *is a free* $\mathbb{Z}(p^m)$*-module.*
4. *If* $\mathcal{T}$ *is compact then the Pontryagin dual* $D := \widehat{(A, \mathcal{T})}$ *is a (discrete) free* $\mathbb{Z}(p^m)$*-module. Consequently, the group* A *is isomorphic to* $\mathbb{Z}(p^m)^d$, *where* d *is the dimension of* D *over* $\mathbb{Z}(p^m)$.

Lemma 8.3. *Let* l, m *be natural numbers, and let* d_l *and* d_m *be cardinals. Consider elements* x *and* y *of* $\mathbb{Z}(p^m)^{d_m}$ *and of* $\mathbb{Z}(p^l)^{d_l}$, *respectively. There is a continuous homomorphism* φ *from* $\mathbb{Z}(p^m)^{d_m}$ *to* $\mathbb{Z}(p^l)^{d_l}$ *mapping* x *to* y *if, and only if, we have* $\operatorname{ord}(y) \le \operatorname{ord}(x)$ *and* $\operatorname{ht}(y) \ge \operatorname{ht}(x)$.

Proof. The inequalities $\operatorname{ord}(g^\varphi) \le \operatorname{ord}(g)$ and $\operatorname{ht}(g^\varphi) \ge \operatorname{ht}(g)$ are obvious for any group homomorphism φ. Assuming $\operatorname{ord}(y) \le \operatorname{ord}(x)$ and $\operatorname{ht}(y) \ge \operatorname{ht}(x)$, we are now going to construct a continuous homomorphism.

By 2.5, the elements of order $\operatorname{ord}(x)$ form a single orbit under $\operatorname{Aut}(\mathbb{Z}(p^m)^{d_m})$, and those of order $\operatorname{ord}(y)$ form a single orbit under $\operatorname{Aut}(\mathbb{Z}(p^l)^{d_l})$. Therefore, it is possible to pick elements $\widetilde{x} \in \mathbb{Z}(p^m)^{d_m}$ and $\widetilde{y} \in \mathbb{Z}(p^l)^{d_l}$ with $\operatorname{ord}(\widetilde{x}) = p^m$ and $\operatorname{ord}(\widetilde{y}) = p^l$, such that $x = k_m\widetilde{x}$ and $y = k_l\widetilde{y}$. Here $k_m = \frac{p^m}{\operatorname{ord}(x)} = p^{\operatorname{ht}(x)}$ divides $p^{\operatorname{ht}(y)} = \frac{p^l}{\operatorname{ord}(y)} = k_l$.

Let C be a closed subgroup of $\mathbb{Z}(p^m)^{d_m}$ such that $\mathbb{Z}(p^m)^{d_m} = \langle\widetilde{x}\rangle \oplus C$; again, we use 2.5 to secure the existence of C. Now the projection modulo C from $\mathbb{Z}(p^m)^{d_m}$ onto $\langle\widetilde{x}\rangle \cong \mathbb{Z}(p^m)$, followed by the homomorphic extension of $\widetilde{x} \mapsto \frac{k_l}{k_m}\widetilde{y}$, is a continuous homomorphism with the required properties. The assumption on the orders is needed to secure that the extension is defined unambiguously. $\square$

If $m \ne l$ we obtain an automorphism of $\mathbb{Z}(p^l)^{d_l} \times \mathbb{Z}(p^m)^{d_m}$ mapping the pair (a_l, a_m) to $(a_l - a_m^\varphi, a_m)$. In particular, the elements $y+x$ and x belong to the same orbit under the group of all automorphisms of $\mathbb{Z}(p^l) \times \mathbb{Z}(p^m)$. For each $N \subseteq \mathbb{N}$ with $m, l \in N$ and each family $(d_k)_{k \in N}$ of cardinals, the automorphism just described extends to an automorphism of $\mathsf{X}_{k \in N}\, \mathbb{Z}(p^k)^{d_k}$. These automorphisms are just those that are used in [18] to recognize the orbits. As in 2.3, we obtain that $\omega\left(\bigtimes_{k\in N}\mathbb{Z}(p^k)^{d_k}\right)$ equals $\omega\left(\bigtimes_{k\in N}\mathbb{Z}(p^k)\right)$ if $d_k \neq 0$ holds for each $k \in N$. This yields the following:

Theorem 8.4. *For each compact bounded abelian group A, we have the equality $\omega(A) = \omega(\widehat{A})$.*

9. Open Problems

Problem 9.1. Determine all locally compact connected groups H satisfying $\omega(H) \leq 5$. (Equivalently, one has to determine the nilpotent real Lie algebras $\mathfrak{h}$ with $\omega(\mathfrak{h}) \leq 5$; see 6.6.)

Problem 9.2. Does there exist a locally compact connected group G such that $\omega(G) = \aleph_0$? (Recall that such a group would have to be a nilpotent, simply connected Lie group.)

Problem 9.3. Does there exist an example of a real Heisenberg group H with $\omega(H) = 2^{\aleph_0}$ such that H has smaller dimension than the example J_H given in 3.7 ?

Problem 9.4. Is it possible to give a simple formula for $\omega(H^n)$ in terms of $\omega(H)$, where H is an arbitrary Heisenberg group ? (See 6.8 for the case where H is almost homogeneous.)

Problems 9.5. Does there exist a locally compact group G which is not totally disconnected and satisfies $(C, Z) = (G, G_1)$? If such groups exist, is it also possible to have $\omega(G) = 3$?

Examples of homogeneous torsion-free simple groups have been constructed in [7]. It is conceivable that a suitable modification of this construction leads to examples of type $(C, Z) = (G, G_1)$.

Problems 9.6. G. Higman, B. Neumann and H. Neumann [7] have constructed an infinite simple group G_{HNN} containing exactly two conjugacy classes.

1. Is there a nontrivial group topology on G_{HNN} ?
2. Is there any embedding of G_{HNN} into a locally compact (non-discrete) group G ?
3. If such an embedding exists, is it possible to preserve $\omega(G_{\text{HNN}}) = 2$?
4. Is it possible to find G in such a way that $\omega(G) = 2$?

Problems 9.7. A.Yu. Ol'shanskiĭ and A.A. Ivanov [15] have proved that, for every sufficiently large prime p ($> 10^{75}$ say), there exists a (necessarily simple) infinite group G_{OI} such that every nontrivial subgroup has order p, and that the elements of any fixed nontrivial subgroup form a set of representatives for the conjugacy classes.

1. What are the orbits of $\mathrm{Aut}(G_{\mathrm{OI}})$?
2. Is it possible to modify the construction of G_{OI} such that $\omega\,(G_{\mathrm{OI}}) = 2$?
3. Is there a non-trivial group topology on G_{OI} ?
4. Is it possible to embed G_{OI} into a locally compact group G ?
5. Is it possible to embed G_{OI} into a locally compact group G such that $\omega\,(G) = 2$?

Problems 9.8.

1. At present, the only examples of LCA groups C with $C = \mathrm{Comp}\,(C)$ and $\omega\,(C) < \aleph_0$ that the author is aware of are those of the form $\mathsf{X}_{p\in F}\,\mathrm{D}(p, e_p) \times T$, where F is a finite set of primes, and T is a bounded LCA group.

Does $\mathrm{Tor}\,(A)$ always have a *closed* complement in $\mathrm{Comp}\,(A)$ (irrespective of the fact whether F_A contains any prime divisors of the exponent of $\mathrm{Tor}\,(A)$, cf. 8.1) ?

2. Clearly a group of the form $A = \mathbb{R}^n \times \mathbb{Q}^{(d)} \times \mathsf{X}_{p\in F}\,\mathrm{D}(p, d_p) \times T$ satisfies $\omega\,(A) < \aleph_0$ if, and only if, we have $\omega\,(T) < \aleph_0$. It remains open whether each bounded LCA group has this property. At least, we know from 2.5 that there are many different topologies turning a given infinite bounded abelian group into an LCA group T with $\omega\,(T) < \infty$.

Problem 9.9. Determine the finite non-solvable groups G with $\omega\,(G) \leq 6$.

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