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Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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A SHORT PROOF OF A CLASSICAL RESULT OF M.G. TKACHENKO

V.V. TKACHUK*

Abstract. We give a comparatively short proof of a theorem which states that any compact space that is a continuous image of a dense subspace of a Σ-product of spaces with countable network, is metrizable. This is a very deep and non-trivial result of M.G. Tkachenko obtained in 1982. The original proof consists of eight pages of a very compressed text which involves construction of an inverse system whose elements are also constructed using additonal inverse systems. We give a transparent proof on less than three pages hoping to contribute to a better understanding of the features of dense subspaces of Σ-products responsible for metrizability of their compact continuous images.

0. Introduction

Given a product $N = \prod\{N_t : t \in T\}$ of topological spaces and a point $u \in N$, let $\Sigma(N, u) = \{x \in N :$ the set $\{t \in T : x_t \neq u_t\}$ is at most countable$\}$. The space $\Sigma(N, u)$ is called *a Σ-product of the spaces* $\{N_t : t \in T\}$. Analogously, the space $\sigma(N, u) = \{x \in N : |\{t \in T : x_t \neq u_t\}| < \omega\}$ is called *a σ-product of the spaces* $\{N_t : t \in T\}$. If $N_t = \mathbb{R}$ for each $t \in T$ then $\Sigma(N, u)$ ($\sigma(N, u)$) is called *a Σ-product (σ-product) of real lines.*

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The first one to study Σ-products was H.H. Corson [Co] who established quite a few important properties of Σ-products and dense subspaces of products of second countable spaces. He proved, in particular, that if Z is a continuous image of a dense subspace of a product of second countable spaces and $Z \times Z$ is normal then Z is collectionwise normal. Later Efimov proved in [Ef] that every compact space that is a continuous image of a Σ-product of metrizable compact spaces, is metrizable.

However, Efimov's method was not applicable to prove that any compact continuous image of a Σ-product of real lines is also metrizable. This was an open problem until 1979 when Tkachenko proved in [Tk1] that any compact continuous image of any σ-product of metrizable compact spaces is metrizable. He also obtained a more general result under the Luzin's Axiom ($2^\omega < 2^{\omega_1}$): if S is a dense subspace of a Σ-product of spaces with countable network and K is a compact continuous image of S then K is metrizable.

Finally, in 1982, Tkachenko showed in [Tk2] that a stronger result is true without assuming the Luzin's Axiom. He proved that any space of pointwise countable type has a countable network if it is a continuous image of a dense subspace of a Σ-product of spaces with countable network. Up to the present day this is the strongest result known about dense subspaces of "nice" spaces.

Unfortunately, the proof given in [Tk2] is very difficult to read; it consists of eight pages of a very compressed text. A very sophisticated inverse system is constructed and the elements of this inverse system are obtained as limits of a family of (also very complicated) inverse systems. That is why the author presents this paper to the public; it has no new results but provides a much shorter and absolutely transparent proof of the main result of Tkachenko. This proof gives a clear idea of the properties of dense subspaces of Σ-products which account for metrizability of their compact images. For the sake of brevity we use E. Michael's term *cosmic* (see [Mi]) for the spaces with countable network.

1. Notation and terminology

All spaces are assumed to be Tychonoff. If X is a space then $\tau(X)$ is its topology; if $A \subset X$ then $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$. We write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. A family $\mathcal{B} \subset \tau(A, X)$ is called *a(n external) base* of A in X if, for any $U \in \tau(A, X)$, there is $V \in \mathcal{B}$ such that $V \subset U$. If a set $F \subset X$ has a countable external base, we say that *F has countable external character*. In general, $\chi(A, X)$ is the minimum of the cardinalities of external bases of A in X. Given any $x \in X$, we write $\chi(x, X)$ instead of $\chi(\{x\}, X)$. A space X is *of pointwise countable type* if X can be covered by its compact subspaces of countable external character.

A family $\mathcal{N}$ of subsets of a space X is called *a network* of X if every $U \in \tau(X)$ is a union of some subfamily of $\mathcal{N}$. Thus a network is like a base, only its elements are not necessarily open in X. A space is called *cosmic* if it has a countable network. Given a product $N = \prod\{N_t : t \in T\}$ of spaces and an arbitrary point $u \in N$, let $\Sigma(N, u) = \{x \in N : |\{t \in T : x_t \neq u_t\}| \leqslant \omega\}$. If X is a space and $f_t : X \to N_t$ is a map for every $t \in T$ then *the Δ-product $f = \Delta\{f_t : t \in T\} : X \to \prod\{N_t : t \in T\}$ of the family $\{f_t : t \in T\}$* is defined by $(f(x))_t = f_t(x)$ for each $x \in X$ and $t \in T$. The symbol $\square$ is used to indicate the end of a proof. The rest of our notation is standard and follows [En].

2. Tkachenko's Theorem and its proof

Throughout our proof we are going to use the following fact which is simple and well-known. Its proof can be left to the reader as an easy exercise.

Proposition 2.1. *Every Lindelöf space L is normally placed in any larger space Z, i.e., $Z \backslash L$ is a union of closed G_δ-subsets of Z. In particular, any cosmic space is normally placed in any larger space.*

Proposition 2.2. *Suppose that K is a non-empty compact space with no points of countable character. Then K cannot be represented as a union of $\leqslant \omega_1$-many cosmic subspaces.*

Proof. To get a contradiction assume that $K = \bigcup\{M_\alpha : \alpha < \omega_1\}$ where $nw(M_\alpha) \leqslant \omega$ for each $\alpha < \omega_1$. Let $F_0 = K$; suppose that $0 < \alpha < \omega_1$ and we have a family $\{F_\beta : \beta < \alpha\}$ of non-empty closed G_δ-subsets of K with the following properties:

(1) $F_\gamma \subset F_\beta$ whenever $\beta < \gamma < \alpha$;
(2) if $\beta < \alpha$ then $F_\beta \cap M_\gamma = \emptyset$ for any $\gamma < \beta$.

It is evident that $F'_\alpha = \bigcap\{F_\beta : \beta < \alpha\}$ is a non-empty closed G_δ-subset of K and hence $\chi(x, F'_\alpha) > \omega$ for any $x \in F'_\alpha$ for otherwise $\chi(x, K) \leqslant \chi(x, F'_\alpha) \cdot \chi(F'_\alpha, K) = \omega$ which is a contradiction. In particular, F'_α is not cosmic and therefore we can pick a point $x \in F'_\alpha \backslash M_\alpha$. By Proposition 2.1 there is a closed G_δ-set G such that $x \in G \subset X \backslash M_\alpha$. It is clear that $F_\alpha = F'_\alpha \cap G$ is a non-empty closed G_δ-subset of K such that (1) and (2) are fulfilled for the family $\{F_\beta : \beta \leqslant \alpha\}$. Consequently, we can continue our inductive construction to obtain a family $\{F_\alpha : \alpha < \omega_1\}$ of closed non-empty G_δ-subsets of K with the properties (1)-(2) for each $\alpha < \omega_1$. Since K is compact, the property (1) implies that $F = \bigcap\{F_\alpha : \alpha < \omega_1\} \neq \emptyset$. It follows from (2) that $x \notin \bigcup\{M_\alpha : \alpha < \omega_1\}$ for any $x \in F$, which is a contradiction. □

Proposition 2.3. *Suppose that X is a space and K is a non-metrizable compact subspace of X with $\chi(K, X) = \omega$. Then there exists a space Y and a continuous onto map $f : X \to Y$ such that $f(K) \cap f(X \backslash K) = \emptyset$, $w(f(K)) = w(Y) = \omega_1$ and $\chi(f(K), Y) = \omega$.*

Proof. Let $\{O_n : n \in \omega\}$ be an external base of the set K in X. Take any continuous function $f_n : X \to \mathbb{I} = [0, 1]$ such that $f_n(K) = \{1\}$ and $f_n(X \backslash O_n) = \{0\}$ for every $n \in \omega$. It is a well-known (and easy to prove) fact that any non-metrizable compact space can be continuously mapped onto a (compact) space of weight ω_1 [Ar2, Proposition IV.8.11]. Let $g : K \to K_1$ be a continuous onto map such that $w(K_1) = \omega_1$. We can consider that $K_1 \subset \mathbb{I}^{\omega_1}$; let $\pi_\alpha : \mathbb{I}^{\omega_1} \to \mathbb{I}$ be the natural projection onto the α-th factor. For each $\alpha < \omega_1$, the map $g_\alpha = \pi_\alpha \circ g : K \to \mathbb{I}$ can be continuously extended to a continuous function $h_\alpha : X \to \mathbb{I}$; let $h = \Delta\{h_\alpha : \alpha < \omega_1\}$. Then $f = h\Delta(\Delta\{f_n : n \in \omega\})$ and $Y = f(X)$ are as promised. □

Proposition 2.4. *Suppose that N_t is a cosmic space for each $t \in T$ and take any point $u \in N = \prod\{N_t : t \in T\}$. If $|T| \leqslant \omega_1$ then any subspace $E \subset \Sigma(N, u)$ is a union of $\leqslant \omega_1$-many cosmic spaces.*

Proof. Given any point $x \in E$ let $\operatorname{supp}(x) = \{t \in T : x_t \neq u_t\}$. Choose an enumeration $\{t_\alpha : \alpha < \omega_1\}$ of the set T and let $T_\alpha = \{t_\beta : \beta < \alpha\}$ for every $\alpha < \omega_1$. If $E_\alpha = \{x \in E : \operatorname{supp}(x) \subset T_\alpha\}$ then $nw(E_\alpha) \leqslant \omega$ for each $\alpha < \omega_1$ and $E = \bigcup\{E_\alpha : \alpha < \omega_1\}$. □

The following result of Arhangel'skii [Ar1] is of crucial importance for the proof of Tkachenko's Theorem. In fact, we formulate only a consequence of Arhangel'skii's Theorem which suffices for our purposes.

Theorem 2.5 ([Ar1]). *Suppose that N_t is a cosmic space for each $t \in T$ and S is a dense subspace of the space $N = \prod\{N_t : t \in T\}$. If Z is any space and $\varphi : S \to Z$ is a continuous onto map then, for any infinite cardinal κ,*
1) the space $Y = \{z \in Z : \chi(z, Z) \leqslant \kappa\}$ has network weight at most κ; in particular, if $\chi(Z) = \omega$ then Z is cosmic.
2) if $\chi(Z) = \kappa$ then there is a set $A \subset T$ with $|A| \leqslant \kappa$ and a continuous map $h : p_A(S) \to Z$ such that $h \circ (p_A|S) = \varphi$. Here $p_A : N \to N_A = \prod\{N_t : t \in A\}$ is the natural projection onto the face N_A.

Now we are ready to formulate and prove Tkachenko's Theorem.

Theorem 2.6 ([Tka2]). *Suppose that a space N_t is cosmic for each $t \in T$. Given any point $u \in N = \prod\{N_t : t \in T\}$ and any dense subset $S \subset \Sigma(N, u)$, assume that a space X of pointwise countable type is a continuous image of S. Then X is cosmic. In particular, if X is compact then X is metrizable.*

Proof. Fix a continuous onto map $\varphi : S \to X$ and denote by C the set of points of countable character of X; we have $nw(C) = \omega$ by Theorem 2.5. If $C = X$ then there is nothing to prove so assume that there is $x \in X \backslash C$. By Proposition 2.1, there is a closed G_δ-set P such that $x \in P \subset X \backslash C$. Since X is of pointwise countable type, there is compact subspace $F \subset X$ such that $\chi(F, X) = \omega$ and $x \in F$. It is clear that $K = F \cap P$ is a compact subspace of X and $\chi(K, X) \leqslant \chi(K, F) \cdot \chi(F, X) \leqslant \omega$. No point $y \in K$ can be a G_δ-set in K because otherwise $\chi(y, X) \leqslant \chi(y, K) \cdot \chi(K, X) \leqslant \omega$ which contradicts $y \in X \backslash C$. Thus K is not metrizable and hence we can apply Proposition 2.3 to find a continuous onto map $\delta : X \to Y$ such that

(3) if $K_1 = f(K)$ then $w(Y) = w(K_1) = \omega_1$;
(4) $\chi(K_1, Y) = \omega$ and $K_1 \cap f(X \backslash K) = \emptyset$.

Since the space Y is a continuous image of S and $w(Y) \leqslant \omega_1$, we can apply Arhangel'skii's theorem again to find a subset $A \subset T$ such that $|A| \leqslant \omega_1$ and there is a continuous map $h : p_A(S) \to Y$ such that $\delta \circ \varphi = h \circ (p_A|S)$. Observe that $S_A = \pi_A(S)$ is a dense subset of $\Sigma(N_A, p_A(u))$ which maps continuously onto the space Y so we can apply Arhangel'skii's theorem once more to conclude that the space $M = \{y \in Y : \chi(y,Y) \leqslant \omega\}$ is cosmic. Since $nw(K_1) = w(K_1) = \omega_1$, the set K_1 is not covered by M; take any $z \in K_1 \backslash M$. Applying Propostion 2.1 again we can find a closed G_δ-set H in Y such that $z \in H \subset Y \backslash M$. It is evident that $K_2 = H \cap K_1$ is a compact subspace of Y with $\chi(K_2,Y) \leqslant \chi(K_2,K_1) \cdot \chi(K_1,Y) \leqslant \omega$. For any point $y \in K_2$ we have $\chi(y,K_2) > \omega$ because otherwise $\chi(y,Y) \leqslant \chi(y,K_2) \cdot \chi(K_2,Y) = \omega$ which contradicts $y \notin M$.

Now apply Proposition 2.4 to conclude that S_A is a union of $\leqslant \omega_1$-many cosmic subspaces. Being cosmic is preserved by continuous maps so the space Y is a union of $\leqslant \omega_1$-many cosmic subspaces. Being cosmic is also preserved by subspaces so the space $K_2 \subset Y$ is a union of $\leqslant \omega_1$-many cosmic subspaces which contradicts Proposition 2.2. □

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Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco, 186, Col. Vicentina, Iztapalapa, C.P. 09340, México D.F.

Current address: Department of Mathematics, Queens College, The City University of New York, Flushing, N.Y., 11367, U.S.A.

E-mail address: vova@xanum.uam.mx