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TWO-COMPLEXES AS GRAPH COMPLEMENT CONE COMPLEXES

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Abstract. We define a *graph complement cone complex* (i.e., a *gccc*) to be a CW 3-complex which consists of a graph complement in a cube with handles, together with the cone over the boundary of the cube with handles. We show that every finite 2-complex 3-deforms to a gccc. We then define the dual of a gccc, a natural companion to the gccc. The dual of a gccc is obtained by trading cones: the cone in the gccc description is replaced with the cone over the boundary component associated with the removed graph. We show that if the data for a gccc consists of a knot in a solid torus and the gccc is simply connected, then its dual 3-deforms to a point.

1. Preliminaries

We work in the category of finite, p.l. CW-complexes. In particular, all attaching and characteristic maps are taken to be p.l. and and all closed cells and skeleta are subpolyhedra of a given p.l. structure. This restriction is not severe, as every finite n-dimensional CW-complex $(n+1)$-deforms to a finite p.l. CW-complex [2][4]. Throughout, $N(\cdot)$ denotes a regular neighborhood of a space in the appropriate ambient space. When mentioning a complement, say of a subcomplex X_0 in X, we write $X \setminus X_0$ in place of what we mean more precisely: $X \setminus \mathrm{int}(N(X_0))$. For a 2-complex K with one 0-cell we denote by $\mathcal{P}_K$ the presentation for

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$\pi_1(K)$ whose generators correspond to the 1-cells of K and whose relators are the attaching words for the 2-cells of K. The complex K is referred to as the *standard complex* associated with the presentation $\mathcal{P}_K$. Every 2-complex 3-deforms to a 2-complex with one 0-cell [2] (also see [6]) and thus every 2-complex may be associated with a presentation.

Define a *graph complement cone complex* (i.e., a *gccc*) to be a 3-complex constructed as follows:

(1) Start with a cube with handles, U, in S^3.
(2) Remove from U the interior of a regular neighborhood of a graph, Γ, which resides in $\text{int}(U)$.
(3) Add the abstract cone over $\text{Bd}(U)$.

We do not insist that U be unknotted in S^3, though we may, without loss of generality, assume this. Altering the embedding of U in S^3 may alter the embedding of Γ in S^3 as well. We introduce notation whose purpose is to make our exposition more efficient. Given the data for a gccc X, let $V = N(\Gamma)$ and let W denote the 3-manifold $U \setminus V$. The gccc X may then be written as $X = W \bigcup c * \text{Bd}(U)$. We define the *dual*, X^*, of X to be the 3-complex $X^* = W \bigcup c * \text{Bd}(V)$. Note that the 3-manifold W has two boundary components: the boundary of the cube with handles and the boundary associated with the removed graph. Coning over the outer boundary component produces a gccc while coning over the inner boundary component produces its dual.

If U is simply a cube, then the corresponding gccc is the graph complement $S^3 \setminus \Gamma$. It will be seen that, in full generality, gccc's account for all finite 2-complexes up to 3-deformation. This is the substance of our main result:

Theorem 2.1. *Every finite 2-complex 3-deforms to a gccc.*

In Section 3, we investigate a class of contractible gccc's whose duals are particularly well-behaved. We prove, in particular, the following result:

Theorem 3.1. *Suppose the data for a gccc X consists of a solid torus U and an interior knot Γ. If $\pi_1(X) = \{1\}$, then the dual X^* 3-deforms to a point.*

A connection between gccc's and their duals may thus contribute to the investigation of contractible 2-complexes. We conclude with a description of gccc spines for 3-manifolds obtained by performing Dehn surgery on a knot in S^3. A member, M, of this class of 3-manifolds satisfies $M = \left[S^3 \setminus \text{int}(N(k))\right] \bigcup_h T$, where k is a knot in S^3, T is a solid torus, and $h : \text{Bd}(T) \longrightarrow \text{Bd}(N(k))$ is a homeomorphism which takes the meridian of $\text{Bd}(T)$ to a specified simple closed curve, say s, in $\text{Bd}(N(k))$. By a *spine* of M we mean a 2-complex to which M collapses upon the removal of an open 3-cell. Given a knot k in S^3 and surgery instructions, we produce a gccc which is, up to 3-deformation, a spine of the surgery manifold M. (See Theorem 3.2.) The data for this gccc consists of a solid torus containing k whose meridian reads the surgery curve s. Our motivation is the study of this spine when M is a homotopy 3-sphere. It is noted in Corollary 3.1 that if M is a homotopy 3-sphere, then the dual of the gccc spine 3-deforms to a point.

We deem two proofs in this paper to be highly visual and thus provide synchronized visual proofs in the Appendix.

Acknowledgment. We are grateful to Robert Craggs at the University of Illinois-Urbana for suggesting the definitions of a gccc and its dual. The notion of a gccc follows naturally from his description of ribbon disk complements in the 4-ball [3].

2. Realizing Two-complexes as GCCC's

We begin by proving a lemma which will be a key component in the proof of the main result. A visual proof of Lemma 2.1 is provided in the Appendix.

Lemma 2.1. *Suppose the data for a gccc X consists of a cube with handles U in S^3 and a graph Γ in its interior. Suppose further that U is unknotted and $\{\mu_i\}$ is a complete set of meridians for $Bd(U)$. Then X 3-deforms to $(S^3 \setminus \Gamma) \bigcup \{E_i\}$, where each E_i is a 2-cell attached abstractly along the curve μ_i .*

Proof: Let W denote the 3-manifold $U \setminus \Gamma$. We take the longitudes $\{\lambda_i\}$ and meridians $\{\mu_i\}$ for $Bd(U)$ to be simple closed curves which meet in a single point, the base point of W. The 3-deformation will be witnessed as a sequence of deformations.

(0) Start with X itself.

(1) Choose an arc α in W from a point on $Bd(N(\Gamma))$ to a point on $Bd(U)$ whose interior misses $N(\Gamma) \bigcup Bd(U)$. A

regular neighborhood $N(\alpha)$, chosen sufficiently small, meets $Bd(W)$ in two disks, a disk D^+ on $Bd(N(\Gamma))$ and a disk D^- on $Bd(U)$. We may assume that D^- misses the longitudes and meridians of $Bd(U)$. Collapse the 3-cell $N(\alpha)$ through the free face D^+.

(2) Next, collapse $c * Bd(U)$ to $[Bd(U) \setminus \text{int}(D^-)] \bigcup c * \{\lambda_i, \mu_i\}$. This event begins with a collapse through the free face D^- and is characterized by the collapse of all 3-cells in $c * \text{Bd}(U)$.

(3) Replace the cone over $\{\lambda_i, \mu_i\}$ with a collection of disks $\{D_i, E_i\}$ attached (respectively) along these curves. For each longitude and meridian the procedure is the same: The cone over the simple closed curve is a disk and along this disk a 3-cell is attached. A side collapse is then performed which removes the interior of the 3-cell. The remnants of this move are a disk attached to the simple closed curve and the line segment from the base point to the cone point. When all candidates have undergone the procedure, collapse the line segment through the cone point to complete the promised replacement.

(4) Since U is unknotted by hypothesis, the subcomplex $U \bigcup \{D_i\}$ is a 3-ball in S^3 and thus its complement in S^3 is a 3-ball as well. Attach this complementary 3-ball to our complex. Recall that U lacks $N(\alpha)$ due to earlier collapses and thus this move is indeed a 3-deformation. An inventory of our complex shows that we have the graph complement $S^3 \setminus \Gamma$ together with the disks $\{E_i\}$. □

We pause for two examples before proving the main result.

Example 2.1. The data for the gccc in Figure 1 consists of a solid torus U and a simple closed curve Γ in its interior. Note that Γ is the unknot in S^3 and the meridian of $\text{Bd}(U)$ reads the generator of $\pi_1(S^3 \setminus \Gamma)$ twice. The gccc, therefore, 3-deforms to a 2-complex with presentation $\{x \,|\, x^2\}$.

Example 2.2. In Figure 2 we offer a more complicated gccc. Standard, Wirtinger-like calculations show that this gccc 3-deforms to a 2-complex with presentation $\{x, y \,|\, x^{-1}y^{-1}xyx^{-1}y\}$. For the reader familiar with ribbon disks (see [1] for definitions), one notes that

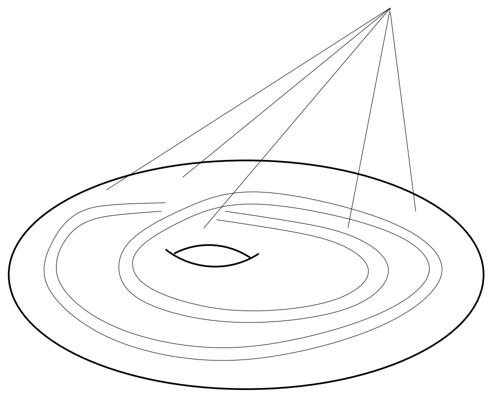


FIGURE 1. A gccc which realizes 2-dimensional projective space.

U is a regular neighborhood of a ribbon disk with two singularities and that Γ is the corresponding ribbon knot (with the addition of two arcs, one for each singularity). It is not a coincidence that the ribbon disk complement in B^4 has a 2-spine whose presentation is precisely the presentation above. It is the study of ribbon disk complements, and complements in the 4-ball in general, that led to the definition of gccc's.

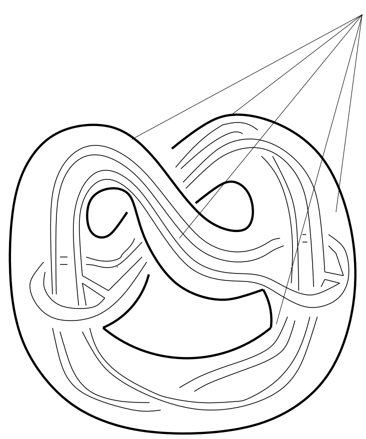


FIGURE 2. A gccc with deeper meaning.

We now prove the main result.

Theorem 2.1. *Every finite 2-complex K 3-deforms to a gccc.*

Proof: Let $\mathcal{P}_K = \{x_1, x_2, \cdots, x_n \mid r_1, r_2, \cdots, r_p\}$. We begin our construction of a gccc by taking Γ to be the core of an unknotted cube with n handles (Figure 3).

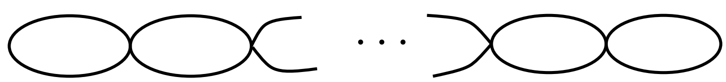


FIGURE 3. The graph Γ.

Note that $\pi_1(S^3 \setminus \Gamma)$ is a free group on n generators, which we denote by $\langle x_1, x_2, \cdots, x_n \mid \rangle$. For each relator r_i of $\mathcal{P}_K$ let s_i be a simple closed curve in $S^3 \setminus \Gamma$ which represents the conjugacy class of r_i in $\pi_1(S^3 \setminus \Gamma)$. By changing crossings among the s_i's without changing crossings with Γ, we may assume that each s_i is unknotted and that the curves s_i are unlinked. There is a tree T in $S^3 \setminus \Gamma$ which intersects each s_i in a single point such that the graph $G = T \bigcup \{s_i\}$ is an unknotted graph in S^3 (Figure 4).

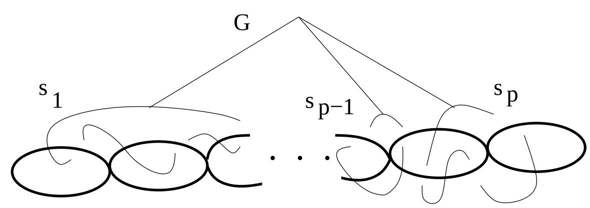


FIGURE 4. The graph G in the complement of Γ.

In particular, the complement of G in S^3 is an unknotted cube with handles which contains the graph Γ. We set $U = S^3 \setminus G$ and consider the gccc $X = (U \setminus \Gamma) \bigcup c * \mathrm{Bd}(U)$. By construction, the meridians of $\mathrm{Bd}(U)$ correspond to the curves s_i and thus represent the same conjugacy classes in $\pi_1(S^3 \setminus \Gamma)$, namely the conjugacy classes of the relators r_i. By Lemma 2.1, X 3-deforms to the union of $S^3 \setminus \Gamma$ and 2-cells with attaching words $\{r_i\}$. This complex, in turn, 3-deforms to a 2-complex associated with $\mathcal{P}_K$. This 2-complex shares the 3-deformation class of our complex K and the proof is complete. □

If we regard the 4-ball B^4 as the cone over S^3, $B^4 = c * S^3$, we notice that gccc's are natural residents of B^4. In particular, a gccc $W \bigcup c * Bd(U)$ may be placed in B^4 such that W resides in $Bd(B^4) = S^3$ and $c * Bd(U) \setminus Bd(U)$ resides in $int(B^4)$. We have as a consequence the following well-known fact. (See [4] and [5].)

Corollary 2.1. *Up to 3-deformation, every finite 2-complex embeds in the 4-ball B^4.*

3. The Dual of a GCCC

Given data U and Γ for a gccc X, recall that we set $V = N(\Gamma)$, $W = U \setminus V$, and $X = W \bigcup c * \mathrm{Bd}(U)$. Define the dual X^* of X to be the 3-complex $X^* = W \bigcup c * \mathrm{Bd}(V)$. We will prove that the dual is particularly well-behaved for a certain class of gccc's. First, we prove the following lemma.

Lemma 3.1. *Suppose the data for a gccc X consists of an unknotted solid torus U and a knot Γ in its interior. If $\pi_1(X) = \{1\}$, then Γ has winding number 1 in U.*

Proof: By Lemma 2.1, X 3-deforms to a 2-complex with presentation $\{x_1, \ldots, x_n | r_1, \ldots, r_{n-1}, s\}$, where $\{x_1, \ldots, x_n | r_1, \ldots, r_{n-1}\}$ is the Wirtinger presentation for $\pi_1(S^3 \setminus \Gamma)$ and s represents a meridian of $Bd(U)$. This complex, and thus X as well, is contractible, as it is modeled on a finite, balanced presentation for the trivial group. Let σ_i denote the exponent sum of x_i in s, $1 \leq i \leq n$. We may assume that the relator r_1 conjugates x_1 into x_2, the relator r_2 conjugates x_2 into x_3, and so on.

Now, $H_1(X) = 0 \implies$

$$\det \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_{n-1} & \sigma_n \end{bmatrix} = \pm 1 \implies \sum_{i=1}^{n} \sigma_i = \pm 1.$$

It follows that Γ has winding number 1 in U. □

We will now use Lemma 3.1 to show that certain duals 3-deform to a point. The words in the upcoming proof deliberately echo those in the written proof of Lemma 2.1. As before, a synchronized visual proof is provided in the Appendix.

Theorem 3.1. *Suppose, as in Lemma 3.1, that the data for a gccc X consists of an unknotted solid torus U and a knot Γ in its interior. If $\pi_1(X) = \{1\}$, then the dual X^* 3-deforms to a point.*

Proof: We will effect the 3-deformation of X^* to a point in a sequence of steps. Assume that the longitude λ and meridian μ of $\mathrm{Bd}(V)$ are simple closed curves which meet in a single point.

(0) We have X^* itself.
(1) Choose an arc β in W from a point on $Bd(V)$ to a point on $Bd(U)$ whose interior misses $Bd(U) \bigcup Bd(V)$. A regular neighborhood $N(\beta)$, chosen sufficiently small, meets $Bd(W)$ in two disks, a disk D^+ on $Bd(V)$ and a disk D^- on $Bd(U)$. We may assume that D^+ misses the longitude and meridian of $Bd(V)$. Collapse $\mathrm{int}(N(\beta))$ through the free face D^-.
(2) Next, collapse $c{*}Bd(V)$ to $[Bd(V) \backslash \mathrm{int}(D^+)] \bigcup c{*}\{\lambda, \mu\}$. This event begins with a collapse through the free face D^+ and is characterized by the collapse of all 3-cells in $c{*}\mathrm{Bd}(V)$.
(3) Replace the cone over $\{\lambda, \mu\}$ with disks $\{D_0, E_0\}$ attached (respectively) to these curves. The procedure here is the same as in step (3) in the proof of Lemma 2.1. The details are omitted.
(4) Fill the tunnel $\mathrm{int}((V \bigcup N(\beta)) \setminus N(E_0))$ with a 3-cell. Recall that $\mathrm{Bd}(U)$ lacks $\mathrm{int}(D^-)$ due to an earlier collapse and thus this move is indeed a 3-deformation. The result of this deformation is the 3-complex $U \bigcup D_0$.
(5) By Lemma 3.1, λ has winding number 1 in U and thus is freely homotopic to a longitude of $Bd(U)$. We may therefore, by a 3-deformation, replace D_0 with a disk attached to a longitude of $Bd(U)$. Attaching such a disk to a solid torus produces a 3-ball.
(6) Collapse this 3-ball to a point. □

We conclude with an application to the spines of surgery 3-manifolds. The emphasis in Theorem 3.2 is on the construction of the gccc spine rather than its existence. It is known that every closed, connected 3-manifold has a 2-spine. By Theorem 2.1, this 2-spine can surely be realized by some gccc. The particular gccc spine described in Theorem 3.2, however, is appealing in that the data is simple and the construction follows closely the surgery instructions.

Theorem 3.2. *Let M be a 3-manifold obtained by surgery on a knot k in S^3. Then M has a 2-spine which 3-deforms to a gccc with the following properties: The data for the gccc consists of a solid torus whose meridian reads the surgery curve. The knot k serves as the graph in the solid torus.*

Proof: Given the knot k and a surgery curve s on $\mathrm{Bd}(N(k))$, push s slightly off $\mathrm{Bd}(N(k))$ into $S^3 \setminus N(k)$. For convenience, we continue to call this curve s. We may assume that s is unknotted in S^3 without changing the class it represents in $\pi_1(S^3 \setminus k)$. Let $U = S^3 \setminus \mathrm{int}(N(s))$, where $N(s)$ is taken sufficiently small, and consider the gccc $X = [U \setminus \mathrm{int}(N(k))] \bigcup c * \mathrm{Bd}(U)$. Note that U is an unknotted solid torus that contains k, as k misses $N(s)$. By Lemma 2.1, X 3-deforms to the 3-complex $X_0 = (S^3 \setminus k) \bigcup E$, where E is a 2-cell whose attaching map reads s in $\pi_1(S^3 \setminus k)$. Now, X_0 collapses to a 2-spine for the surgery manifold and our proof is complete. □

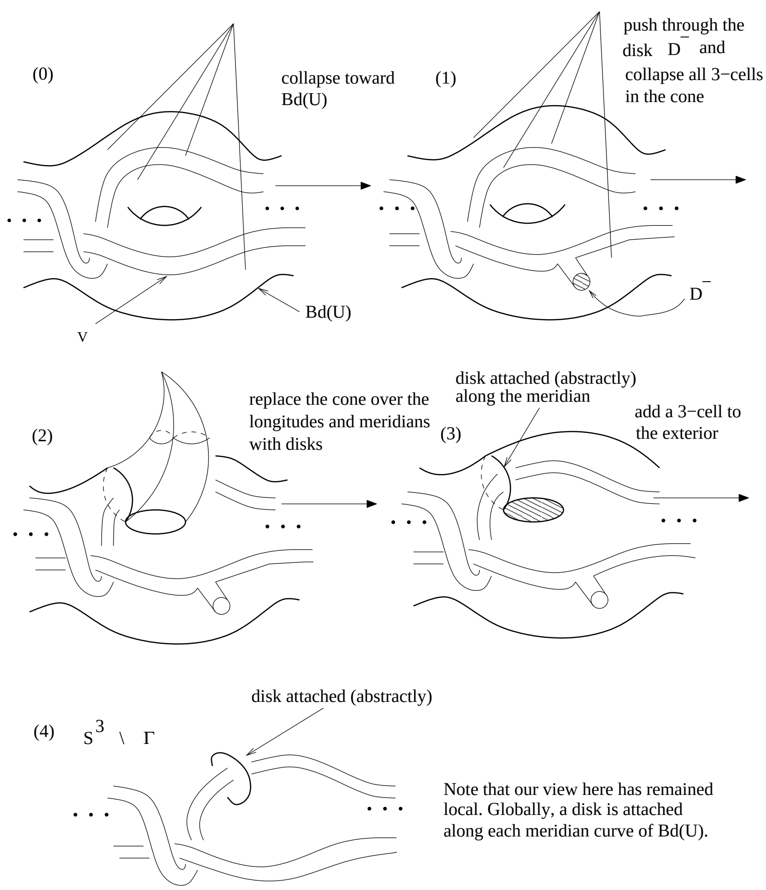
Corollary 3.1. *If M is a homotopy 3-sphere and X is the gccc spine constructed in Theorem 3.2, then X^* 3-deforms to a point.*

We need only note that X is contractible and apply Theorem 3.1.

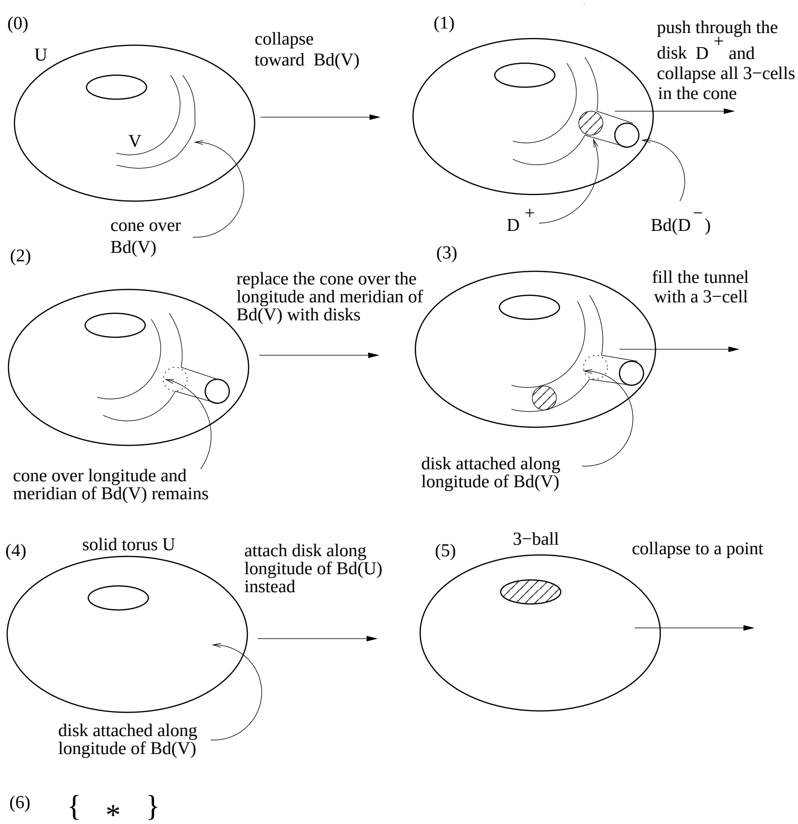
4. Appendix

Here we provide visual proofs of Lemma 2.1 and Theorem 3.1.

Lemma 2.1. *Suppose the data for a gccc X consists of a cube with handles U in S^3 and a graph Γ in its interior. Suppose further that U is unknotted and $\{\mu_i\}$ is a complete set of meridians in $Bd(U)$. Then X 3-deforms to $(S^3 \setminus \Gamma) \bigcup \{E_i\}$, where each E_i is a 2-cell attached along the curve μ_i .*



Theorem 3.1. *Suppose that the data for a gccc X consists of an unknotted solid torus U and a knot Γ in its interior. If $\pi_1(X) = \{1\}$, then the dual X^* 3-deforms to a point.*



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