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COVERING PROPERTIES OF INVERSE LIMITS, II

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Abstract. Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection $\pi_\alpha : X \to X_\alpha$ being a pseudo-open map. Suppose that X is λ-paracompact, where λ is the cardinality of Λ. The first author asked whether X is $\delta\theta$-refinable if each X_α is. She raised the same question for collectionwise δ-normality and collectionwise subnormality. In the present paper, we give a partial answer to the first question assuming the subnormality of X, and we solve the remaining two questions giving affirmative answers. Moreover, we solve two other similar questions also raised by her.

1. Introduction

Throughout this paper, all spaces are topological spaces without any separation axiom, and all maps are continuous. For an inverse system $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ and its limit X, let Λ be a directed set with an order $<$ and its cardinality λ, where $\lambda \geq \omega$, and let π_α be the projection from X into X_α for each $\alpha \in \Lambda$.

First recall the following result of A. Bešlagić, which is a motivation of the study for the covering properties of inverse limits.

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(A) [2] (also see [6]). Let $X = \prod_{\alpha \in \Gamma} X_\alpha$ be a product space such that $\prod_{\alpha \in F} X_\alpha$ is normal for each finite $F \subset \Gamma$. Then X is normal if and only if it is λ-paracompact, where λ is the cardinality of Γ.

However, since a product space is the limit of the inverse system consisting of its finite subproducts and their projections, the "if" part of (A) had been already extended by Y. Aoki as follows:

(B) [1]. Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α is normal, then so is X.

Moreover, Aoki and K. Chiba proved many analogous results to (B) for several covering properties and the other separation properties. The following are some which relate to this paper.

(C) [1], [7], [9]. Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α satisfies one of the following properties, then X has the corresponding property.

(1) Paracompactness.
(2) Collectionwise normality.
(3) Subparacompactness.
(4) Metacompactness.
(5) Submetacompactness (= θ-refinability).
(6) Subnormality.

In addition, Chiba also obtained the analogies of (B) for hereditary properties of the above ones without any assumption of the projections.

(D) [7], [9]. Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Suppose that X is hereditarily λ-paracompact. If each X_α satisfies one of the following properties, then X has the corresponding property.

(0) Hereditary normality.
(1) Hereditary paracompactness.
(2) Hereditary collectionwise normality.
(3) Hereditary subparacompactness.
(4) Hereditary metacompactness.
(5) Hereditary submetacompactness (= hereditary θ-refinability).
(6) Hereditary subnormality.

The purpose of this study is to prove that (C) and (D) hold for all main covering properties and all main separation properties. By [7, theorems 1 (vi) and 2 (v)], (C) and (D) also hold for metaLindelöfness and hereditary metaLindelöfness, respectively. Moreover, observe that both submetacompact spaces and metaLindelöf spaces are $\delta\theta$-refinable (= submetaLindelöf). It is natural from this point of view to raise the following question:

Question 1 [9]**.** (i) Does (C) hold for $\delta\theta$-refinability?
(ii) Does (D) hold for hereditary $\delta\theta$-refinability?

On the other hand, as separation properties, collectionwise δ-normality and collectionwise subnormality are between subnormality and subparacompactness. So it is also natural from C (3), C (6), D (3) and D (6) to raise the following two questions:

Question 2 [9]**.** (i) Does (C) hold for collectionwise δ-normality?
(ii) Does (D) hold for hereditary collectionwise δ-normality?

Question 3 [9]**.** (i) Does (C) hold for collectionwise subnormality?
(ii) Does (D) hold for hereditary collectionwise subnormality?

For Question 1 (i), Chiba [8], [9] gave partial answers under the assumption of Λ being countable or the normality of X. In Section 2, we extend the latter result assuming the subnormality of X. Second, we show that (C) holds for weak θ-refinability and weak $\delta\theta$-refinability without any additional condition. In Section 3, we give affirmative answers to both questions 2 (i) and 3 (i) by paying attention to the subnormality of X. In Section 4, we give a partial answer to Question 1 (ii) by replacing λ-paracompactness of X with its λ-subparacompactness. Moreover, we also give affirmative answers to both questions 2 (ii) and 3 (ii). In Section 5, we show that the condition of hereditary λ-paracompactness of X in (D) can be generalized in terms of hereditary λ-subparacompactness, hereditary λ-metacompactness, and hereditary λ-submetacompactness, according to each case.

From the results of the present paper, we can say that (C) and (D) hold for all main separation properties and for almost all main covering properties. However, we have to recognize that only $\delta\theta$-refinability and hereditary $\delta\theta$-refinability may be exceptional covering properties for (C) and (D), respectively. This seems to be a quite curious phenomenon because these have been not so noteworthy as covering properties.

2. $\delta\theta$-refinability and weak θ-refinability

A map f from X onto Y is *pseudo-open* if $y \in \operatorname{Int} f(U)$ holds for each $y \in Y$ and each open set U in X with $f^{-1}(y) \subset U$. Note that both open and onto maps and closed and onto maps are pseudo-open.

As in [7, Corollary 2], we first consider the relation between the projections and the bonding maps of inverse limits for pseudo-openness. The following lemma may be well-known.

Lemma 2.1. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let $\alpha, \beta \in \Lambda$ with $\beta \le \alpha$. If the projection π_α is onto, then $\pi^\alpha_\beta(A) = \pi_\beta \pi_\alpha^{-1}(A)$ for each $A \subset X_\alpha$.*

Proof: Let $A \subset X_\alpha$ and pick a $y \in \pi^\alpha_\beta(A)$. We find $x_\alpha \in A$ with $\pi^\alpha_\beta(x_\alpha) = y$. Since π_α is onto, we also find $x \in X$ with $\pi_\alpha(x) = x_\alpha$. Then, we have $x \in \pi_\alpha^{-1}(A)$ and $\pi_\beta(x) = \pi^\alpha_\beta \pi_\alpha(x) = \pi^\alpha_\beta(x_\alpha) = y$. Hence, it follows that $y \in \pi_\beta \pi_\alpha^{-1}(A)$. Conversely, pick a $y \in \pi_\beta \pi_\alpha^{-1}(A)$. We find $x \in \pi_\alpha^{-1}(A)$ with $\pi_\beta(x) = y$. Then we have $\pi_\alpha(x) \in A$ and $\pi^\alpha_\beta \pi_\alpha(x) = \pi_\beta(x) = y$. Hence, it follows that $y \in \pi^\alpha_\beta(A)$. □

Proposition 2.2. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. If each projection π_α is a pseudo-open map, then so is π^α_β.*

Proof: Let $\alpha, \beta \in \Lambda$ with $\beta \le \alpha$. Pick a $y \in X_\beta$ and take an open set U in X_α with $(\pi^\alpha_\beta)^{-1}(y) \subset U$. Since π_β is onto, we find $x \in X$ with $\pi_\beta(x) = y$. Then we have $x \in \pi_\beta^{-1}(y) = (\pi^\alpha_\beta \pi_\alpha)^{-1}(y) = \pi_\alpha^{-1}(\pi^\alpha_\beta)^{-1}(y) \subset \pi_\alpha^{-1}(U)$. It follows from the pseudo-openness of π_β and Lemma 2.1 above that $y \in \operatorname{Int} \pi_\beta(\pi_\alpha^{-1}(U)) = \operatorname{Int} \pi^\alpha_\beta(U)$. □

Now, we proceed the main results of this section.

A space X is λ-*paracompact* if every open cover of X with cardinality $\le \lambda$ has a locally finite open refinement. A cover $\mathcal{A}$ of a space X is *directed* if for any $A_0, A_1 \in \mathcal{A}$, there is $A_2 \in \mathcal{A}$ with $A_0 \cup A_1 \subset A_2$.

Lemma 2.3 [13]**.** *A space X is λ-paracompact if and only if for every directed open cover $\mathcal{U}$ of X with cardinality $\leq \lambda$, there is a locally finite open cover $\mathcal{V}$ of X such that $\{\overline{V} : V \in \mathcal{V}\}$ refines $\mathcal{U}$.*

A space X is *λ-subparacompact* if every open cover of X with cardinality $\leq \lambda$ has a σ-locally finite closed refinement. Let X be a space and $\mathcal{V}$ a collection of subsets in X. For each $x \in X$, we denote by $\text{ord}(x, \mathcal{V})$ the cardinality of $\{V \in \mathcal{V} : x \in V\}$.

Burke actually proved the following.

Lemma 2.4 [3], [4]**.** *For a space X, the following are equivalent.*

(a) *X is λ-subparacompact.*
(b) *Every open cover of X with cardinality $\leq \lambda$ has a σ-discrete closed refinement.*
(c) *Every open cover of X with cardinality $\leq \lambda$ has a σ-closure-preserving closed refinement.*
(d) *For every open cover $\mathcal{U}$ of X with cardinality $\leq \lambda$, there is a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$, one can find $n_x \in \omega$ with $\text{ord}(x, \mathcal{V}_{n_x}) = 1$.*

A space X is *subnormal* if for any disjoint closed sets A and B in X, there are disjoint G_δ-sets G and H such that $A \subset G$ and $B \subset H$. Note that X is subnormal if and only if every finite (or binary) open cover of X has a countable closed refinement.

Lemma 2.5. *Every λ-paracompact and subnormal space is λ-subparacompact.*

Proof: Let X be a λ-paracompact and subnormal space. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Omega\}$ be an open cover of X with cardinality $\leq \lambda$. Let $\mathcal{U}^*$ be the collection consisting of all unions of finitely many members of $\mathcal{U}$. Then $\mathcal{U}^*$ is a directed open cover of X with cardinality $\leq \lambda$. By Lemma 2.3, there is a locally finite closed refinement $\mathcal{K}$ of $\mathcal{U}^*$. For each $K \in \mathcal{K}$, find $\mu_K \in [\Omega]^{<\omega}$ with $K \subset \bigcup_{\alpha \in \mu_K} U_\alpha$. Since K is subnormal, there is a countable closed cover $\{F_{\alpha,n} : \alpha \in \mu_K \text{ and } n \in \omega\}$ of K such that $F_{\alpha,n} \subset U_\alpha$ for each $\alpha \in \mu_K$ and each $n \in \omega$. Let $\mathcal{F}_n = \{F_{\alpha,n} : \alpha \in \mu_K \text{ and } K \in \mathcal{K}\}$ for each $n \in \omega$. Then $\bigcup_{n \in \omega} \mathcal{F}_n$ is a σ-locally finite closed refinement of $\mathcal{U}$. □

Remark. As is well-known, paracompactness implies subparacompactness. However, for each $\lambda \geq \omega$, λ-paracompactness does not imply λ-subparacompactness. In fact, let $X_\lambda = \lambda^+ \times (\lambda^+ + 1)$.

Since X_λ is the product of a λ-paracompact space and a compact space, it is λ-paracompact. However, X_λ is not subnormal (because, by the pressing down lemma, $\{(\alpha,\alpha) \in X_\lambda : \alpha \in \lambda^+\}$ and $\lambda^+ \times \{\lambda^+\}$ cannot be separated by disjoint G_δ-sets). Since every ω-subparacompact space is subnormal, X_λ is not ω-subparacompact, hence not λ-subparacompact.

A space X is *$\delta\theta$-refinable (= submetaLindelöf)* if for every open cover $\mathcal{U}$ of X, there is a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$ one can find $n_x \in \omega$ with $\mathrm{ord}(x, \mathcal{V}_{n_x}) \leq \omega$. Obviously, every submetacompact space is $\delta\theta$-refinable.

First, we give a partial answer to Question 1 (i) above. This is also a generalization of [9, Theorem 1 (i)].

Theorem 2.6 *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact and subnormal. If each X_α is $\delta\theta$-refinable, then so is X.*

Proof: Let $\mathcal{G} = \{G_\xi : \xi \in \Xi\}$ be an open cover of X. For each $\alpha \in \Lambda$ and each $\xi \in \Xi$, let $U_{\alpha,\xi}$ be the maximal open set in X_α such that $\pi_\alpha^{-1}(U_{\alpha,\xi}) \subset G_\xi$. Moreover, for each $\alpha \in \Lambda$, let $U_\alpha = \bigcup\{U_{\alpha,\xi} : \xi \in \Xi\}$. Then $\{\pi_\alpha^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a directed open cover of X such that $\pi_\beta^{-1}(U_\beta) \subset \pi_\alpha^{-1}(U_\alpha)$ if $\beta \leq \alpha$. Since X is λ-paracompact, it follows from Lemma 2.3 that there is an open cover $\{P_\alpha : \alpha \in \Lambda\}$ of X such that

(i) $\overline{P_\alpha} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$,
(ii) $P_\beta \subset P_\alpha$ if $\beta \leq \alpha$.

Then, as is shown in [1, 7], there is an open set V_α in X_α for each $\alpha \in \Lambda$ such that

(iii) $\overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in \Lambda$,
(iv) $\{\pi_\alpha^{-1}(V_\alpha) : \alpha \in \Lambda\}$ covers X.

In fact, let V_α be the maximal open set in X_α such that $\pi_\alpha^{-1}(V_\alpha) \subset P_\alpha$ (as in the proof of [1, Lemma 2.3]), or let $V_\alpha = \mathrm{Int}\big((X_\alpha \setminus \pi_\alpha(X \setminus \overline{P_\alpha})\big)$ (as in the proof of [7, Theorem 1 (i)]). Then it follows from (i), (ii), and the assumption of π_α that each V_α is a desired one.

Since X is subnormal, it follows from Lemma 2.5 that X is λ-subparacompact. By Lemma 2.4 and (iv), there is a closed cover $\{F_{\alpha,n} : \alpha \in \Lambda \text{ and } n \in \omega\}$ of X such that

(v) $F_{\alpha,n} \subset \pi_\alpha^{-1}(V_\alpha)$ for each $\alpha \in \Lambda$ and each $n \in \omega$,
(vi) $\{F_{\alpha,n} : \alpha \in \Lambda\}$ is discrete in X for each $n \in \omega$.

For each $\alpha \in \Lambda$ and each $n \in \omega$, let $D_{\alpha,n} = \bigcup\{F_{\beta,n} : \beta \in \Lambda \text{ with } \beta \neq \alpha\}$ and let $E_n = \bigcup\{F_{\alpha,n} : \alpha \in \Lambda\}$.

Take an $\alpha \in \Lambda$. By (iii), $\{U_{\alpha,\xi} : \xi \in \Xi\}$ covers $\overline{V_\alpha}$. So it follows from the $\delta\theta$-refinability of X_α that there is a sequence $\mathcal{W}_{\alpha,k} = \{W_{\alpha,\xi,k} : \xi \in \Xi\}, k \in \omega$, of collections of open sets in X_α such that

(vii) $W_{\alpha,\xi,k} \subset U_{\alpha,\xi}$ for each $\xi \in \Xi$ and each $k \in \omega$,
(viii) $\mathcal{W}_{\alpha,k}$ covers $\overline{V_\alpha}$ for each $k \in \omega$,
(ix) for each $y \in \overline{V_\alpha}$, one can find $k_y \in \omega$ with $\operatorname{ord}(y, \mathcal{W}_{\alpha,k_y}) \leq \omega$.

For each $n, k \in \omega$, we put

$$\mathcal{H}_k^n = \{\pi_\alpha^{-1}(W_{\alpha,\xi,k}) \setminus D_{\alpha,n} : \alpha \in \Lambda \text{ and } \xi \in \Xi\} \cup \mathcal{G} \restriction (X \setminus E_n).$$

Take any $n, k \in \omega$. Since each $D_{\alpha,n}$ and E_n is closed in X, each member of $\mathcal{H}_k^n$ is open in X. By (vii) and the choice of $U_{\alpha,\xi}$, each member of $\mathcal{H}_k^n$ is contained in some member of $\mathcal{G}$. Pick an $x \in E_n$. Find δ with $x \in F_{\delta,n}$. By (v), note that $\pi_\delta(x) \in \pi_\delta(F_{\delta,n}) \subset \pi_\delta\pi_\delta^{-1}(V_\delta) = V_\delta \subset \overline{V_\delta}$. It follows from (viii) that there is $\mu \in \Xi$ with $\pi_\delta(x) \in W_{\delta,\mu,k}$. By (vi), we have $x \in \pi_\delta^{-1}(W_{\delta,\mu,k}) \setminus D_{\delta,n} \in \mathcal{H}_k^n$. Hence, $\mathcal{H}_k^n$ covers X. Thus each $\mathcal{H}_k^n$ is an open refinement of $\mathcal{G}$. Pick an $x \in X$. There are some $m \in \omega$ and $\gamma \in \Lambda$ with $x \in F_{\gamma,m} \subset E_m$. As above, we have $\pi_\gamma(x) \in \overline{V_\gamma}$. By (ix), there is $\ell \in \omega$ with $\operatorname{ord}(\pi_\gamma(x), \mathcal{W}_{\gamma,\ell}) \leq \omega$. We can let $\{\xi \in \Xi : \pi_\gamma(x) \in W_{\gamma,\xi,\ell}\} = \{\xi_0, \xi_1, \cdots\}$. It is easily verified that $x \in H \in \mathcal{H}_\ell^m$ implies $H = \pi_\gamma^{-1}(W_{\gamma,\xi_i,\ell}) \setminus D_{\gamma,m}$ for some $i \in \omega$. So we have $\operatorname{ord}(x, \mathcal{H}_\ell^m) \leq \omega$. Hence, $\{\mathcal{H}_k^n\}$ is a desired sequence which witnesses the $\delta\theta$-refinability of X. □

The following is an immediate consequence of Theorem 2.6 and C (6) in the Introduction.

Corollary 2.7. *Let $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α is $\delta\theta$-refinable and subnormal, then so is X.*

Recall that a space X is *weakly θ-refinable* (respectively, *weakly $\delta\theta$-refinable*) if for every open cover $\mathcal{U}$ of X, there is an open refinement $\bigcup_{n\in\omega} \mathcal{V}_n$ of $\mathcal{U}$ such that for each $x \in X$ one can find $n_x \in \omega$ with $\operatorname{ord}(x, \mathcal{V}_{n_x}) = 1$ (respectively, $1 \leq \operatorname{ord}(x, \mathcal{V}_{n_x}) \leq \omega$).

Theorem 2.8. *Let* $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ *be an inverse system and* X *its inverse limit with each projection* π_α *being a pseudo-open map. Suppose that* X *is* λ*-paracompact. If each* X_α *is weakly* θ*-refinable (weakly* $\delta\theta$*-refinable), then so is* X*.*

Proof: Let $\mathcal{G} = \{G_\xi : \xi \in \Xi\}$ be an open cover of X. For each $\alpha \in \Lambda$ and each $\xi \in \Xi$, let $U_{\alpha,\xi}$ and U_α be the same ones as described in the proof of Theorem 2.6. As in that proof, there is an open set V_α in X_α for each $\alpha \in \Lambda$ such that

(i) $\overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in \Lambda$,
(ii) $\{\pi^{-1}_\alpha(V_\alpha) : \alpha \in \Lambda\}$ covers X.

By λ-paracompactness of X and (ii), there is a locally finite open cover $\{O_\alpha : \alpha \in \Lambda\}$ of X such that $O_\alpha \subset \pi^{-1}_\alpha(V_\alpha)$ for each $\alpha \in \Lambda$.

For each $\alpha \in \Lambda$, it follows from the weak θ-refinability (respectively, weak $\delta\theta$-refinability) of X_α that there is a sequence $\mathcal{W}_{\alpha,n} = \{W_{\alpha,\xi,n} : \xi \in \Xi\}, n \in \omega$, of collections of open sets in X_α such that

(iii) $W_{\alpha,\xi,n} \subset U_{\alpha,\xi}$ for each $\xi \in \Xi$,
(iv) for each $y \in \overline{V_\alpha}$, one can find $n_y \in \omega$ with $\mathrm{ord}(y, \mathcal{W}_{\alpha,n_y}) = 1$ (respectively, $1 \leq \mathrm{ord}(y, \mathcal{W}_{\alpha,n_y}) \leq \omega$).

Let Λ well-order by $\prec$, which may be different from the directed order $<$ on Λ. For each $\sigma = (n_0, \cdots, n_{k-1}) \in \omega^k, k \in \omega$, let

$$\mathcal{H}(\sigma) = \Big\{ \bigcap_{i<k} \big(\pi^{-1}_{\alpha_i}(W_{\alpha_i,\xi_i,n_i}) \cap O_{\alpha_i}\big) : \alpha_0, \cdots, \alpha_{k-1} \in \Lambda \text{ with } a_0 \prec \cdots \prec \alpha_{k-1} \text{ and } \xi_0, \cdots, \xi_{k-1} \in \Xi \Big\}.$$

We show that $\bigcup\{\mathcal{H}(\sigma) : \sigma \in \omega^{<\omega}\}$ is a desired open refinement of $\mathcal{G}$. Obviously, each member of $\mathcal{H}(\sigma)$ is open in X. Moreover, it is easily seen by (iii) that each member of $\mathcal{H}(\sigma)$ is contained in some member of $\mathcal{G}$. Pick an $x \in X$. Let $\Lambda(x) = \{\alpha \in \Lambda : x \in O_\alpha\}$. Since $\Lambda(x)$ is finite, we can let $\Lambda(x) = \{\delta_0, \cdots, \delta_{k-1}\}$ with $\delta_0 \prec \cdots \prec \delta_{k-1}$. Take any $i < k$. Since $x \in O_{\delta_i} \subset \pi^{-1}_{\delta_i}(V_{\delta_i})$, we have $\pi_{\delta_i}(x) \in V_{\delta_i} \subset \overline{V_{\delta_i}}$. It follows from (iv) that there is $m_i \in \omega$ with $\mathrm{ord}(\pi_{\delta_i}(x), \mathcal{W}_{\delta_i,m_i}) = 1$ (respectively, $1 \leq \mathrm{ord}(\pi_{\delta_i}(x), \mathcal{W}_{\delta_i,m_i}) \leq \omega$). So we can find some $\mu_i \in \Xi$ (respectively, $\{\mu_{ij} : i, j \in \omega\} \subset \Xi$) such that

$\pi_{\delta_i}(x) \in W_{\delta_i,\mu_i,m_i} \setminus \bigcup\{W \in \mathcal{W}_{\delta_i,m_i} : W \neq W_{\delta_i,\mu_i,m_i}\}$
$\Big(\pi_{\delta_i}(x) \in \bigcap_{j\in\omega} W_{\delta_i,\mu_{ij},m_i} \setminus \bigcup\{W \in \mathcal{W}_{\delta_i,m_i} : W \neq W_{\delta_i,\mu_{ij},m_i}\}$, respectively$\Big)$.

We first take care of the case where X is weakly θ-refinable. Let $\sigma_0 = (m_0, \cdots, m_{k-1}) \in \omega^{<\omega}$ and let $H_0 = \bigcap_{i<k} \pi_{\delta_i}^{-1}(W_{\delta_i,\mu_i,m_i}) \cap O_{\delta_i}$. Then we have $x \in H_0 \in \mathcal{H}(\sigma_0)$. Take any $H \in \mathcal{H}(\sigma_0)$ with $H \neq H_0$. Let $H = \bigcap_{i<k}(\pi_{\alpha_i}^{-1}(W_{\alpha_i,\xi_i,m_i}) \cap O_{\alpha_i})$, where $\alpha_0 \prec \cdots \prec \alpha_{k-1}$. First, assume $(\alpha_0, \cdots, \alpha_{k-1}) \neq (\delta_0, \cdots, \delta_{k-1})$. Since $\Lambda(x) = \{\delta_0, \cdots, \delta_{k-1}\}$, we can find $\ell < k$ with $\alpha_\ell \notin \Lambda(x)$. Hence, we have $H \subset O_{\alpha_\ell} \not\ni x$. Second, assume $(\alpha_0, \cdots, \alpha_{k-1}) = (\delta_0, \cdots, \delta_{k-1})$. By $H \neq H_0$, we can find $j < k$ with $\xi_j \neq \mu_j$. By the choice of μ_j, we have $\pi_{\alpha_j}(x) = \pi_{\delta_j}(x) \notin W_{\delta_j,\xi_j,m_j} = W_{\alpha_j,\xi_j,m_j}$. So it follows that $H \subset \pi_{\alpha_j}^{-1}(W_{\alpha_j,\xi_j,m_j}) \not\ni x$. Hence, we obtain that $\mathrm{ord}(x, \mathcal{H}(\sigma_0)) = 1$.

The weak $\delta\theta$-refinable case of X is similar to the above. □

3. Collectionwise δ-normality and collectionwise subnormality

A space X is *collectionwise δ-normal* if for every discrete collection $\{C_\xi : \xi \in \Xi\}$ of subsets in X, there is a disjoint collection $\{U_\xi : \xi \in \Xi\}$ of G_δ-sets in X such that $C_\xi \subset U_\xi$ for each $\xi \in \Xi$. It is clear that every collectionwise δ-normal space is subnormal.

The following is an affirmative answer to Question 2 (i) (= [9, Question 3]).

Theorem 3.1. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α is collectionwise δ-normal, then so is X.*

Proof: Let $\mathcal{C} = \{C_\xi : \xi \in \Xi\}$ be a discrete collection of subsets in X. For each $\alpha \in \Lambda$, let $U_\alpha = \bigcup\{U : U$ is an open set in X_α such that $\pi_\alpha^{-1}(U)$ meets at most one member of $\mathcal{C}\}$. Then $\{\pi_\alpha^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a directed open cover of X such that $\pi_\beta^{-1}(U_\beta) \subset \pi_\alpha^{-1}(U_\alpha)$ if $\beta \leq \alpha$. As in the proof of Theorem 2.6, there is an open set V_α in X_α for each $\alpha \in \Lambda$ such that

(i) $\overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in \Lambda$,
(ii) $\{\pi_\alpha^{-1}(V_\alpha) : \alpha \in \Lambda\}$ covers X.

By λ-paracompactness of X and (ii), there is a locally finite open cover $\{O_\alpha : \alpha \in \Lambda\}$ of X such that $O_\alpha \subset \pi_\alpha^{-1}(V_\alpha)$ for each $\alpha \in \Lambda$. It follows from C (6) in the Introduction that X is subnormal. So it follows from Lemma 2.5 that X is λ-subparacompact. By Lemma 2.4, there is a closed cover $\{F_{\alpha,n} : \alpha \in \Lambda \text{ and } n \in \omega\}$ of X such that

(iii) $F_{\alpha,n} \subset O_\alpha$ for each $\alpha \in \Lambda$ and each $n \in \omega$,
(iv) $\{F_{\alpha,n} : \alpha \in \Lambda\}$ is discrete in X for each $n \in \omega$.

For each $\alpha \in \Lambda$ and each $n \in \omega$, let $D_{\alpha,n} = \bigcup\{F_{\beta,n} : \beta \in \Lambda \text{ with } \beta \neq \alpha\}$. Here, we may assume without loss of generality that

(v) $\{n \in \omega : x \in \bigcup\{F_{\alpha,n} : \alpha \in \Lambda\}\}$ is infinite for each $x \in X$.

Otherwise, we may infinitely repeat each $\{F_{\alpha,n} : \alpha \in \Lambda\}$. Take an $\alpha \in \Lambda$. Since $\{\pi_\alpha(C_\xi) \cap U_\alpha : \xi \in \Xi\}$ is discrete in U_α, it follows from (i) that $\{\pi_\alpha(C_\xi) \cap \overline{V_\alpha} : \xi \in \Xi\}$ is discrete in X_α. By the collectionwise δ-normality of X_α, there is a collection $\{W_{\alpha,\xi,n} : \xi \in \Xi \text{ and } n \in \omega\}$ of open sets in X_α such that

(vi) $\pi_\alpha(C_\xi) \cap \overline{V_\alpha} \subset W_{\alpha,\xi,n}$ and
(vii) $W_{\alpha,\xi,n+1} \subset W_{\alpha,\xi,n}$ for each $\xi \in \Xi$ and each $n \in \omega$,
(viii) $(\bigcap_{n\in\omega} W_{\alpha,\xi,n}) \cap (\bigcap_{n\in\omega} W_{\alpha,\xi',n}) = \emptyset$ if $\xi \neq \xi'$.

Letting α range over Λ, we put

$$H_{\xi,n} = \bigcup\{\pi_\alpha^{-1}(W_{\alpha,\xi,n}) \cap (O_\alpha \setminus D_{\alpha,n}) : \alpha \in \Lambda\}$$

for each $\xi \in \Xi$ and each $n \in \omega$. Then each $H_{\xi,n}$ is open in X. So it suffices to show the following two claims.

Claim 1. $C_\xi \subset H_{\xi,n}$ for each $\xi \in \Xi$ and each $n \in \omega$.

Proof of Claim 1. Take a $\xi \in \Xi$ and an $n \in \omega$. Pick an $x \in C_\xi$. By (iv), we can find $\delta \in \Lambda$ with $x \notin D_{\delta,n}$. First, assume $x \in F_{\delta,n}$. By (iii), we have $x \in O_\delta \setminus D_{\delta,n}$. Since $x \in O_\delta \subset \pi_\delta^{-1}(V_\delta)$, it follows from (vi) that $\pi_\delta(x) \in \pi_\delta(C_\xi) \cap \overline{V_\delta} \subset W_{\delta,\xi,n}$. Hence, we conclude that $x \in \pi_\delta^{-1}(W_{\delta,\xi,n}) \cap (O_\delta \setminus D_{\delta,n}) \subset H_{\xi,n}$. On the other hand, assume $x \notin F_{\delta,n}$. Find $\gamma \in \Lambda$ with $x \in O_\gamma$. Then we have $x \in O_\gamma \setminus (F_{\delta,n} \cup D_{\delta,n}) \subset O_\gamma \setminus D_{\gamma,n}$. Similarly, it follows that $\pi_\gamma(x) \in \pi_\gamma(C_\xi) \cap \overline{V_\gamma} \subset W_{\gamma,\xi,n}$. Hence, we conclude that $x \in \pi_\gamma^{-1}(W_{\gamma,\xi,n}) \cap (O_\gamma \setminus D_{\gamma,n}) \subset H_{\xi,n}$.

Claim 2. $(\bigcap_{n\in\omega} H_{\xi,n}) \cap (\bigcap_{n\in\omega} H_{\xi',n}) = \emptyset$ if $\xi \neq \xi'$.

Proof of Claim 2. Assume that there is $p \in (\bigcap_{n\in\omega} H_{\xi,n}) \cap (\bigcap_{n\in\omega} H_{\mu,n})$ for some different $\xi, \mu \in \Xi$. By (iv), there is $\alpha_n \in \Lambda$

with $p \notin D_{\alpha_n,n}$ for each $n \in \omega$. By (v), $p \in F_{\alpha_n,n} \subset O_{\alpha_n}$ for infinitely many n's. Since $\{O_\alpha : \alpha \in \Lambda\}$ is point-finite at p, there is $\alpha^* \in \Lambda$ such that $p \in F_{\alpha^*,n}$ for infinitely many n's. Let $N^* = \{n \in \omega : p \in F_{\alpha^*,n}\}$. Then it follows that $p \in O_{\alpha^*} \setminus D_{\alpha^*,n}$ for each $n \in N^*$, and that $p \notin X \setminus F_{\alpha^*,n} \supset X \setminus D_{\alpha,n} \supset O_\alpha \setminus D_{\alpha,n}$ for each $\alpha \in \Lambda$ with $\alpha \neq \alpha^*$ and each $n \in N^*$. By the choice of p and $H_{\xi,n}$, we have $p \in \pi_{\alpha^*}^{-1}(W_{\alpha^*,\xi,n}) \cap \pi_{\alpha^*}^{-1}(W_{\alpha^*,\mu,n})$ for each $n \in N^*$. Since N^* is infinite, it follows from (vii) that

$$\pi_{\alpha^*}(p) \in (\textstyle\bigcap_{n\in N^*} W_{\alpha^*,\xi,n}) \cap (\bigcap_{n\in N^*} W_{\alpha^*,\mu,n}) = (\bigcap_{n\in\omega} W_{\alpha^*,\xi,n}) \cap (\bigcap_{n\in\omega} W_{\alpha^*,\mu,n}).$$

This contradicts (viii). □

A space X is *collectionwise subnormal (= discretely subexpandable)* if for every discrete collection $\{C_\xi : \xi \in \Xi\}$ of subsets in X, there is a sequence $\{U_{\xi,n} : \xi \in \Xi\}, n \in \omega$, of collections of open sets in X such that $C_\xi \subset U_{\xi,n}$ for each $\xi \in \Xi$ and each $n \in \omega$, and for each $x \in X$, one can find an $n_x \in \omega$ with $\operatorname{ord}(x, \{U_{\xi,n_x} : \xi \in \Xi\}) \leq 1$.

Note that every subparacompact space is collectionwise subnormal. It is clear that every collectionwise subnormal space is collectionwise δ-normal. Moreover, the following plays an auxiliary role in combining collectionwise subnormality with collectionwise δ-normality.

A space X is *discretely θ-expandable* (see [12]) if for every discrete collection $\{C_\xi : \xi \in \Xi\}$ of subsets in X, there is a sequence $\{U_{\xi,n} : \xi \in \Xi\}, n \in \omega$, of collections of open sets in X such that $C_\xi \subset U_{\xi,n}$ for each $\xi \in \Xi$ and each $n \in \omega$, and for each $x \in X$, one can find an $n_x \in \omega$ with $\operatorname{ord}(x, \{U_{\xi,n_x} : \xi \in \Xi\}) < \omega$.

Lemma 3.2. *A space X is collectionwise subnormal if and only if it is collectionwise δ-normal and discretely θ-expandable.*

Proof: The "only if" part is obvious. Let X be collectionwise δ-normal and discretely θ-expandable. Let $\{C_\xi : \xi \in \Xi\}$ be a discrete collection of subsets in X. There is a disjoint collection $\{U_\xi : \xi \in \Xi\}$ of G_δ-sets in X such that $C_\xi \subset U_\xi$ for each $\xi \in \Xi$. Then we can let $U_\xi = \bigcap_{n\in\omega} U_{\xi,n}$ for each $\xi \in \Xi$, where $U_{\xi,n}$ is an open set in X with $U_{\xi,n+1} \subset U_{\xi,n}$ for each $n \in \omega$. Moreover, there is a sequence $\{V_{\xi,n} : \xi \in \Xi\}, n \in \omega$ of collections of open sets in X such that

(i) $C_\xi \subset V_{\xi,n+1} \subset V_{\xi,n}$ for each $\xi \in \Xi$ and each $n \in \omega$,

(ii) for each $x \in X$, one can find $n_x \in \omega$ with $\mathrm{ord}(x, \{V_{\xi,n_x} : \xi \in \Xi\}) < \omega$.

Let $W_{\xi,n} = U_{\xi,n} \cap V_{\xi,n}$ for each $\xi \in \Xi$ and each $n \in \omega$. Then each $W_{\xi,n}$ is an open set in X such that $C_\xi \subset W_{\xi,n}$. For each $x \in X$, it is easily verified that one can find $m \in \omega$ such that $\mathrm{ord}(x, \{W_{\xi,m} : \xi \in \Xi\}) \leq 1$. □

The following has been already shown by the first author in [10]. However, we state the proof again for the reader's convenience.

Lemma 3.3 [10]**.** *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α is discretely θ-expandable, then so is X.*

Proof: Let $\mathcal{C} = \{C_\xi : \xi \in \Xi\}$, each U_α, each V_α, and $\{O_\alpha : \alpha \in \Lambda\}$ be the same ones as in the proof of Theorem 3.1. Take an $\alpha \in \Lambda$. Since $\{\pi_\alpha(C_\xi) \cap \overline{V_\alpha} : \xi \in \Xi\}$ is discrete in X_α, it follows from the assumption of X_α that there is a sequence $\{W_{\alpha,\xi,n} : \xi \in \Xi\}, n \in \omega$, of collections of open sets in X_α such that

(i) $\pi_\alpha(C_\xi) \cap \overline{V_\alpha} \subset W_{\alpha,\xi,n+1} \subset W_{\alpha,\xi,n}$ for each $\xi \in \Xi$ and each $n \in \omega$,

(ii) for each $y \in X_\alpha$, one can find $n_y \in \omega$ with $\mathrm{ord}(y, \{W_{\alpha,\xi,n_y} : \xi \in \Xi\}) < \omega$.

Here, we put $H_{\xi,n} = \bigcup\{\pi_\alpha^{-1}(W_{\alpha,\xi,n}) \cap O_\alpha : \alpha \in \Lambda\}$ for each $\xi \in \Xi$ and each $n \in \omega$. Then, it is not difficult to verify that the sequence $\{H_{\xi,n} : \xi \in \Xi\}, n \in \omega$, of collections of open sets in X witnesses the discrete θ-expandability of X. □

The following is an immediate consequence of Theorem 3.1 and lemmas 3.2 and 3.3, which is an affirmative answer to Question 2 (i) (= [9, Question 2]).

Theorem 3.4. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit with each projection π_α being a pseudo-open map. Suppose that X is λ-paracompact. If each X_α is collectionwise subnormal, then so is X.*

Remark. Recall that a space X is *finitely subparacompact* if every finite open cover of X has a σ-discrete closed refinement, and that a space X is *boundedly subexpandable* if X is collectionwise subnormal (= discretely subexpandable) and finitely subparacompact (see [12]). However, note that finite subparacompactness is equivalent to subnormality, and that collectionwise subnormality implies subnormality. Hence, collectionwise subnormality is exactly equivalent to bounded subexpandability. So [9, Theorem 1 (vii)] would be an affirmative answer to our Question 3 (= [9, Question 2]) if the proof were correct. However, there is a gap in this proof (more precisely, the part of "Proof of (1)" is not correct). Consequently, our Theorem 3.4 means that [9, Theorem 1 (vii)] is correct.

4. Hereditary subnormality and related properties

Recall that a space X is *hereditarily subnormal* if every subspace of X is subnormal. Note that X is hereditarily subnormal if and only if every open subspace of X is subnormal.

The first part of Proposition 4.1 was actually stated in [9]; however, there was no proof. Here, we give the proof of Proposition 4.1 for completeness.

Proposition 4.1 [9]**.** *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be an open set of X. Suppose that G is either λ-paracompact or λ-subparacompact. If each X_α is hereditarily subnormal, then G is subnormal.*

Proof: Let $\{G_0, G_1\}$ be a binary open cover of G. For each $\alpha \in \Lambda$ and each $i \in 2$, let $U_{\alpha,i}$ be the maximal open set in X_α such that $\pi_\alpha^{-1}(U_{\alpha,i}) \subset G_i$, and let $U_\alpha = U_{\alpha,0} \cup U_{\alpha,1}$. Then $\{\pi_\alpha^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a directed open cover of G such that $\pi_\beta^{-1}(U_\beta) \subset \pi_\alpha^{-1}(U_\alpha)$ if $\beta \leq \alpha$.

We first prove the case for G being λ-paracompact. It follows from Lemma 2.3 that there is a locally finite open cover $\{O_\alpha : \alpha \in \Lambda\}$ of G such that $\overline{O_\alpha}^G \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$, where $\overline{O_\alpha}^G$ denotes the closure of O_α in G. Take an $\alpha \in \Lambda$. Since $\pi_\alpha(\overline{O_\alpha}^G)$ is subnormal and $\pi_\alpha(\overline{O_\alpha}^G) \subset U_\alpha$, there is a countable collection $\{F_{\alpha,i,n} : i \in 2 \text{ and } n \in \omega\}$ of closed sets in X_α such that

(i) $F_{\alpha,i,n} \cap \pi_\alpha(\overline{O_\alpha}^G) \subset U_{\alpha,i}$ for each $i \in 2$ and each $n \in \omega$,

(ii) $\{F_{\alpha,i,n} : i \in 2 \text{ and } n \in \omega\}$ covers $\pi_\alpha(\overline{O_\alpha}^G)$.

Let $E_{i,n} = \bigcup\{\pi_\alpha^{-1}(F_{\alpha,i,n}) \cap \overline{O_\alpha}^G : \alpha \in \Lambda\}$ for each $i \in 2$ and each $n \in \omega$.Then it is easily seen that $\{E_{i,n} : i \in 2 \text{ and } n \in \omega\}$ is a countable closed cover of G such that $E_{i,n} \subset G_i$ for each $i \in 2$ and each $n \in \omega$.

For the case of G being λ-subparacompact, it follows from Lemma 2.4 that there is a closed cover $\{F_{\alpha,n} : \alpha \in \Lambda \text{ and } n \in \omega\}$ of G such that

(i) $F_{\alpha,n} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$ and $n \in \omega$,
(ii) $\{F_{\alpha,n} : \alpha \in \Lambda\}$ is discrete in G for each $n \in \omega$.

Take an $\alpha \in \Lambda$ and an $n \in \omega$. Since $\pi_\alpha(F_{\alpha,n})$ is subnormal and $\pi_\alpha(F_{\alpha,n}) \subset U_\alpha$, there is a countable collection $\{K^n_{\alpha,0,k}, K^n_{\alpha,1,k} : k \in \omega\}$ of closed sets in X_α such that

(iii) $K^n_{\alpha,i,k} \cap \pi_\alpha(F_{\alpha,n}) \subset U_{\alpha,i}$ for each $i \in 2$ and each $k \in \omega$,
(iv) $\{K^n_{\alpha,0,k}, K^n_{\alpha,1,k} : k \in \omega\}$ covers $\pi_\alpha(F_{\alpha,n})$.

Let $E^n_{i,k} = \bigcup\{\pi_\alpha^{-1}(K^n_{\alpha,i,k}) \cap F_{\alpha,n} : \alpha \in \Lambda\}$ for each $i \in 2$ and each $n, k \in \omega$. Then, it is easily verified that $\{E^n_{i,k} : i \in 2 \text{ and } n, k \in \omega\}$ is a countable closed cover of G such that $E^n_{i,k} \subset G_i$ for each $i \in 2$ and each $n, k \in \omega$. □

Recall that a space X is *hereditarily $\delta\theta$-refinable* if every (open) subspace of X is $\delta\theta$-refinable.

Theorem 4.2. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be λ-subparacompact open subspace of X. If each X_α is hereditarily $\delta\theta$-refinable, then G is $\delta\theta$-refinable.*

Proof: Let $\mathcal{G} = \{G_\xi : \xi \in \Xi\}$ be an open cover of G. For each $\alpha \in \Lambda$ and each $\xi \in \Xi$, let $U_{\alpha,\xi}$ be the maximal open set in X_α such that $\pi_\alpha^{-1}(U_{\alpha,\xi}) \subset G_\xi$, and let $U_\alpha = \bigcup\{U_{\alpha,\xi} : \xi \in \Xi\}$. Then, $\{\pi_\alpha^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a directed open cover of G such that $\pi_\beta^{-1}(U_\beta) \subset \pi_\alpha^{-1}(U_\alpha)$ if $\beta \leq \alpha$. Since G is λ-subparacompact, it follows from Lemma 2.4 that there is a closed cover $\{F_{\alpha,n} : \alpha \in \Lambda \text{ and } n \in \omega\}$ of G such that

(i) $F_{\alpha,n} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$ and $n \in \omega$,
(ii) $\{F_{\alpha,n} : \alpha \in \Lambda\}$ is discrete in G for each $n \in \omega$.

For each $\alpha \in \Lambda$ and each $n \in \omega$, let $D_{\alpha,n} = \bigcup\{F_{\beta,n} : \beta \in \Lambda \text{ with } \beta \neq \alpha\}$ and let $E_n = \bigcup\{F_{\alpha,n} : \alpha \in \Lambda\}$. Take an $\alpha \in \Lambda$ and an $n \in \omega$. Since $\pi_\alpha(F_{\alpha,n})$ is $\delta\theta$-refinable and $\pi_\alpha(F_{\alpha,n}) \subset U_\alpha$, there is a sequence $\mathcal{W}^n_{\alpha,k} = \{W^n_{\alpha,\xi,k} \cap \pi_\alpha(F_{\alpha,n}) : \xi \in \Xi\}, k \in \omega$, of open covers of $\pi_\alpha(F_{\alpha,n})$ such that

(iii) $W^n_{\alpha,\xi,k}$ is open in X_α with $W^n_{\alpha,\xi,k} \subset U_{\alpha,\xi}$ for each $\xi \in \Xi$ and each $k \in \omega$,

(iv) for each $y \in \pi_\alpha(F_{\alpha,n})$, one can find $k_y \in \omega$ with $\mathrm{ord}(y, \mathcal{W}^n_{\alpha,k_y}) \leq \omega$.

For each $n, k \in \omega$, we put

$$\mathcal{H}^n_k = \{\pi_\alpha^{-1}(W^n_{\alpha,\xi,k}) \setminus D_{\alpha,n} : \alpha \in \Lambda \text{ and } \xi \in \Xi\} \cup \mathcal{G} \restriction (G \setminus E_n).$$

Then, as is shown in the proof of Theorem 2.6, it is verified that $\{\mathcal{H}^n_k\}$ is a sequence of open refinements of $\mathcal{G}$ which witnesses the $\delta\theta$-refinability of G. □

We also obtain a generalization of [8, Theorem 2] as follows.

Proposition 4.3. *Let $\{X_n, \pi^n_k, \omega\}$ be an inverse sequence and X its inverse limit. Let G be a countably metacompact open subspace of X. If each X_n is hereditarily $\delta\theta$-refinable, then G is $\delta\theta$-refinable.*

Proof: Let $\mathcal{G} = \{G_\xi : \xi \in \Xi\}$ be an open cover of G. For each $n \in \omega$ and $\xi \in \Xi$, let $U_{n,\xi}$ be the maximal open set in X_n such that $\pi_n^{-1}(U_{n,\xi}) \subset G_\xi$. Let $U_n = \bigcup\{U_{n,\xi} : \xi \in \Xi\}$ for each $n \in \omega$. Then $\{\pi_n^{-1}(U_n) : n \in \omega\}$ is a countable increasing open cover of G. Since G is countably metacompact, there is a countable increasing closed cover $\{F_n : n \in \omega\}$ of G such that $F_n \subset \pi_n^{-1}(U_n)$ for each $n \in \omega$. Take an $n \in \omega$. Since $\pi_n(F_n)$ is $\delta\theta$-refinable with $\pi_n(F_n) \subset U_n$, there is a sequence $\mathcal{W}_{n,k} = \{W_{n,\xi,k} \cap \pi_n(F_n) : \xi \in \Xi\}$ of open covers of $\pi_n(F_n)$ such that

(i) $W_{n,\xi,k}$ is open in X_n with $W_{n,\xi,k} \subset U_{n,\xi}$ for each $\xi \in \Xi$ and each $k \in \omega$,

(ii) for each $y \in \pi_n(F_n)$, one can find $k_y \in \omega$ with $\mathrm{ord}(y, \mathcal{W}_{n,k_y}) \leq \omega$.

Here, we put $\mathcal{H}_{n.k} = \{\pi_n^{-1}(W_{n,\xi,k}) : \xi \in \Xi\} \cup \mathcal{G} \restriction (G \setminus F_n)$ for each $n, k \in \omega$. Then it is easily verified that $\{\mathcal{H}_{n,k}\}$ is a sequence of open refinement of $\mathcal{G}$ which witnesses the $\delta\theta$-refinability of G. □

A space X is *hereditarily collectionwise δ-normal* if every subspace of X is collectionwise δ-normal.

Theorem 4.4. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-subparacompact open subspace of X. If each X_α is hereditarily collectionwise δ-normal, then G is collectionwise δ-normal.*

Proof: Let $\mathcal{C} = \{C_\xi : \xi \in \Xi\}$ be a discrete collection of subsets in G. For each $\alpha \in \Lambda$, let

$U_\alpha = \bigcup\{U : U$ is an open set in X_α such that $\pi_\alpha^{-1}(U)$ meets at most one member of $\mathcal{C}$ with $\pi_\alpha^{-1}(U) \subset G\}$.

Then $\{\pi_\alpha^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a directed open cover of G such that $\pi_\beta^{-1}(U_\beta) \subset \pi_\alpha^{-1}(U_\alpha)$ if $\beta \leq \alpha$. It follows from Lemma 2.4 that there is a closed cover $\{F_{\alpha,n} : \alpha \in \Lambda$ and $n \in \omega\}$ of G such that

(i) $F_{\alpha,n} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$ and $n \in \omega$,
(ii) $\{F_{\alpha,n} : \alpha \in \Lambda\}$ is discrete in G for each $n \in \omega$.

For each $\alpha \in \Lambda$ and each $n \in \omega$, let $D_{\alpha,n} = \bigcup\{F_{\beta,n} : \beta \in \Lambda$ with $\beta \neq \alpha\}$. By (ii), we may assume without loss of generality that

(iii) for each $x \in G$, there is an $\alpha_x \in \Lambda$ such that $\{n \in \omega : x \in F_{\alpha_x,n}\}$ is infinite.

Since U_α is collectionwise δ-normal and $\{\pi_\alpha(C_\xi) \cap U_\alpha : \xi \in \Xi\}$ is discrete in U_α, there is a collection $\{W_{\alpha,\xi,n} : \xi \in \Xi\}$ of open sets in X_α such that

(iv) $\pi_\alpha(C_\xi) \cap U_\alpha \subset W_{\alpha,\xi,n} \subset U_\alpha$,
(v) $W_{\alpha,\xi,n+1} \subset W_{\alpha,\xi,n}$ for each $\xi \in \Xi$ and $n \in \omega$,
(vi) $(\bigcap_{n\in\omega} W_{\alpha,\xi,n}) \cap (\bigcap_{n\in\omega} W_{\alpha,\xi',n}) = \emptyset$ if $\xi \neq \xi'$.

Let $H_{\xi,n} = \bigcup\{\pi_\alpha^{-1}(W_{\alpha,\xi,n}) \setminus D_{\alpha,n} : \alpha \in \Lambda\}$ for each $\xi \in \Xi$ and each $n \in \omega$. Then, in a way similar to the proof of Theorem 3.1, we can show that $\{\bigcap_{n\in\omega} H_{\xi,n} : \xi \in \Xi\}$ is a disjoint collection of G_δ-sets in G such that $C_\xi \subset \bigcap_{n\in\omega} H_{\xi,n}$ for each $\xi \in \Xi$. □

It is easily seen that a space X is hereditarily collectionwise δ-normal if and only if every open subspace of X is collectionwise δ-normal. Thus, Lemma 2.5, Proposition 4.1, and Theorem 4.4 immediately yield an affirmative answer to Question 2 (ii) (= [9, Question 6]):

Corollary 4.5. *Let $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ be an inverse system and X its inverse limit. Suppose that X is hereditarily λ-paracompact. If each X_α is hereditarily collectionwise δ-normal, then so is X.*

Recall that a space X is *hereditarily collectionwise subnormal* if every subspace of X is collectionwise subnormal. A space X is *hereditarily discretely θ-expandable* if every subspace of X is discretely θ-expandable.

Lemma 4.6. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-subparacompact open subspace of X. If each X_α is hereditarily discretely θ-expandable, then G is discretely θ-expandable.*

Proof: Let $\mathcal{C} = \{C_\xi : \xi \in \Xi\}$ and each U_α be the same ones as in the proof of Theorem 4.4. Since $\{\pi_\alpha^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is an open cover of G with cardinality $\leq \lambda$, it follows from Lemma 2.4 and the assumption of G that there is a sequence $\{O_{\alpha,n} : \alpha \in \Lambda\}, n \in \omega$, of open covers of G such that

(i) $O_{\alpha,n} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$ and each $n \in \omega$,
(ii) for each $x \in G$, one can find $n_x \in \omega$ with $\mathrm{ord}(x, \{O_{\alpha,n} : \alpha \in \Lambda\}) = 1$.

Take an $\alpha \in \Lambda$. Since $\{\pi_\alpha(C_\xi) \cap U_\alpha : \xi \in \Xi\}$ is discrete in U_α, it follows from the discrete θ-expandability of U_α that there is a sequence $\{W_{\alpha,\xi,k} : k \in \omega\}, k \in \omega$, of collections of open sets in U_α (in X_α) such that

(iii) $\pi_\alpha(C_\xi) \cap U_\alpha \subset W_{\alpha,\xi,k+1} \subset W_{\alpha,\xi,k}$ for each $\xi \in \Xi$ and each $k \in \omega$,
(iv) for each $y \in U_\alpha$, one can find $k_y \in \omega$ with $\mathrm{ord}(y, \{W_{\alpha,\xi,k_y} : \xi \in \Xi\}) < \omega$.

Here, we put $H^n_{\xi,k} = \bigcup\{\pi_\alpha^{-1}(W_{\alpha,\xi,k}) \cap O_{\alpha,n} : \alpha \in \Lambda\}$ for each $n, k \in \omega$ and each $\xi \in \Xi$. Then it is not difficult to verify that the sequence $\{H^n_{\xi,k} : \xi \in \Xi\}, n, k \in \omega$, of collections of open sets in X witnesses the discrete θ-expandability of G. □

Remark. The "λ-subparacompact" for G in Lemma 4.6 can be easily generalized by "λ-submetacompact," which is defined in the next section.

By Theorem 4.4 and lemmas 3.2 and 4.6, we have

Theorem 4.7. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-subparacompact open subspace of X. If each X_α is hereditarily collectionwise subnormal, then G is collectionwise subnormal.*

Since a space X is hereditarily collectionwise subnormal if and only if every open subspace of X is collectionwise subnormal, Lemma 2.5, Proposition 4.1, and Theorem 4.7 immediately yield an affirmative answer to Question 3 (ii) (= [9, Question 5]):

Corollary 4.8. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Suppose that X is hereditarily λ-paracompact. If each X_α is hereditarily collectionwise subnormal, then so is X.*

5. Hereditary subparacompactness and hereditary submetacompactness

Proposition 5.1. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-subparacompact open subspace of X. If each X_α is hereditarily subparacompact, then G is subparacompact.*

Proof: Let $\mathcal{G} = \{G_\xi : \xi \in \Xi\}$ be an open cover of G. For each $\alpha \in \Lambda$ and each $\xi \in \Xi$, let $U_{\alpha,\xi}$ be the maximal open set in X_α such that $\pi_\alpha^{-1}(U_{\alpha,\xi}) \subset G_\xi$. Moreover, for each $\alpha \in \Lambda$, let $U_\alpha = \bigcup\{U_{\alpha,\xi} : \xi \in \Xi\}$. Then, $\{\pi_\alpha^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a directed open cover of G such that $\pi_\beta^{-1}(U_\beta) \subset \pi_\alpha^{-1}(U_\alpha)$ if $\beta \leq \alpha$.

Since G is λ-subparacompact, it follows from Lemma 2.4 that there is a closed cover $\{F_{\alpha,n} : \alpha \in \Lambda \text{ and } n \in \omega\}$ of G such that

(i) $F_{\alpha,n} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$ and $n \in \omega$,
(ii) $\{F_{\alpha,n} : \alpha \in \Lambda\}$ is discrete in G for each $n \in \omega$.

Take an $\alpha \in \Lambda$ and an $n \in \omega$. Since $\pi_\alpha(F_{\alpha,n})$ is subparacompact and $\pi_\alpha(F_{\alpha,n}) \subset U_\alpha$, there is a closed cover $\{K^n_{\alpha,\xi,k} \cap \pi_\alpha(F_{\alpha,n}) : \xi \in \Xi \text{ and } k \in \omega\}$ of $\pi_\alpha(F_{\alpha,n})$ such that

(iii) each $K^n_{\alpha,\xi,k}$ is closed in X_α with $K^n_{\alpha,\xi,k} \cap \pi_\alpha(F_{\alpha,n}) \subset U_{\alpha,\xi}$,
(iv) $\{K^n_{\alpha,\xi,k} \cap \pi_\alpha(F_{\alpha,n}) : \xi \in \Xi\}$ is discrete in $\pi_\alpha(F_{\alpha,n})$ for each $k \in \omega$.

For each $n, k \in \omega$, let

$$\mathcal{C}^n_k = \{\pi_\alpha^{-1}(K^n_{\alpha,\xi,k}) \cap F_{\alpha,n} : \alpha \in \Lambda \text{ and } \xi \in \Xi\}.$$

Then, each member of $\mathcal{C}^n_k$ is closed in G and is contained in some member of $\mathcal{G}$. Moreover, it is verified that $\mathcal{C} = \bigcup\{\mathcal{C}^n_k : n, k \in \omega\}$ is a closed cover of G such that each $\mathcal{C}_{n,k}$ is discrete in G. Hence, $\mathcal{C}$ is a σ-discrete closed refinement of $\mathcal{G}$. □

For the submetacompact case, we need the following lemma.

Lemma 5.2 [11]**.** *There is a filter $\mathcal{F}$ on ω satisfying the following condition: For every submetacompact space X and every open cover*

$\mathcal{U}$ of X, there is a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$, $\{n \in \omega : \operatorname{ord}(x, \mathcal{V}_n) < \omega\} \in \mathcal{F}$.

A space X is *λ-submetacompact* if for every open cover $\mathcal{U}$ of X with cardinality $\leq \lambda$, there is a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$ one can find an $n_x \in \omega$ with $\operatorname{ord}(x, \mathcal{V}_{n_x}) < \omega$. By Lemma 2.4, λ-subparacompactness implies λ-submetacompactness.

Theorem 5.3. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-submetacompact open subspace of X. If each X_α is hereditarily submetacompact, then G is submetacompact.*

Proof: Let $\mathcal{G} = \{G_\xi : \xi \in \Xi\}$ be an open cover of G. For each $\alpha \in \Lambda$ and each $\xi \in \Xi$, let $U_{\alpha,\xi}$ and U_α be the same ones as described in the proof of Proposition 5.1. Since G is λ-submetacompact, there is a sequence $\mathcal{O}_n = \{O_{\alpha,n} : \alpha \in \Lambda\}, n \in \omega$, of open covers of G such that

(i) $O_{\alpha,n} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$ and each $n \in \omega$,
(ii) for each $x \in G$, one can find $n_x \in \omega$ with $\operatorname{ord}(x, \mathcal{O}_{n_x}) < \omega$.

Take an $\alpha \in \Lambda$ and an $n \in \omega$. Since $\pi_\alpha(O_{\alpha,n})$ is submetacompact and $\pi_\alpha(O_{\alpha,n}) \subset U_\alpha$, it follows from Lemma 5.2 that there is a sequence $\mathcal{V}^n_{\alpha,k} = \{V^n_{\alpha,\xi,k} \cap \pi_\alpha(O_{\alpha,n}) : \xi \in \Xi\}, k \in \omega$, of open covers of $\pi_\alpha(O_{\alpha,n})$ such that

(iii) each $V^n_{\alpha,\xi,k}$ is open in X_α with $V^n_{\alpha,\xi,k} \subset U_{\alpha,\xi}$,
(iv) $\{k \in \omega : \operatorname{ord}(y, \mathcal{V}^n_{\alpha,k}) < \omega\} \in \mathcal{F}$ for each $y \in \pi_\alpha(O_{\alpha,n})$, where $\mathcal{F}$ is a filter on ω described in Lemma 5.2.

For each $\xi \in \Xi$ and each $k \in \omega$, let $H^n_{\alpha,\xi,k} = \pi_\alpha^{-1}(V^n_{\alpha,\xi,k}) \cap O_{\alpha,n}$. Letting α and n range over Λ and ω, respectively, we put $\mathcal{H}^n_k = \{H^n_{\alpha,\xi,k} : \alpha \in \Lambda \text{ and } \xi \in \Xi\}$ for each $n, k \in \omega$. Take any $k, n \in \omega$. Pick an $x \in G$. Find $\delta \in \Lambda$ with $x \in O_{\delta,n}$. Since $\mathcal{V}^n_{\delta,k}$ covers $\pi_\delta(O_{\delta,n})$, there is $\mu \in \Xi$ with $\pi_\delta(x) \in V^n_{\delta,\mu,k} \cap \pi_\delta(O_{\delta,n})$. Then we have $x \in \pi_\delta^{-1}(V^n_{\delta,\mu,k}) \cap O_{\delta,n} = H^n_{\delta,\mu,k} \in \mathcal{H}^n_k$. So $\mathcal{H}^n_k$ covers G. By (i) and (iii), each $H \in \mathcal{H}^n_k$ is open in X (hence in G) and is contained in some member of $\mathcal{G}$.

Pick an $x \in G$ again. By (ii), find $m \in \omega$ with $\operatorname{ord}(x, \mathcal{O}_m) < \omega$. Let $\{\alpha \in \Lambda : x \in O_{\alpha,m}\} = \{\delta_0, \cdots, \delta_{k-1}\}$. For each $i < k$, by $\pi_{\delta_i}(x) \in \pi_{\delta_i}(O_{\delta_i,m})$ and (iv), we have $\{k \in \omega : \operatorname{ord}(\pi_{\delta_i}(x), \mathcal{V}^m_{\delta_i,k}) <$

$\omega\} \in \mathcal{F}$. Since $\mathcal{F}$ is a filter on ω, there is $\ell \in \omega$ such that $\operatorname{ord}(\pi_{\delta_i}(x), \mathcal{V}^m_{\delta_i,\ell}) < \omega$ for each $i < k$. So we can put

$$\{\xi \in \Xi : \pi_{\delta_i}(x) \in V^m_{\delta_i,\xi,\ell} \cap \pi_{\delta_i}(O_{\delta_i,m})\} = \{\mu_{is} : s < r_i\}$$

for each $i < k$. Take any $H^m_{\alpha,\xi,\ell} \in \mathcal{H}^m_\ell$ with $x \in H^m_{\alpha,\xi,\ell}$. Since $H^m_{\alpha,\xi,\ell} \subset O_{\alpha,m}$, it follows that $\alpha = \delta_j$ for some $j < k$. Since $\pi_\alpha(x) = \pi_{\delta_j}(x) \in \pi_\alpha(H^m_{\alpha,\xi,\ell}) \subset \pi_{\delta_j}\pi^{-1}_{\delta_j}(V^m_{\delta_j,\xi,\ell}) = V^m_{\delta_j,\xi,\ell}$ and $\pi_\alpha(x) = \pi_{\delta_j}(x) \in \pi_{\delta_j}(O_{\delta_j,m})$, it follows that $\xi = \mu_{jt}$ for some $t < r_j$. Hence, we obtain

$$\{H \in \mathcal{H}^m_\ell : x \in H\} \subset \{H^m_{\delta_i,\mu_{ij},\ell} : j < r_i \text{ and } i < k\}.$$

This implies that $\operatorname{ord}(x, \mathcal{H}^m_\ell) < \omega$. Thus, $\{\mathcal{H}^n_k\}$ is a sequence of open refinements of $\mathcal{G}$ which witnesses the submetacompactness of G. □

A space X is λ-*metacompact* if every open cover of X with cardinality $\leq \lambda$ has a point-finite open refinement. Clearly, λ-metacompactness implies λ-submetacompactness.

Proposition 5.4. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-metacompact open subspace of X. If each X_α is hereditarily metacompact, then G is metacompact.*

Proof: Let $\mathcal{G} = \{G_\xi : \xi \in \Xi\}, U_{\alpha,\xi}$ and U_α be the same ones as described in the proof of Proposition 5.1. Since G is λ-metacompact, there is a point-finite open cover $\{O_\alpha : \alpha \in \Lambda\}$ of G such that $O_\alpha \subset \pi^{-1}_\alpha(U_\alpha)$ for each $\alpha \in \Lambda$. Since $\pi_\alpha(O_\alpha)$ is metacompact and $\pi_\alpha(O_\alpha) \subset U_\alpha$, there is a point-finite open cover $\{V_{\alpha,\xi} \cap \pi_\alpha(O_\alpha) : \xi \in \Xi\}$ of $\pi_\alpha(O_\alpha)$ such that $V_{\alpha,\xi}$ is open in X_α with $V_{\alpha,\xi} \subset U_{\alpha,\xi}$ for each $\xi \in \Xi$. Then it is easily verified that

$$\mathcal{H} = \{\pi^{-1}_\alpha(V_{\alpha,\xi}) \cap O_\alpha : \alpha \in \Lambda \text{ and } \xi \in \Xi\}$$

is a point-finite open refinement of $\mathcal{G}$. □

A space X is λ-*weakly θ-refinable* if for every open cover $\mathcal{U}$ of X with cardinality $\leq \lambda$, there is an open refinement $\bigcup_{n\in\omega} \mathcal{V}_n$ of $\mathcal{U}$ such that for each $x \in X$ one can find an $n_x \in \omega$ with $\operatorname{ord}(x, \mathcal{V}_{n_x}) = 1$.

Proposition 5.5. *Let $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ be an inverse system and X its inverse limit. Let G be a λ-weakly θ-refinable open subspace of X. If each X_α is hereditarily weakly θ-refinable, then G is weakly θ-refinable.*

Proof: Let $\mathcal{G} = \{G_\xi : \xi \in \Xi\}, U_{\alpha,\xi}$ and U_α be the same ones as described in the proof of Proposition 5.1. Since G is λ-weakly θ-refinable, there is a sequence $\mathcal{O}_n = \{O_{\alpha,n} : \alpha \in \Lambda\}, n \in \omega$, of collections of open sets in G such that

(i) $O_{\alpha,n} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$ and each $n \in \omega$,

(ii) for each $x \in G$, one can find $n_x \in \omega$ with $\mathrm{ord}(x, \mathcal{O}_{n_x}) = 1$.

Since $\pi_\alpha(O_{\alpha,n})$ is weakly θ-refinable and $\pi_\alpha(O_{\alpha,n}) \subset U_\alpha$, there is a sequence $\mathcal{V}^n_{\alpha,k} = \{V^n_{\alpha,\xi,k} \cap \pi_\alpha(O_{\alpha,n}) : \xi \in \Xi\}, k \in \omega$, of collections of open sets in $\pi_\alpha(O_{\alpha,n})$ such that

(iii) each $V^n_{\alpha,\xi,k}$ is open in X_α with $V^n_{\alpha,\xi,k} \subset U_{\alpha,\xi}$,

(iv) for each $y \in \pi_\alpha(O_{\alpha,n})$, one can find $k_y \in \omega$ with $\mathrm{ord}(y, \mathcal{V}^n_{\alpha,k_y})$ $= 1$.

For each $n, k \in \omega$, let

$$\mathcal{H}_{n,k} = \{\pi_\alpha^{-1}(V^n_{\alpha,\xi,k}) \cap O_{\alpha,n} : \alpha \in \Lambda \text{ and } \xi \in \Xi\}.$$

Then it is verified that $\bigcup_{n,k\in\omega} \mathcal{H}_{n,k}$ is a desired open refinement of $\mathcal{G}$ which witnesses the weak θ-refinability of G. □

Remark. Finally, it should be noted that some kind of assumption of X such as λ-paracompactness seems to be always necessary to consider the covering properties of inverse limits. In fact, let ω^{ω_1} be the product of uncountably many copies of ω (= the countable infinite discrete space). Then ω^{ω_1} is the limit of an inverse system of discrete spaces with each projection being open. However, it is known that ω^{ω_1} is not countably paracompact and not subnormal. Moreover, it was proved in [5, 11.4] that ω^{ω_1} is not even weakly $\delta\theta$-refinable.

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