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FINITE GRAPHS X HAVE UNIQUE HYPERSPACES $C_n(X)$

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Abstract. Let X be a metric continuum and let $C_n(X)$ be the hyperspace of nonempty closed subsets of X with at most n components. In this paper we prove that:

Theorem. Let X be a finite graph, $n \in \mathbb{N}$, and Y a continuum such that $C_n(X)$ is homeomorphic to $C_n(Y)$. Then

(a) if $n = 1$ and X is neither an arc nor a simple closed curve, then X is homeomorphic to Y;
(b) if $n \geq 2$, then X is homeomorphic to Y.

1. Introduction

A *continuum* is a nonempty, nondegenerate, compact, connected metric space. We consider the following hyperspaces of a continuum X:

$$2^X = \{A \subset X : A \text{ is closed and nonempty}\},$$
$$C(X) = \{A \in 2^X : A \text{ is connected}\},$$

and if n is a positive integer,

$$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}.$$

All the hyperspaces are endowed with the Hausdorff metric H. The hyperspace $F_n(X)$ is also known as the *n-th symmetric product* of X.

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The continuum X is said to have *unique hyperspace* $C(X)$ (2^X, $C_n(X)$, and $F_n(X)$, respectively) provided that if Y is a continuum and $C(X)$ (2^X, $C_n(X)$, and $F_n(X)$, respectively) is homeomorphic to $C(Y)$ (2^Y, $C_n(Y)$, and $F_n(Y)$, respectively), then X is homeomorphic to Y. A class of continua $\mathcal{C}$ is defined to be *C-determined* provided that if $X, Y \in \mathcal{C}$, and $C(X)$ is homeomorphic to $C(Y)$, then X is homeomorphic to Y.

The continua X belonging to one of the following classes are known to have unique hyperspace $C(X)$:

(a) finite graphs different from an arc or a simple closed curve (R. Duda [8, 9.1]; see also [2, Theorem 1]);
(b) hereditarily indecomposable continua (S. B. Nadler [20, 0.60]; see also [2, Theorem 2]);
(c) indecomposable continua such that all their proper nondegenerate subcontinua are arcs (S. Macías [17]);
(d) metric compactifications of the ray $[0, \infty)$ with nondegenerate remainder (G. Acosta [2, Theorem 4]).

Hereditarily indecomposable continua X have unique hyperspace 2^X (Macías [18]).

Finite graphs X have unique hyperspace $F_n(X)$ [7, Theorem 21].

The author has shown that the classes of chainable continua and fans are not C-determined [12], [13]. C. Eberhart and Nadler proved that the class of smooth fans is C-determined [10, Corollary 3.3].

In [15] some results about uniqueness of the hyperspace $F_2(X)$, when X is a dendrite, are presented.

The following questions remain open.

Question ([20, Question 0.62]). Is the class of circle-like continua C-determined?

Question ([15, p. 77]). Do hereditarily indecomposable continua X have unique hyperspace $F_2(X)$?

In [14], the author proved that finite graphs X have unique hyperspace $C_2(X)$.

The main result of this paper is:

Theorem. *Let X be a finite graph, $n \in \mathbb{N}$, and Y a continuum such that $C_n(X)$ is homeomorphic to $C_n(Y)$.*

(a) *If $n = 1$ and X is neither an arc nor a simple closed curve, then X is homeomorphic to Y.*

(b) *If* $n \geq 2$*, then* X *is homeomorphic to* Y*.*

Therefore, for $n \geq 2$, finite graphs X have unique hyperspace $C_n(X)$. Since this result was shown for $n = 2$ in [14], here we only consider the case $n \geq 3$ (which has a very different proof than the case $n = 2$).

2. Conventions

A *finite (connected) graph* is a continuum which is a finite union of arcs such that each two of them meet at a finite set. If X is a finite graph, in X are defined *edges* and *vertices*. The vertices of X are the end points of the edges of X. We are interested in distinguishing the ramification points of the graph X from the rest of the points, so we assume that each vertex of a graph X, different from a simple closed curve, is either an end point of X or a ramification point of X. With this restriction the two end points of an edge of X may coincide and such an edge is a simple closed curve. These kinds of edges will be called *loops*. Thus, the edges of X are arcs or simple closed curves, and in X there are only three kind of edges: loops, edges that contain some end point of X, and edges joining ramification points. We assume that the metric d in X is the metric of arc length and each edge of X has length equal to one. The set of ramification points of X is denoted by $R(X)$. ($R(X) = \emptyset$ if X is either a simple closed curve or an arc.) Two different vertices p and q of X are said to be *adjacent* provided that there is an edge J of X such that p and q are the end points of J. A *simple n-od* Y is a finite graph which is the union of n arcs $J_1, ..., J_n$ such that there exists a point $p \in Y$ with the property $J_i \cap J_j = \{p\}$, if $i \neq j$, and p is an end point of each one of the arcs J_i. The point p is called the *core of* Y. A simple 3-od is called a *simple triod*.

The set of positive integers is denoted by $\mathbb{N}$. If $n \in \mathbb{N}$, an *n-cell* is a continuum homeomorphic to the product $[0, 1]^n$.

Given a finite graph X, a subset A of X, $p \in X$, and $\varepsilon > 0$, let $B(\varepsilon, p) = \{q \in X : d(p, q) < \varepsilon\}$ and $N(\varepsilon, A) = \{q \in X : \text{there exists } p \in A \text{ such that } d(p, q) < \varepsilon\}$. Given subsets $U_1, ..., U_m$ of X, let $\langle U_1, ..., U_m\rangle_n = \{A \in C_n(X) : A \subset U_1 \cup ... \cup U_m \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, ..., m\}\}$. It is known that the sets of the form

$\langle U_1, ..., U_m \rangle_n$, where the sets $U_1, ..., U_m$ are open in X, form a basis of the topology of $C_n(X)$ (see [20, Theorem 0.13]).

3. Results

We start proving several lemmas that are needed to show the main result of the paper, i.e., Theorem 3.8.

Lemma 3.1. *Let X be a finite graph, $n \in \mathbb{N}$, and $A \in C_n(X)$. If $A \cap R(X) \neq \emptyset$, then for each neighborhood $\mathcal{U}$ of A in $C_n(X)$, $\dim[\mathcal{U}] \geq 2n+1$.*

Proof: Let $\mathcal{U}$ be a neighborhood of A in $C_n(X)$. Let $p \in R(X) \cap A$. It is easy to prove that there exist pairwise disjoint subcontinua $A_1, ..., A_n$ of X such that $A_0 = A_1 \cup ... \cup A_n \in \mathcal{U}$, $p \in A_1$ and $A_1 \neq X$. Let $C_1, ..., C_n$ be pairwise disjoint subcontinua of X such that $A_i \subset \text{int}_X(C_i)$ for each $i \in \{1, ..., n\}$. Let $\mathcal{V} = \langle C_1, ..., C_n \rangle_n$. Then $C(C_1) \times ... \times C(C_n)$ is homeomorphic to $\mathcal{V}$ (by the homeomorphism $\varphi : C(C_1) \times ... \times C(C_n) \to \mathcal{V}$ given by $\varphi(B_1, ..., B_n) = B_1 \cup ... \cup B_n$). By [8, 5.2], for each $i \in \{1, ..., n\}$, there exists an m_i-cell $\mathcal{C}_i$ ($m_i \geq 2$) contained in $C(C_i)$ such that $A_i \in \mathcal{C}_i$ and, since $p \in R(X)$, $m_1 \geq 3$. Thus, $\mathcal{C} = \mathcal{C}_1 \times ... \times \mathcal{C}_n$ is an m-cell for some $m \geq 2n+1$ that contains A_0. So, $A_0 \in \mathcal{U} \cap \mathcal{C}$. Hence, $\mathcal{U} \cap \mathcal{C}$ contains an m-cell $\mathcal{C}_0$. Therefore, $\dim[\mathcal{U}] \geq 2n+1$. □

Given a finite graph X and $n \in \mathbb{N}$, let

$$\mathcal{L}_n(X) = \{A \in C_n(X) : A \text{ has a neighborhood in } C_n(X) \text{ that is a } 2n\text{-cell}\}.$$

If $n \geq 2$, let

$$\mathcal{M}_n(X) = \{A \in C_n(X) : A \notin C_{n-1}(X) \text{ and } A \cap R(X) = \emptyset\}.$$

We also define

$$\mathcal{M}_1(X) = \{A \in C(X) : A \cap R(X) = \emptyset\}.$$

Lemma 3.2. *Let X be a finite graph and $n \in \mathbb{N}$. Then $\mathcal{M}_n(X) \subset \mathcal{L}_n(X)$.*

Proof: Let $A \in C_n(X) - C_{n-1}(X)$ be such that $A \cap R(X) = \emptyset$. Let $A_1, ..., A_n$ be the different components of A. For each $i \in \{1, ..., n\}$, there exists an arc J_i in X such that A_i is contained in the interior of J_i in X and $J_i \cap R(X) = \emptyset$, and $J_1, ..., J_n$ are pairwise disjoint. It is easy to show that $\mathcal{U} = \langle J_1, ..., J_n \rangle_n$ is a neighborhood of A

in $C_n(X)$. Moreover, the Cartesian product $C(J_1) \times ... \times C(J_n)$ is homeomorphic to $\mathcal{U}$. Since each $C(J_i)$ is a 2-cell (see [8, p. 267]), we conclude that $\mathcal{U}$ is a $2n$-cell. Therefore, $A \in \mathcal{L}_n(X)$. □

Lemma 3.3. *Let X be a finite graph and $n \in \mathbb{N}$. Let $\mathcal{W} = \{A \in C_n(X) : A \cap R(X) = \emptyset\}$. Then* $\dim[\mathcal{W}] \leq 2n$.

Proof: We prove this lemma by induction. First, suppose that $n = 1$. If X is a simple closed curve, it is known that $C(X)$ is a 2-cell ([8, p. 267]). Thus $\dim[C(X)] = 2$. So we may assume that X is not a simple closed curve. Given $A \in \mathcal{W}$, there exists a subarc J of X such that $A \subset \text{int}_X(J)$. Hence, $C(J)$ is a neighborhood of A in $C(X)$. Since $C(J)$ is a 2-cell ([8, p. 267]), $\dim_A[C(X)] = 2$. Therefore, $\dim[\mathcal{W}] \leq 2$.

Now, suppose that $n \geq 2$ and that the lemma is true for $n-1$. Let $\mathcal{Z} = \{A \in C_{n-1}(X) \subset C_n(X) : A \cap R(X) = \emptyset\}$. Then $\mathcal{Z}$ is closed in $\mathcal{W}$ and $\dim[\mathcal{Z}] \leq 2n-2$. Observe that $\mathcal{W} = \mathcal{Z} \cup \mathcal{M}_n(X)$. By Lemma 3.2, $\dim[\mathcal{M}_n(X)] \leq 2n$. Since $\mathcal{M}_n(X)$ is open in $\mathcal{W}$, $\mathcal{W}$ is the union of a countable family of closed sets of dimension $\leq 2n$. Therefore, $\dim[\mathcal{W}] \leq 2n$ (see [11, Theorem III 2, Ch. III]). □

Lemma 3.4. *Let X be a finite graph and $n \geq 3$. Let $A \in C_{n-1}(X) - C(X)$ be such that $A \cap R(X) = \emptyset$. Then there exists a neighborhood $\mathcal{V}$ of A in $C_n(X)$ such that each closed neighborhood $\mathcal{W}$ of A in $C_n(X)$ satisfying $\mathcal{W} \subset \mathcal{V}$ can be separated by a closed set $\mathcal{S} \subset C_{n-1}(X)$ such that* $\dim[\mathcal{S}] \leq 2n-2$.

Proof: Take $A \in C_{n-1}(X) - C(X)$. Let $B_1, ..., B_m$ be the components of A, where $2 \leq m < n$. Choose pairwise disjoint subarcs $J_1, ..., J_m$ of X such that for each $i \in \{1, ..., m\}$, $A_i \subset \text{int}_X(J_i)$ and $(J_1 \cup ... \cup J_m) \cap R(X) = \emptyset$, and put $\mathcal{V} = \langle J_1, ..., J_m \rangle_n$.

Let $\mathcal{W}$ be a closed neighborhood of A in $C_n(X)$ such that $\mathcal{W} \subset \mathcal{V}$. Let $\mathcal{P} = \{B \in \mathcal{W} : B \cap J_1$ has at least two components$\}$ and $\mathcal{Q} = \{B \in \mathcal{W} : B \cap (J_2 \cup ... \cup J_m)$ has at least $n-1$ components$\}$. Clearly, $\mathcal{P}$ and $\mathcal{Q}$ are open in $\mathcal{W}$. If we choose a point $p \in J_1 - A_1$ such that p is close enough to A_1, then $A \cup \{p\} \in \mathcal{W}$ and $(A \cup \{p\}) \cap J_1$ has two components. So, $\mathcal{P}$ is nonempty. Now, choose different points $q_{m+1}, ..., q_n$ in $J_2 - A_2$ such that the points q_i are close enough to A_2. Then $A \cup \{q_{m+1}, ..., q_n\} \in \mathcal{Q}$. (Here is the step where we use the assumption $n \geq 3$ since $A \cup \{q_{m+1}, ..., q_n\}$ has at least three components, namely, A_1, A_2, and $\{q_n\}$.) Thus, $\mathcal{Q} \neq \emptyset$. Notice that an element $B \in \mathcal{P} \cap \mathcal{Q}$ contains at least $n+1$ components which is impossible since $\mathcal{P} \cap \mathcal{Q} \subset C_n(X)$. So, we have proved that $\mathcal{P} \cap \mathcal{Q} = \emptyset$. In conclusion, $\mathcal{P}$ and $\mathcal{Q}$ are open nonempty disjoint subsets of $\mathcal{W}$. Let $\mathcal{S} = \mathcal{W} - (\mathcal{P} \cup \mathcal{Q}) = \{B \in \mathcal{W} : B$ has exactly one component contained in J_1 and $B \cap (J_2 \cup ... \cup J_m)$ has at most $n-2$ components$\} \subset C_{n-1}(X)$. Since, for each $B \in \mathcal{S}$, $B \cap R(X) = \emptyset$, we have that $\mathcal{S} \subset \{B \in C_{n-1}(X) : B \cap R(X) = \emptyset\}$. By Lemma 3.3, $\dim[\mathcal{S}] \leq 2n-2$. Therefore, $\mathcal{W}$ can be separated by a closed subset $\mathcal{S}$ of $C_{n-1}(X)$ such that $\dim[\mathcal{S}] \leq 2n-2$. □

Lemma 3.5. *Let X be a finite graph and $n \geq 3$. Then $\mathcal{M}_n(X) = \mathcal{L}_n(X)$.*

Proof: By Lemma 3.2, we only need to prove that $\mathcal{L}_n(X) \subset \mathcal{M}_n(X)$. Take $A \in \mathcal{L}_n(X)$. Then there exists a neighborhood $\mathcal{U}$ of A in $C_n(X)$ such that $\mathcal{U}$ is a $2n$-cell. By Lemma 3.1, $A \cap R(X) = \emptyset$. We need to prove that $A \notin C_{n-1}(X)$. Suppose to the contrary that $A \in C_{n-1}(X)$. It is easy to show that there exists an element $A_0 \in C_{n-1}(X) - C(X)$ such that $A_0 \in \text{int}_{C_n(X)}(\mathcal{U})$ and $A_0 \cap R(X) = \emptyset$. Let $\mathcal{V}$ be a neighborhood of A_0 in $C_n(X)$ as in Lemma 3.4. Then there exists a closed neighborhood $\mathcal{W}$ of A_0 in $C_n(X)$ such that $\mathcal{W} \subset \mathcal{U} \cap \mathcal{V}$ and $\mathcal{W}$ is a $2n$-cell. By the choice of $\mathcal{V}$, $\mathcal{W}$ can be separated by a closed subset $\mathcal{S}$ of $\mathcal{W}$ such that $\dim[\mathcal{S}] \leq n-2$. This contradicts [11, Corollary to Theorem IV.4] and completes the proof of the lemma. □

Given a finite graph X and an integer $n \geq 2$, let $\mathcal{D}_n(X) = \{A \in C_n(X) : A \notin \mathcal{L}_n(X)$ and there is a local basis of open neighborhoods $\mathcal{B}$ of $C_n(X)$ at A such that for each $\mathcal{U} \in \mathcal{B}$, $\dim[\mathcal{U}] \leq 2n$ and $\mathcal{U} \cap \mathcal{L}_n(X)$ is arcwise connected$\}$.

Lemma 3.6. *Let X be a finite graph and $n \geq 3$. Then $\mathcal{D}_n(X) = \{A \in C_n(X) : A$ is connected and $A \cap R(X) = \emptyset\}$.*

Proof: Let $A \in \mathcal{D}_n(X)$. By Lemma 3.1, $A \cap R(X) = \emptyset$. We need to prove that A is connected. Suppose to the contrary that $A \notin C(X)$. By Lemma 3.5, $A \notin \mathcal{M}_n(X)$, so $A \in C_{n-1}(X) - C(X)$. Thus, we can apply Lemma 3.4 to A. Let $\mathcal{V}$ be a neighborhood of A in $C_n(X)$ as in Lemma 3.4. Since $A \in \mathcal{D}_n(X)$, there exists an open neighborhood $\mathcal{U}$ of A in $C_n(X)$ such that $\dim[\mathcal{U}] \leq 2n$, $\mathcal{U} \cap \mathcal{L}_n(X)$ is arcwise connected, each element $B \in \mathcal{U}$ has the property that $B \cap R(X) = \emptyset$ and $\text{cl}_{C_n(X)}(\mathcal{U}) \subset \mathcal{V}$. By the choice of $\mathcal{V}$, $\text{cl}_{C_n(X)}(\mathcal{U})$ can be separated by a closed set $\mathcal{S}$ contained in $C_{n-1}(X)$. Then there exist two nonempty disjoint open subsets $\mathcal{H}$ and $\mathcal{K}$ of $\mathrm{cl}_{C_n(X)}(\mathcal{U})$ such that $\mathrm{cl}_{C_n(X)}(\mathcal{U}) - \mathcal{S} = \mathcal{H} \cup \mathcal{K}$. Then $\mathcal{H} \cap \mathcal{U}$ and $\mathcal{K} \cap \mathcal{U}$ are nonempty open subsets of $C_n(X)$. Since $C_n(X) - C_{n-1}(X)$ is dense in $C_n(X)$, there exist elements $B \in \mathcal{H} \cap \mathcal{U} \cap (C_n(X) - C_{n-1}(X))$ and $C \in \mathcal{K} \cap \mathcal{U} \cap (C_n(X) - C_{n-1}(X))$. Then $B, C \in \mathcal{M}_n(X) \cap \mathcal{U} = \mathcal{L}_n(X) \cap \mathcal{U}$. Since $\mathcal{U} \cap \mathcal{L}_n(X)$ is arcwise connected, there exists a subarc α of $\mathcal{U} \cap \mathcal{L}_n(X)$ joining B and C. By the choice of $\mathcal{S}$, there exists an element $D \in \alpha \cap \mathcal{S}$. Since $D \in \mathcal{L}_n(X) = \mathcal{M}_n(X)$, $D \notin C_{n-1}(X)$. This is impossible since $D \in \mathcal{S} \subset C_{n-1}(X)$. This contradiction proves that A is connected.

To prove the opposite inclusion, take $A \in C_n(X)$ such that A is connected and $A \cap R(X) = \emptyset$. Since $A \in C(X) \subset C_{n-1}(X)$, $A \notin \mathcal{M}_n(X) = \mathcal{L}_n(X)$. Since $A \cap R(X) = \emptyset$ and A is connected, there exists an open connected subset U of X such that $A \subset U$ and $U \cap R(X) = \emptyset$. Let $\mathcal{B} = \{B_{C_n(X)}(\varepsilon, A) \subset C_n(X) : N(\varepsilon, A) \subset U\}$ and take $\mathcal{U} = B_{C_n(X)}(\varepsilon, A) \in \mathcal{B}$. By Lemma 3.3, $\dim[\mathcal{U}] \leq 2n$. Notice that $\mathcal{U} \cap \mathcal{L}_n(X) = \mathcal{U} \cap \mathcal{M}_n(X) = \mathcal{U} \cap (C_n(X) - C_{n-1}(X))$. Given $B \in \mathcal{U} \cap \mathcal{L}_n(X)$, we can enlarge B, by an arc β in $\mathcal{U} \cap \mathcal{L}_n(X)$, in such a way that we arrive to an element C in $\mathcal{U} \cap \mathcal{L}_n(X)$ with the property that each component of $N(\varepsilon, A) - C$ has the same length and this common length η is less that $\frac{\varepsilon}{3n}$. Now, we can connect C, by an arc γ in $\mathcal{U} \cap \mathcal{L}_n(X)$, to an element D such that each component of $N(\varepsilon, A) - D$ has the same length η and they are evenly distributed in $N(\varepsilon, A)$. Finally, enlarge all the components of $N(\varepsilon, A) - D$ in such a way that D is connected, by an arc σ in $\mathcal{U} \cap \mathcal{L}_n(X)$, to an element E with the property that each component of $N(\varepsilon, A) - E$ has the same length $\frac{\varepsilon}{3n}$ and they are evenly distributed in $N(\varepsilon, A) - E$. Since E does not depend on B, we have shown that $\mathcal{U} \cap \mathcal{L}_n(X)$ is arcwise connected. This completes the proof of the lemma. □

Lemma 3.7. *Let X be a finite graph and $n \in \mathbb{N}$. Then* $\dim[C_n(X)]$ *is finite.*

Proof: We are going to prove, inductively, that $\dim[C_n(X)] \leq n \dim[C(X)]$ and this will be enough since $\dim[C(X)]$ is finite (see [8, 1.1]).

For $n = 1$, there is nothing to prove. Suppose now that $n \geq 2$ and the claim holds for $n-1$. Then $\dim[C_{n-1}(X)] \leq (n-1) \dim[C(X)]$.

Since $C_n(X) - C_{n-1}(X)$ is open in $C_n(X)$, by [11, Theorem III 2] it is enough to show that for each $B \in C_n(X) - C_{n-1}(X)$, $\dim_B[C_n(X)] \leq n \dim[C(X)]$. Take $B \in C_n(X) - C_{n-1}(X)$. Let $B_1, ..., B_n$ be the different components of B. Take pairwise disjoint subcontinua $A_1, ..., A_n$ of X such that $B_i \subset \text{int}_X(A_i)$ for each $i \in \{1, ..., n\}$. Since $\langle A_1, ..., A_n \rangle_n$ is a closed neighborhood of B in $C_n(X)$ and $C(A_1) \times ... \times C(A_n)$ is homeomorphic to $\langle A_1, ..., A_n \rangle_n$ (by the homeomorphism $(D_1, ..., D_n) \to D_1 \cup ... \cup D_n$), we conclude that $\dim \langle A_1, ..., A_n \rangle_n = \dim[C(A_1) \times ... \times C(A_n)] \leq \dim[C(A_1)] + ... + \dim[C(A_n)] \leq n \dim[C(X)]$. The lemma is proved. □

Theorem 3.8. *Let X be a finite graph, $n \in \mathbb{N}$, and Y a continuum such that $C_n(X)$ is homeomorphic to $C_n(Y)$.*

(a) *If $n = 1$ and X is neither an arc nor a simple closed curve, then X is homeomorphic to Y.*
(b) *If $n \geq 2$, then X is homeomorphic to Y.*

Proof: (a) was proved by Duda (see [8, 9.1]). For $n = 2$ this result was proved in [14, Theorem 4.1]. Thus, we can assume that $n \geq 3$. By Lemma 3.7, $C_n(X)$, $C_n(Y)$, and then $C(Y)$ are finite dimensional. Since X is locally connected, by [19, Theorem 3.2], $C_n(X)$, $C_n(Y)$, Y, and $C(Y)$ are locally connected. Thus, $C(Y)$ is locally connected and finite dimensional. By [8, 1.1], Y is a finite graph. Hence, we can define $\mathcal{L}_n(Y)$ and $\mathcal{D}_n(Y)$.

Let $h : C_n(X) \to C_n(Y)$ be a homeomorphism. Clearly, $h(\mathcal{L}_n(X)) = \mathcal{L}_n(Y)$ and $h(\mathcal{D}_n(X)) = h(\mathcal{D}_n(Y))$.

According to Lemma 3.6, $\mathcal{D}_n(X) = \{A \in C_n(X) : A$ is connected and $A \cap R(X) = \emptyset\}$. Let $\mathcal{E}_n(X) = \{A \in \mathcal{D}_n(X) :$ there exists a neighborhood $\mathcal{U}$ of A in $\mathcal{D}_n(X)$ and a homeomorphism $f : \mathcal{U} \to [0,1]^2$ with the property $f(A) = (0,0)\}$. It is easy to show that $\mathcal{E}_n(X) = \{\{p\} \in C_n(X) : p \in X - R(X)\} \cup \{E \in C(X) : E$ is a subarc of $X - R(X)$ containing an end point of $X\}$. Thus, $\text{cl}_{C_n(X)}(\mathcal{E}_n(X))$ is homeomorphic to X (and $\text{cl}_{C_n(X)}(\mathcal{E}_n(Y))$ is homeomorphic to Y). Since the definition of $\mathcal{E}_n(X)$ is topological and $h(\mathcal{D}_n(X)) = \mathcal{D}_n(Y)$, we have that $h((\mathcal{E}_n(X)) = \mathcal{E}_n(Y)$ and $h(\text{cl}_{C_n(X)}(\mathcal{E}_n(X))) = \text{cl}_{C_n(X)}(\mathcal{E}_n(Y))$. Therefore, the following continua are homeomorphic: X, $\text{cl}_{C_n(X)}(\mathcal{E}_n(X))$, $\text{cl}_{C_n(X)}(\mathcal{E}_n(Y))$, and Y. Hence, X is homeomorphic to Y. □

Corollary 3.9. *Let X be a finite graph, $n, m \in \mathbb{N}$, and Y a continuum such that $C_n(X)$ is homeomorphic to $C_m(Y)$.*

(a) *If* $n = 1$ *and* X *is neither an arc nor a simple closed curve, then* X *is homeomorphic to* Y.
(b) *If* $n \geq 2$, *then* X *is homeomorphic to* Y.

Proof: Proceeding as in Theorem 3.8, we obtain that Y is a finite graph. According to [16, Theorem 12], $n = m$, so Corollary 3.9 is immediate. □

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