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Department of Mathematics & Statistics
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FAMILIES OF INVERSE LIMITS ON $[0,1]$

W. T. INGRAM

Abstract. In this paper we prove that certain rays to continua with the Property of Kelley also have the Property of Kelley. Then, we survey recent results on inverse limits with a single bonding mapping chosen from one of the following families of mappings: Markov maps determined by permutations, unimodal maps consisting of two linear pieces (including the tent family), and the logistic family. The focus of the survey is on indecomposability, end points, arc continua, and the Property of Kelley.

1. Introduction

Parameterized families of mappings have attracted the attention of dynamicists over the years. Such families include the *logistic family*, $\{f_\lambda \mid f_\lambda(x) = 4\lambda x(1-x), \text{ where } 0 \leq x \leq 1 \text{ and } 0 \leq \lambda \leq 1\}$, and the *tent family*, $\{T_\lambda \mid T_\lambda(x) = 2\lambda x \text{ if } 0 \leq x \leq 1/2 \text{ and } T_\lambda(x) = 2\lambda(1-x) \text{ if } 1/2 \leq x \leq 1 \text{ with } 0 \leq \lambda \leq 1\}$. Both of these are families of *unimodal maps*. The tent family and certain other unimodal families of piecewise linear unimodal maps inspired the author to investigate a *two-parameter family of unimodal maps*, $\{g_{bc} \mid g_{bc}(x) = \frac{1-b}{c}x + b \text{ if } 0 \leq x \leq c \text{ and } g_{bc}(x) = \frac{x-1}{c-1} \text{ if } c \leq x \leq 1 \text{ with } 0 \leq b \leq 1 \text{ and } 0 < c < 1\}$.

Markov maps have also proved to be valuable in the study of inverse limits on intervals and the author has identified a family of

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Markov maps of $[0,1]$ based on permutations. A permutation on n elements is a function from $\{1,2,\ldots,n\}$ onto itself. We denote the set of all permutations on n elements by S_n. If σ is in S_n, we associate a mapping f_σ of $[0,1]$ onto itself with σ in a natural way. For $1 \leq i \leq n$, let $a_i = \frac{i-1}{n-1}$. Then, let f_σ be the linear extension of the map which takes a_i to $a_{\sigma(i)}$. We shall denote the set of all maps so defined by $\mathcal{S}_n$. The family of maps determined by permutations will be called the *family of permutation maps*.

If f is a mapping chosen from one of the families identifed above, one can ask about the nature of the continuum that results from the inverse limit on $[0,1]$ using f as a single bonding mapping. In the final four sections of this paper, we survey results concerning such inverse limits related to the following properties: being an indecomposable continuum, containing an indecomposable continuum, having only n end points for some positive integer n, having every proper subcontinuum an arc, and having the Property of Kelley. Section 2 is devoted to a proof that continua which are the closure of certain rays with remainder having the Property of Kelley also have the Property of Kelley.

Of course, a complete understanding of the topology of the subcontinua of inverse limits using a single bonding map requires one to consider inverse limits using non-constant sequences of maps. An investigation of inverse limits on $[0,1]$ using non-constant sequences of bonding maps is beyond the scope of this paper. Some results about inverse limits using non-constant sequences of bonding maps chosen from the two-parameter family of maps are found in Brian Raines's Master's Thesis [13].

By a *continuum*, we mean a compact, connected subset of a metric space. By a *mapping*, we mean a continuous function. If M is a continuum, a subcontinuum H of M is said to be *irreducible about* a closed subset E of M if H contains E but no proper subcontinuum of H contains E. If M is a continuum, and E is a subset of M, we denote the diameter of E by $\text{diam}(E)$. A continuum M with metric d is said to have the *Property of Kelley* (or *Property* κ) provided, if $\varepsilon > 0$, there is a positive number δ such that if p and q are points of M such that $d(p,q) < \delta$ and H is a subcontinuum of M containing p, then there is a subcontinuum K of M containing q such that $\mathcal{H}(H,K) < \varepsilon$, where $\mathcal{H}$ denotes the Hausdorff metric. For more on the Hausdorff metric or the Property of Kelley, the reader is referred to [5]. If $X_1, X_2, X_3, \cdots$ is a sequence of topological spaces and $f_1, f_2, f_3, \cdots$ is a sequence of mappings such that, for each positive integer i, $f_i : X_{i+1} \to X_i$, then by the *inverse limit* of the inverse sequence $\{X_i, f_i\}$, we mean the subset of $\prod_{i>0} X_i$ to which the point x belongs if and only if $f_i(x_{i+1}) = x_i$ for $i = 1, 2, 3, \cdots$. The inverse limit of the inverse limit sequence $\{X_i, f_i\}$ is denoted $\varprojlim\{X_i, f_i\}$. In case we have a single factor space, X, and a single bonding map, f, we denote the inverse limit by $\varprojlim\{X, f\}$. It is well known [6] that if each factor space, X_i, is a continuum, the inverse limit is a continuum. We shall denote the Hausdorff metric on the space of subcontinua of the factor space X_i by $\mathcal{H}_i$ and the Hausdorff metric on the subcontinua of the inverse limit by $\mathcal{H}$. If f is a mapping of a space X into itself, we denote the n-fold composition of f with itself by f^n, and, for convenience, we let f^0 denote the identity map on X. If i is a positive integer, we denote the projection of the inverse limit into the ith factor space by π_i. If K is a subcontinuum of the inverse limit, we denote $\pi_i[K]$ by K_i. If $f : X \to Y$ is a mapping and $f[X] = Y$, we write $f : X \twoheadrightarrow Y$. A map of continua is called *monotone* provided each point inverse is connected. A map f of the interval $[0, 1]$ into itself is called *unimodal* provided it is not monotone and there is a point c of $(0, 1)$ such that $f|[0, c]$ and $f|[c, 1]$ are monotone. A map of an interval into itself is called *Markov* if there is a finite invariant set $\mathcal{A}$ containing the end points of the interval such that if p and q are consecutive members of $\mathcal{A}$ then the restriction of the map to $[p, q]$ is monotone. We shall refer to the set $\mathcal{A}$ as a Markov partition. For permutation maps, we shall refer to the partition $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ where $a_i = \frac{i-1}{n-1}$ as *the Markov partition associated with n.*

2. A Ray Theorem

In this section, we prove a theorem concerning continua which are closures of rays limiting to continua with the Property of Kelley. One way to get a ray limiting to a continuum by using inverse limits is through Bennett's Theorem. For the reader's convenience we state Bennett's Theorem.

Theorem 2.1. (H. Bennett, see [7, Theorem 2.16]) *Suppose f is a mapping of the interval $[a, b]$ onto itself and d is a number between a and b such that (1) $f[d, b]$ is a subset of $[d, b]$, (2) $f|[a, d]$ is* *monotone, and (3) there is a positive integer j such that $f^j[a,d] = [a,b]$. Then $\varprojlim\{[a,b],f\}$ is the union of a topological ray R and a continuum $K = \varprojlim\{[d,b],f|[d,b]\}$ such that $\overline{R} - R = K$.*

Bennett's Theorem applies to the maps in the logistic family for $\frac{1}{4} < \lambda < 1$ and the tent family for $1/2 < \lambda < 1$. We call the continuum K in Bennett's Theorem the *core* of the inverse limit. Specifically, for these two families, the core is the inverse limit on $[f(\lambda), \lambda]$ with the restriction of the bonding map to this interval as a single bonding map. In general, a continuum which is the closure of a ray limiting to a core continuum with the Property of Kelley does not have to have the Property of Kelley. This may be seen from the following example.

Example 1. Take the inverse limit on $[0,1]$ using a single bonding map $f : [0,1] \twoheadrightarrow [0,1]$ given by

$$f(x) = \begin{cases} 4x & \text{if } 0 \le x \le \frac{1}{4}, \\ \frac{3}{2} - 2x & \text{if } \frac{1}{4} \le x \le \frac{3}{8}, \\ 2x & \text{if } \frac{3}{8} \le x \le \frac{1}{2}, \\ \frac{3}{2} - x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

In this example, we see from Bennett's Theorem using $a = 0, b = 1$ and $d = \frac{1}{4}$ that we get a ray limiting to an arc. (Since no point of $[\frac{1}{4}, \frac{1}{2})$ has an inverse in $[\frac{1}{4}, 1]$, the limit arc is actually the inverse limit on $[\frac{1}{2}, 1]$ using $f|[\frac{1}{2}, 1]$.) Unlike the regular $\sin(\frac{1}{x})$-curve which has the Property of Kelley, this continuum does not have the Property of Kelley. In fact, this continuum is homeomorphic to the continuum indicated in Figure 1. See also [5, Figure 27, page 168]. Using the choices of a, d, and b mentioned above, this example satisfies all the hypotheses of Theorem 2.3 except the condition that f be surjective on $[d, b]$.

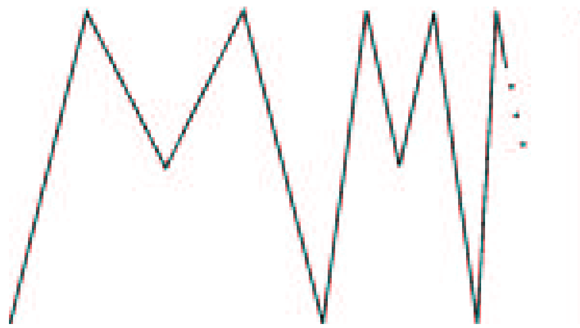

Figure 1

When the author was writing the papers [10] and [11], he proved a version of Lemma 2.2 and Theorem 2.3 appropriate for those papers. He thought at the time that Theorem 2.3 should hold, but did not see how to prove it. A weaker version of Theorem 2.3 specific to permutation maps appears in [11, Theorem 1].

Lemma 2.2. *If $M = \varprojlim\{X_i, f_i\}$ where, for each i, X_i is a continuum, and $\varepsilon > 0$, there exist a positive integer N and a positive number δ such that if H and K are subcontinua of M such that $\mathcal{H}_N(H_N, K_N) < \delta$, then $\mathcal{H}(H, K) < \varepsilon$.*

Proof: Suppose $\varepsilon > 0$. Using $\varepsilon/2$ in Theorem 1.15 of [7], there are a positive integer N and a positive number δ such that if C is a subset of X_N and the diameter of C is less than δ then $\text{diam}(\pi_N^{-1}(C)) < \varepsilon/2$. If H and K are subcontinua of M such that $\mathcal{H}_N(H_N, K_N) < \delta$ then H_N is a subset of the δ neighborhood of K_N and K_N is a subset of the delta neighborhood of H_N; thus, H is a subset of the $\varepsilon/2$ neighborhood of K and K is a subset of the $\varepsilon/2$ neighborhood of H. It follows that $\mathcal{H}(H, K) \leq \varepsilon/2 < \varepsilon$.

Theorem 2.3. *Suppose $f : [a, b] \twoheadrightarrow [a, b]$ is a mapping and d is a number between a and b such that (1) $f[d, b] = [d, b]$, (2) $f|[a, d]$ is monotone, (3) there is a positive integer m such that $f^m[a, d] = [a, b]$, and (4) $C = \varprojlim\{[d, b], f|[d, b]\}$ has the Property of Kelley. Then, $M = \varprojlim\{[a, b], f\}$ has the Property of Kelley.*

Proof: Let ε be a positive number. As a matter of convenience of notation, we denote the interval $[a, b]$ by X_i for $i = 1, 2, 3, \ldots$ in order to be able to keep track of the factor spaces when we use Lemma 2.2. Since C has the Property of Kelley, there is a positive number δ_1 such that if A is a subcontinuum of C, r is a point of A and s is a point of C such that $d(r,s) < \delta_1$, then there is a subcontinuum B of C such that s is in B and $\mathcal{H}(A,B) < \varepsilon/3$.

Let ε_1 be the smaller of $\varepsilon/3$ and δ_1. Using ε_1 in Lemma 2.2, there are a positive number $\delta_2 < \delta_1$ and a positive integer N such that if H and K are subcontinua of M such that if $\mathcal{H}_N(H_N, K_N) < \delta_2$ then $\mathcal{H}(H,K) < \varepsilon_1$.

Since each f^i, $1 \leq i \leq m$, is uniformly continuous, there exists a positive number δ_3 such that if j is a positive integer, $0 \leq j \leq m$, and x and y are points of X_{N+j} such that $|x - y| < \delta_3$, then $|f^i(x) - f^i(y)| < \delta_2/2$.

Let $\delta = \delta_3/2^{N+m}$ and suppose H is a subcontinuum of M, p is a point of H and q is a point of M such that $d(p,q) < \delta$. Let J be the interval irreducible about $H_{N+m} \cup \{q_{N+m}\}$. Since $d(p,q) < \delta$, $|p_{N+m} - q_{N+m}| < \delta_3$, so $\mathcal{H}(H_N, f^m[J]) \leq \delta_2/2 < \delta_2$.

We consider two cases: (I) $q_{N+m+1} < d$ and (II) $q_{N+m+1} \geq d$.

In case (I), since $f^m[a,d] = [a,b]$ and f^m is weakly confluent, there is an interval $[x,y]$ lying in $[a,d]$ such that $f^m[x,y] = J$. There is a point t of $[x,y]$ such that $f^m(t) = q_{N+m}$. The point q_{N+2m} is a point of $[a,d]$ since q_{N+m+1} is in $[a,d]$, so q_{N+2m} and t are points of $(f|[a,d])^{-m}(f^m(t))$. Let $K_{N+2m} = [x,y] \cup (f|[a,d])^{-m}(f^m(t))$. Then, K_{N+2m} is an interval containing q_{N+2m} and lying in $[a,d]$. Let $K_{N+2m+1} = (f|[a,d])^{-1}(K_{N+2m}), K_{N+2m+2} = (f|[a,d])^{-1}$ $(K_{N+2m+1}), \ldots$ and if $i < N + 2m$, let $K_i = f^{N+2m-i}[K_{N+2m}]$. It follows that, $K = \varprojlim\{K_i, f|K_{i+1}\}$ is a continuum containing q such that $K_N = f^m[J]$. Since $\mathcal{H}_N(H_N, K_N) < \delta_2$, $\mathcal{H}(H,K) < \varepsilon/3$.

In case (II), where $q_{N+m+1} \geq d$, there are two possibilities: (i) J is a subset of $[d,b]$, and (ii) J contains a point of $[a,d)$.

In case (II)(i), H_{N+m} is a subset of $[d,b]$ so there is a subcontinuum A of C such that $A_i = H_i$ for each positive integer i for which H_i is a subset of $[d,b]$. Further, there are points r and s of C such that $r_i = p_i$ for each i which p_i is in $[d,b]$ and $s_i = q_i$ for each i for which q_i is in $[d,b]$. Since $|r_{N+m} - s_{N+m}| < \delta_3$, we have $|r_N - s_N| < \delta_2$, so by the choice of δ_2, $d(r,s) < \delta_1$. Thus, there exists a subcontinuum B of C such that s is a point of B and $\mathcal{H}(A,B) < \varepsilon/3$. Since $H_{N+m} = A_{N+m}$, we have $H_N = A_N$ and, consequently, $\mathcal{H}(H,A) < \varepsilon/3$. Therefore, $\mathcal{H}(H,B) < 2\varepsilon/3$. If q is a point of B, then let $K = B$. If q is not in B, there is an integer $i > N + m + 1$ such that $q_i < d$. Denote by $j + 1$ the least integer $i > N + m + 1$ such that $q_i < d$. We now proceed to construct K in the manner of case (I). Since $f^m[a,d] = [a,b]$, there is an interval $[x,y]$ lying in $[a,d]$ such that $f^m[x,y] = B_j$. Let $K_{j+m} = [x,y] \cup f^{-m}(q_j), K_{j+m+1} = (f|[a,d])^{-1}(K_{j+m}), \ldots$ and if $i < j+m, K_i = f^{j+m-i}[K_{j+m}]$. Let $K = \varprojlim\{K_i, f|K_{i+1}\}$, and note that q is in K. Since $K_j = B_j$, $K_N = B_N$, so $\mathcal{H}(B,K) < \varepsilon/3$. It follows that $\mathcal{H}(H,K) < \varepsilon$.

In case (II)(ii) where J contains a point of $[a,d)$, if for some $i \leq m$, $f^i[J]$ is a subset of $[d,b]$ we may construct K as we did in case (II)(i) using $f^i[J]$ in place of J. On the other hand, if $f^i[J]$ contains a point of $[a,d)$ for each $i \leq m$, then $f^m[J]$ contains the interval $[a,d]$. To see this, observe that there is a point w of $[a,d]$ such that $f^m(w) = b$. Since f is monotone on $[a,d]$, there is a first term of $\{w, f(w), f^2(w), \ldots, f^m(w)\}$ greater than d. Let z be that first term and note that z is in $(d, f(d)]$ and $z = f^r(w)$ for some $r \geq 1$. Then, z is in $f[J]$. Since $f^l(z) = b$ for some $l \leq m-1$, we have $f^{l+1}[J]$ contains $[d,b]$, so $f^m[J]$ contains $[d,b]$. Let $K_N = f^m[J], K_{N+1} = f^{-1}[K_N], \ldots$ and if $i < N$, let $K_i = f^{N-i}[K_N]$ and $K = \varprojlim\{K_i, f|K_{i+1}\}$. Since q is in K and $\mathcal{H}_N(H_N, f^m[J]) < \delta_2$ and $K_N = f^m[J]$, we have $\mathcal{H}(H,K) < \varepsilon$.

3. Indecomposability

If f is in the logistic family, the inverse limit on $[0,1]$ using f as a single bonding map contains an indecomposable continuum if and only if $\lambda > \lambda_c$ where $\lambda_c \approx 0.89249\ldots$ is the Feigenbaum limit [1]. In the two-parameter family, indecomposability occurs in the inverse limit only for $b < \frac{1}{2-c}$, and the inverse limit contains an indecomposable continuum only for $b < c^2 - c + 1$ [6]. In the tent family, the core contains an indecomposable continuum for $\lambda > 1/2$. For the permutation family we have the following theorem from [8, Theorem 11].

Theorem 3.1. *Suppose n is a positive integer, $n \geq 3$, σ is a permutation on $\{1,2,\ldots n\}$, and $f = f_\sigma$. Suppose further, 1 is in the orbit of 0 under f, and $k < n$ is a positive integer such that $f^k(0) = \frac{1}{n-1}$. If n and k are relatively prime, then $\varprojlim\{[0,1], f\}$ is an indecomposable continuum.*

The problem of deciding whether a continuum contains an indecomposable subcontinuum is the problem of deciding when a continuum is hereditarily decomposable. David Ryden [15, Theorem 8] has addressed this question nicely for inverse limits on intervals using Markov maps as bonding maps.

More can be found on indecomposability arising in inverse limits on $[0, 1]$ in a survey article [6], as well as in its references.

4. End Points

A point p of a continuum M is called an *end point* of M provided it is true that if H and K are subcontinua of M containing p then H is a subset of K or K is a subset of H. Counting end points can be a way of distinguishing between continua. In general for maps of the interval, we have the following theorem from [8, Theorem 14].

Theorem 4.1. *Suppose* $f : [0, 1] \to [0, 1]$ *is a mapping such that* 0 *(1, respectively) is periodic and* p *is a point of* $\varprojlim\{[0, 1], f\}$ *such that for each positive integer* i*,* p_i *is in* $\mathcal{O}(0, f)$ *(*$\mathcal{O}(1, f)$*, respectively). Then* p *is an end point of* $\varprojlim\{[0, 1], f\}$*.*

To be sure of the count of the number of end points of an inverse limit, we have the following theorem giving a sufficient condition for a point not to be an end point of a continuum produced by a Markov map [8, Theorem 15].

Theorem 4.2. *If* $f : [0, 1] \to [0, 1]$ *is a Markov map with Markov partition* $\mathcal{A} = \{0 = a_1 < a_2 < \ldots < a_n = 1\}$ *and* p *is a point of* $\varprojlim\{[0, 1], f\}$ *such that* p_i *is not in* $\mathcal{A}$ *for infinitely many positive integers* i*, then* p *is not an end point of* $\varprojlim\{[0, 1], f\}$*.*

Using Theorem 4.1, we see that if f is in the logistic family, $(0, 0, 0, \ldots)$ is always an end point of $\varprojlim\{[0, 1], f\}$. When the critical point $1/2$ is in or is attracted to a periodic orbit of period n, there are n more end points occurring in the core. When the critical point is in a periodic orbit of period n, the logistic map is Markov and Theorem 4.2 yields that there are no other end points. See [1] for additional information on the logistic family.

For the two-parameter family, there are n end points if c is in (or is attracted to) a periodic orbit of period n. Period three occurs along the curve $b = c$ for the two-parameter family, while period four occurs along the curve $b = c^2 - c + 1$ and also along the part of the curve $b^2 - (c+1)b + c^2 =$ lying in the unit square. Period five occurs along the curves in the unit square determined by (i) $b^3-(c+2)b^2+(1+c+c^2)b-c^3=0$, (ii) $b^2-(c+1)b+c^3-c^2+c=0$, and (iii) $b=c^3-2c^2+2c$. The equation (i) determines the stair-step period five, while (iii) holds in case $b > c$. When $c = \frac{3}{4}$, equation (i) holds for $b = 1/4$, while equation (ii) holds for $b \approx 0.479715\ldots$, and equation (iii) holds for $b = \frac{51}{64}$. For parameter pairs along any one of the period five curves, the continua produced are homeomorphic. In fact, the maps are Markov maps with equivalent Markov partitions (see Raines's Theorem, [14, Theorem 3.1]). On the other hand, continua produced by parameter pairs chosen along different period five curves are topologically distinct [3]. See the remarks immediately following Theorem 6.3 in Section 6 for more information on these continua with n end points arising in the two-parameter family.

For the permutation family, when the map is unimodal and arises from an n-cycle (we call a permutation σ in S_n an *n-cycle* provided the orbit of 1 under σ is $\{1, 2, \ldots, n\}$), it produces a continuum homeomorphic to one produced by a map in the two-parameter family with the critical point in a periodic orbit of period n. The three period five continua in cases (i), (ii), and (iii) above are produced by the 5-cycles $(1\,2\,3\,4\,5)$, $(1\,2\,4\,3\,5)$, and $(1\,3\,4\,2\,5)$, respectively. See reference [8] as well as Theorem 6.3 for additional information on continua with n end points arising in the permutation family.

Of course, a point does not have to have any of its coordinates in the two element set $\{0, 1\}$ to be an end point of an inverse limit on $[0, 1]$. The pseudo-arc can be obtained as an inverse limit on $[0, 1]$ using a single bonding map which fixes 0 and 1 (see [4]), but every point of the pseudo-arc is an end point of it. Much more on end points (including a characterization) may be found in [2] where M. Barge and J. Martin provide an interesting example of a map which has $\frac{1}{2}$ as a fixed point, but the inverse limit using that map as a single bonding map has $(\frac{1}{2}, \frac{1}{2}, \ldots)$ as an end point [2, Example 2, page 169].

5. Arc Continua

A continuum such that every proper subcontinuum is an arc is called an *arc continuum*. For the logistic family, only the BJK continuum, produced when $\lambda = 1$, and the arc, produced for $\frac{1}{4} < \lambda \leq \frac{3}{4}$, are arc continua. Often the core in the logistic family is an arc continuum as is the case when $\frac{3}{4} < \lambda \leq \lambda_2$ ($\lambda_2 \approx 0.87464$) where the core is an arc or when $\lambda \approx 0.9564\ldots$ where the core is a three end point indecomposable arc continuum [1]. For the cores of the logistic family as well as the two-parameter family and the permutation family, we get an arc continuum when the map is a Markov map satisfying the hypothesis of the following theorem, see [8, Theorem 13].

Theorem 5.1. *Suppose $f : [0,1] \to [0,1]$ is a Markov map with Markov partition $0 = a_1 < a_2 < \ldots < a_n = 1$ for $n \geq 3$ and $\mathcal{O}(0,f) = \{a_1, a_2, \ldots, a_n\}$. If k is an integer, $k < n$, such that $f^k(0) = a_2$ and n and k are relatively prime, then $\varprojlim\{[0,1], f\}$ is an arc continuum.*

For the two-parameter family, the conditions of Theorem 5.1 are normally satisfied when 0 is in a periodic orbit of period n, but there are exceptions when n is even. An exception that shows most clearly occurs along the curve $b = c^2 - c + 1$ where 0 is in a periodic orbit of period 4. This is where the inverse limit is a pair of $\sin(\frac{1}{x})$-curves. Another exception occurs along the curve $b = 1 - c\sqrt{1-c}$ for $c > \frac{-1+\sqrt{5}}{2}$. Here, the inverse limit is an indecomposable continuum with three $\sin(\frac{1}{x})$-curves as end continua such that if the three limit arcs of the $\sin(\frac{1}{x})$-curves are shrunk to points the result is the familiar three end point indecomposable arc continuum. In general, exceptions occur along the lower boundary of regions of stable periodicity (see [9, Figure 1]). The cores of the tent family, when viewed as arising from maps in the two-parameter family of maps of $[0,1]$, occur well away from the region where these exceptions occur. Consequently, for the cores of the tent family, the maps always satisfy the conditions of Theorem 5.1 when 0 is in a periodic orbit of period n.

For the family of permutation maps we can say more. Theorem 5.2 appears as Theorem 16 in [8].

Theorem 5.2. *If σ is an n-cycle and k is a positive integer, $k < n$, such that $f_\sigma^k(0) = \frac{1}{n-1}$ and n and k are relatively prime, then $\varprojlim\{[0,1], f_\sigma\}$ is an indecomposable arc continuum with only n end points.*

6. Property of Kelley

The Property of Kelley is the final topological property we consider for inverse limits on $[0,1]$. In Section 1, we saw an example of a ray limiting to an arc which does not have the Property of Kelley (Figure 1). Additionally, the union of two BJK continua intersecting only at a common end point is another example of a continuum which does not have the Property of Kelley. This continuum is the core of the continuum arising from the logistic family for $\lambda \approx .91964\dots$.

Our first theorem on the Property of Kelley is that the heritarily decomposable continua arising as inverse limits in the logistic family all have the Property of Kelley. Here we denote by λ_c the Feigenbaum limit ($\lambda_c \approx 0.89249\dots$). Theorem 6.1 (below) summarizes [11, theorems 5 and 6] although it is a reasonably direct consequence of Theorem 2.3 except for $\lambda = \lambda_c$. For additional information on the continuum occurring when $\lambda = \lambda_c$, see [1], [12].

Theorem 6.1. *If $\lambda \leq \lambda_c$, then $\varprojlim\{[0,1], f_\lambda\}$ has the Property of Kelley.*

Our next theorem on the Property of Kelley for the logistic family and the tent family follows immediately from Theorem 2.3.

Theorem 6.2. *If f is in the logistic family or if f is in the tent family, then $\varprojlim\{[0,1], f\}$ has the Property of Kelley if and only if the core has the Property of Kelley.*

The core in the logistic family as well as in the tent family may or may not have the Property of Kelley. For example when $\lambda \approx 0.91964\dots$, the core for the logistic family is the union of two BJK continua intersecting at a common end point and, as noted above, does not have the Property of Kelley. This phenomenon occurs in the tent family for $\lambda = \frac{\sqrt{2}}{2}$. On the other hand, for $\lambda \approx 0.9564\dots$, the core in the logistic family is the familiar three end point indecomposable continuum every proper subcontinuum of which is an arc and does have the Property of Kelley. (That this core continuum has the Property of Kelley follows from the fact that it is homeomorphic to the continuum which occurs in the two-parameter family whenever $b = c$ and the map g_{bc} is a permutation map based on a 3-cycle and satisfies the conditions of Theorem 6.3 below.) This phenomenon occurs in the tent family for $\lambda = \frac{1+\sqrt{5}}{4}$ (yes, at half of the golden mean).

For permutation maps, we know the most about those which are based on n-cycles. For these maps, we summarize the information into the following theorem. This combines results from [8, Theorem 16] and [10, Theorem 3.3].

Theorem 6.3. *If σ is an n-cycle in S_n and $f_\sigma^k(0) = \frac{1}{n-1}$ where n and k are relatively prime, then $\varprojlim\{[0,1], f_\sigma\}$ is an indecomposable continuum with the Property of Kelley which has only n end points and is such that every non-degenerate proper subcontinuum of it is an arc.*

In the two-parameter family, for $c = \frac{n-2}{n-1}$ and $b = \frac{1}{n-1}$, the map g_{bc} is a permutation map based on the n-cycle $\sigma = (1\,2\,\ldots\,n)$ and, consequently, by Theorem 5.2, we see that it produces an indecomposable arc continuum having n and only n end points and the Property of Kelley.

Every map based on permutations on 3, 4, or 5 elements produces a continuum having the Property of Kelley, see [11, Section 5]. Additional information on the topology of these continua appears in [8, Section 3].

Problem 6.4. It would be interesting to know more about the topology of the continua produced by permutation maps based on permutations on n elements with $n \geq 6$.

Problem 6.5 (W. J. Charatonik). Does every permutation map produce a continuum having the Property of Kelley?

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Department of Mathematics and Statistics, University of Missouri–Rolla, Rolla, MO 65409-0020

E-mail address: ingram@umr.edu