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HOMOMORPHISMS AND PROTOPOLOGICAL GROUPS

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Abstract. A *protopological group* is a group G with a topology τ such that there exists a collection $\mathcal{N}$ of normal subgroups, called a *normal system*, satisfying (1) For every neighborhood U of the identity e, there exists an $N \in \mathcal{N}$ with $N \subseteq U$ and (2) G/N with the quotient topology is a topological group for every $N \in \mathcal{N}$. The collection of quotient topologies on G/N for each $N \in \mathcal{N}$ is called the *quotient system*. From the definition, it is clear that every topological group is a protopological group since we can take $\{e\}$ to be an element of the normal system. In this paper, we will show that two results concerning homomorphisms in the category of topological groups hold in the category of protopological groups.

1. Introduction

In the early 1990's, J. L. Covington [1] defined a *protopological group* as a group G with a topology τ such that there exists a collection $\mathcal{N}$ of normal subgroups, called a *normal system*, satisfying (1) For every neighborhood U of the identity e, there exists an $N \in \mathcal{N}$ with $N \subseteq U$ and (2) G/N with the quotient topology is a topological group for every $N \in \mathcal{N}$. The collection of quotient topologies on G/N for each $N \in \mathcal{N}$ is called the *quotient system*. From the definition, it is clear that every topological group is a protopological group since we can take $\{e\}$ to be an element of the

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normal system. One property that is often useful in proving theorems about topological groups is the following: *If* (G, τ) *is a topological group and* G/N *has the quotient topology where* $N \lhd G$*, then the natural map* $\eta_N : G \to G/N$ *is an open map.* Protopological groups do not have this property, in general. Consequently, while protopological groups possess many of the properties of topological groups, the proofs concerning these properties often differ from the proofs used in the study of topological groups. Two well-known facts concerning topological groups are:

(1) *If* (G, τ) *is a topological group and* $h : (G, \tau) \to (G', \tau')$ *is a continuous, open, onto homomorphism, then* (G', τ') *is a topological group.*

(2) *Let* G *be a group and* $h : G \to G'$ *an onto homomorphism. If* G' *has a topology* τ' *such that* (G', τ') *is a topological group, then* $(G, h^{-1}(\tau'))$ *is a topological group where* $h^{-1}(\tau')$ *is the topology given by* $\{h^{-1}(U) | U \in \tau'\}$.

In this paper, I will show that the analogous results hold in the category of protopological groups, and thereby obtain as a corollary a major result of a 1994 paper by Covington [3].

If G is a group with a topology τ and $N \lhd G$, then we will denote the quotient topology on G/N as $q_N(\tau)$. When (G, τ) is a protopological group with normal system $\mathcal{N}$, $(G/N, q_N(\tau))$ is a topological group, whose pullback topology is a group topology on G for each $N \in \mathcal{N}$. We will call the join of these pullback topologies the *complete pullback topology* or the *weak topology* and denote it as τ_p. Covington [4] has shown that τ_p is the Graev topology, the finest group topology no finer than τ. For each $N \in \mathcal{N}$, we will denote the natural map by η_N. That is, $\eta_N : G \to G/N$ is given by $\eta_N(g) = gN$. In the study of protopological groups, saturated sets play an important role. If G is a group and $N \lhd G$, we say that a subset $U \subseteq G$ is *saturated with respect to* N if for each $x \in U$, $xN \subseteq U$.

2. The Homomorphic Image of A Protopological Group

Theorem 2.1. *Let* (G, τ) *be a protopological group with normal system* $\mathcal{N}$*. If* $h : (G, \tau) \to (G', \tau')$ *is a continuous, open, onto homomorphism, then* (G', τ') *is a protopological group with normal system* $\mathcal{N}' = \{h(N) | N \in \mathcal{N}\}$.

Proof: If U' is a neighborhood of the identity $e' \in G'$, then $e \in h^{-1}(U')$ and $h^{-1}(U') \in \tau$ where e is the identity element of G. Since (G,τ) is protopological, there exists $N \in \mathcal{N}$ with $e \in N \subseteq h^{-1}(U')$, and it follows that $e' \in h(N) \subseteq U'$.

Now consider the diagram:

$$\begin{array}{ccc} (G,\tau) & \overset{h}{\longrightarrow} & (G',\tau') \\ \eta_N \downarrow & & \downarrow \eta_{h(N)} \\ (G/N, q_N(\tau)) & \underset{\bar{h}}{\longrightarrow} & (G'/h(N), q_{h(N)}(\tau')) \end{array}$$

Clearly, $\bar{h}$ is an induced continuous function that makes the diagram commute.

If $U \in q_N(\tau)$, then $\eta_N^{-1}(U) \in \tau$ and is saturated with respect to N. It is easy to show that $h(\eta_N^{-1}(U))$ is saturated with respect to $h(N)$ and since h is an open map, $h(\eta_N^{-1}(U)) \in \tau'$. Therefore, $\eta_{h(N)}(\ h(\eta_N^{-1}(U))) \in q_{h(N)}(\tau')$, but since the diagram commutes, $\bar{h}(U) = \eta_{h(N)}(h(\eta_N^{-1}(U)))$. Hence, $\bar{h}$ is an open map. A straightforward argument shows that $\bar{h}$ is an onto homomorphism. Since $\bar{h} : (G/N, q_N(\tau)) \to (G'/h(N), q_{h(N)}(\tau'))$ is an onto, open, continuous homomorphism, it follows that $(G'/h(N), q_{h(N)}(\tau'))$ is a topological group. Hence, (G',τ') is a protopological group with normal system $\mathcal{N}' = \{h(N)|N \in \mathcal{N}\}$. □

We note that a slightly stronger result holds: h does not have to be an open map. We actually used the following property: If $U \in \tau$ is saturated with respect to some $N \in \mathcal{N}$, then $h(U) \in \tau'$. That is, h does not have to map open unsaturated sets to open sets.

Covington [3] studied a special class of protopological groups, which she called t-protopological groups. She defines a *t−protopo−logical group* as a protopological group with the additional property that the natural map $\eta_N : G \to G/N$ is an open map for all $N \in \mathcal{N}$. She shows that Theorem 2.1 holds when the word "protopological" is replaced with "t-protopological." We will obtain this result in the following corollary.

Corollary 2.2. *If (G,τ) is a t-protopological group with normal system $\mathcal{N}$ and $h : (G,\tau) \longrightarrow (G',\tau')$ is a continuous, open, onto homomorphism, then (G',τ') is a t-protopological group with normal system $\mathcal{N}' = \{h(N)|N \in \mathcal{N}\}$.*

Proof: By Theorem 2.1, we know that (G', τ') is a protopological group with normal system $\mathcal{N}' = \{h(N) | N \in \mathcal{N}\}$. If $U \in \tau'$, then $h^{-1}(U) \in \tau$ by the continuity of h. It follows from the fact that (G, τ) is t-protopological that $\eta_N(h^{-1}(U)) \in q_N(\tau)$. We saw in the previous theorem that $\bar{h}$ is an open map. Therefore, $\bar{h}(\eta_N(h^{-1}(U))) \in q_{h(N)}(\tau')$, but $\bar{h}(\eta_N(h^{-1}(U))) = \eta_{h(N)}(U)$. Thus, $\eta_{h(N)}$ is an open map. □

3. The Pullback of A Protopological Topology

In the remainder of this paper, we will show that the result concerning the pullback topology holds in the category of protopological groups. The following Characterization Theorem was published in [5].

Theorem 3.1 (Characterization Theorem)**.** *Let (G, τ) be a protopological group with normal system $\mathcal{N}$ and quotient system $\mathcal{Q}$. Let τ^* be a topology on G. (G, τ^*) is a protopological group with normal system $\mathcal{N}$ and quotient system $\mathcal{Q}$ if and only if τ^* has the following properties:*

(1) $\tau_p \subseteq \tau^*$;

(2) if $U \in \tau^*$ is saturated with respect to some $N \in \mathcal{N}$, then $U \in \tau_p$;

(3) if $U \in \tau^*$ is a neighborhood of e, then there exists $N \in \mathcal{N}$ such that $N \subseteq U$.

For the sake of completeness, we give the following lemma, which was proved by Covington in [1].

Lemma 3.2. *Let G and G' be groups with $U' \subseteq G'$ and $N' \lhd G'$. If $h : G \to G'$ is an onto homomorphism, then $h^{-1}(U'N') = h^{-1}(U')h^{-1}(N')$.*

Proof: If $g \in h^{-1}(U')h^{-1}(N')$, then $g = un$ where $n \in h^{-1}(N')$ and $u \in h^{-1}(U')$. Then $h(g) = h(u)h(n) \in U'N'$ and therefore, $g \in h^{-1}(U'N')$. Thus, $h^{-1}(U')h^{-1}(N') \subseteq h^{-1}(U'N')$. Conversely, if $g \in h^{-1}(U'N')$, then there exists $n \in N'$ and $u \in U'$ with $h(g) = un$ and also $z \in h^{-1}(N')$ and $y \in h^{-1}(U')$ with $h(z) = n$ and $h(y) = u$. So, we have $h(g) = un = h(yz)$. Then $h(gz^{-1}y^{-1}) = e'$ and $h(gz^{-1}) = h(gz^{-1}y^{-1}y) = h(gz^{-1}y^{-1})h(y) = e'u = u \in U'$. So, $gz^{-1} \in h^{-1}(U')$ and $g = gz^{-1}z \in h^{-1}(U')h^{-1}(N')$. Thus, $h^{-1}(U'N') \subseteq h^{-1}(U')h^{-1}(N')$. □

Theorem 3.3. *Let G be a group and $h : G \to G'$ an onto homomorphism. If G' has a topology τ' such that (G', τ') is a protopological group with normal system $\mathcal{N}'$, then $(G, h^{-1}(\tau'))$ is a protopological group with normal system $\mathcal{N} = \{h^{-1}(N') | N' \in \mathcal{N}'\}$ where $h^{-1}(\tau')$ is the topology given by $\{h^{-1}(U) | U \in \tau'\}$.*

Proof: Let τ'_p be the complete pullback topology for (G', τ'). If $U \in h^{-1}(\tau'_p)$ is a neighborhood of e, then there exists a $U' \in \tau'_p$ with $U = h^{-1}(U')$. Since h is a homomorphism and (G', τ') is protopological, there exists $N' \in \mathcal{N}'$ with $e' \in N' \subseteq U'$. Now, using the preimage of these sets, we have $e \in h^{-1}(N') \subseteq U$. So, $(G, h^{-1}(\tau'_p))$ is a protopological group with normal system $\mathcal{N} = \{h^{-1}(N') | N' \in \mathcal{N}'\}$ and quotient system $\mathcal{Q} = \{q_{h^{-1}(N')}(h^{-1}(\tau'_p)) | N' \in \mathcal{N}'\}$. Using a result from Covington [3], we have that the complete pullback topology τ_p for $(G, h^{-1}(\tau'_p))$ is $\tau_p = h^{-1}(\tau'_p)$ since $h^{-1}(\tau'_p)$ is the Graev topology.

Now we consider $(G, h^{-1}(\tau'))$. By definition of $h^{-1}(\tau')$, it is clear that $\tau_p \subseteq h^{-1}(\tau')$ and that $h : (G, h^{-1}(\tau')) \to (G', \tau')$ is an open map. If $U \in h^{-1}(\tau')$ is saturated with respect to $h^{-1}(N')$, then $h(U) \in \tau'$ and $h(U)$ is saturated with respect to N'. Since (G', τ') is a protopological group with normal system $\mathcal{N}'$, $h(U) \in \tau'_p$ by the Characterization Theorem, and hence, $h^{-1}(h(U)) \in \tau_p$.

But $h^{-1}(h(U)) = h^{-1}(h(U) \cdot N')$ since $h(U)$ is saturated with respect to N'

$= h^{-1}(h(h^{-1}(U')) \cdot N')$ where $U' \in \tau'$ and $U = h^{-1}(U')$

$= h^{-1}(U' \cdot N')$ since $U' = h(h^{-1}(U'))$

$= h^{-1}(U')h^{-1}(N')$ by Lemma 3.2

$= Uh^{-1}(N')$ since $U = h^{-1}(U')$

$= U$ since U is saturated with respect to $h^{-1}(N')$.

So, we have that if $U \in h^{-1}(\tau')$ is saturated with respect to $h^{-1}(N')$, then $U = h^{-1}(h(U)) \in \tau_p$.

Now, if V is a neighborhood of e, then there exists $V' \in \tau'$ with $V = h^{-1}(V')$. Since h is an open map, V' is a neighborhood of e'. But since (G', τ') is a protopological group with normal system $\mathcal{N}'$, there exists an $N' \in \mathcal{N}'$ with $e' \in N' \subseteq V'$. But then, $e \in h^{-1}(N') \subseteq V$. Hence, by the Characterization Theorem, $(G, h^{-1}(\tau'))$ is a protopological group with normal system $\mathcal{N} = \{h^{-1}(N') | N' \in \mathcal{N}'\}$. □

Covington [2] has shown that Theorem 3.3 holds when the word "protopological" is replaced with "t-protopological." We will obtain her result as a corollary to this theorem.

Corollary 3.4. *Let G be a group and $h : G \to G'$ an onto homomorphism. If G' has a topology τ' such that (G', τ') is a t-protopological group with normal system $\mathcal{N}'$, then $(G, h^{-1}(\tau'))$ is a t-protopological group with normal system $\mathcal{N} = \{h^{-1}(N')|N' \in \mathcal{N}'\}$ where $h^{-1}(\tau')$ is the topology given by $\{h^{-1}(U)|U \in \tau'\}$.*

Proof: From Theorem 3.3, we know that $(G, h^{-1}(\tau'))$ is a protopological group with normal system $\mathcal{N} = \{h^{-1}(N')|N' \in \mathcal{N}'\}$. Now we wish to show that for every $h^{-1}(N') \in \mathcal{N}$, $\eta_{h^{-1}(N')} : G \to G/h^{-1}(N')$ is an open map.

Consider the diagram:

$$\begin{array}{ccc} (G, h^{-1}(\tau') & \xrightarrow{h} & (G', \tau') \\ \eta_{h^{-1}(N')} \downarrow & & \downarrow \eta_N \\ (G/h^{-1}(N'), q_{h^{-1}(N')}(h^{-1}(\tau')) & \xrightarrow[\bar{h}]{} & (G'/N', q_N(\tau')) \end{array}$$

Clearly, $\bar{h}$ makes the diagram commute and is continuous.

If $U \in h^{-1}(\tau')$, then $h(U) \in \tau'$ since h is an open map. Since (G', τ') is t-protopological, it follows that $\eta_{N'}(h(U)) \in q_{N'}(h(U))$. By the continuity of $\bar{h}$, $\bar{h}^{-1}(\eta_{N'}(h(U))) = \bar{h}^{-1}(\bar{h}(\eta_{h^{-1}(N')}(U))) \in q_{h^{-1}(N')}(h^{-1}(\tau'))$. Clearly, $\eta_{h^{-1}(N')}(U) \subseteq \bar{h}^{-1}(\bar{h}(\eta_{h^{-1}(N')}(U)))$. If there exists an $xh^{-1}(N') \in \bar{h}^{-1}(\bar{h}(\eta_{h^{-1}(N')}(U)))$ with $xh^{-1}(N') \notin \eta_{h^{-1}(N')}(U)$, then $\bar{h}(xh^{-1}(N')) = h(x)N' \in \bar{h}(Uh^{-1}(N')) = h(U)N'$. Then, there must exist $y \in U$ with $h(x) = h(y)$. Now we consider $\eta_{h^{-1}(N')}(x)$ and $\eta_{h^{-1}(N')}(y)$. $\eta_{h^{-1}(N')}(x) = \eta_{h^{-1}(N')}(y)$ if and only if $y^{-1}x \in h^{-1}(N')$. Because $h(x) = h(y)$, we have $h(y^{-1}x) = h(e) = e' \in N'$ for all $N' \in \mathcal{N}'$, and therefore, $y^{-1}x \in h^{-1}(N')$. Now, since $\eta_{h^{-1}(N')}(x) = \eta_{h^{-1}(N')}(y)$ and $y \in U$, $yh^{-1}(N') = xh^{-1}(N') \in \eta_{h^{-1}(N')}(U)$. This is a contradiction since we chose $xh^{-1}(N') \notin \eta_{h^{-1}(N')}(U)$. Hence, $\eta_{h^{-1}(N')}(U) \subseteq \bar{h}^{-1}(\bar{h}(\eta_{h^{-1}(N')}(U)))$, and $\eta_{h^{-1}(N')}$ is an open map. □

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