Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 27, No. 1, 2003
Pages 211–216

A NOTE ON SPECIAL MEASURE PRESERVING DYNAMICAL SYSTEMS IN METRIC SPACES

MARCIN KULCZYCKI

Abstract. We prove that if a dynamical system on a separable metric space X is measure preserving (with respect to some complete Borel measure μ that is finite on balls) and every orbit is bounded, then almost all points (in the sense of measure μ) have their orbits contained in their ω-limit sets.

1. Introduction

An example of a dynamical system in $\mathbb{R}^3$ without fixed points with the diameters of the trajectories uniformly bounded is known (see [2] for the example itself and a discussion of its relation to Problem 110 from [3]).

A paper by G. Kuperberg [1] shows the existence of a volume-preserving dynamical system without fixed points or closed trajectories on any smooth 3-dimensional manifold with an empty boundary (compact or not) giving another class of counterexamples to the Seifert Conjecture. (The conjecture stated that every fixed point free dynamical system on the three-sphere has a periodic orbit.)

2000 *Mathematics Subject Classification.* Primary 37A05; Secondary 58C30, 37C25.

Key words and phrases. Dynamical system, measure, ω-limit set.

The author was supported in part by the Baskerville Fellowship at Auburn University.

The following question has been posed by G. Kuperberg: Is it possible to obtain a dynamical system in $\mathbb{R}^3$ that combines several of these properties: a system that is fixed point free, volume-preserving, AND has the diameters of the trajectories bounded by the same constant?

The theorem below was motivated by a search for specific properties that such a system would have to exhibit, but is presented here in a much more general setting; the dynamical system is assumed to be defined on a general metric space and the assumptions about the measure are weak.

2. Main result

To introduce some notations used later we begin with a handful of basic definitions.

Definition 2.1. A dynamical system is a continuous mapping $\Phi : \mathbb{R} \times X \to X$ such that $\Phi(0, x) = x$ and $\Phi(t, \Phi(s, x)) = \Phi(s + t, x)$ for any x, s and t.

Definition 2.2. Let $\overline{A}$ denote the closure of the set A. The ω-limit set of a point x is defined as $\bigcap_{n=1}^{\infty} \overline{\Phi([n, \infty), x)}$.

Definition 2.3. A minimal set of a dynamical system Φ is a set that is closed, nonempty, invariant with respect to Φ, and does not contain a smaller set with such properties.

Definition 2.4. A dynamical system Φ is measure preserving if for any measurable $A \subset X$ and for any t, the set $\Phi(t, A)$ is also measurable and has the same measure as A.

Theorem 2.5. *Let X be a separable metric space with metric d. Let μ be a complete Borel measure on X that is finite on balls. Let Φ be a measure preserving dynamical system on X that has diameters of all its orbits bounded (not necessarily uniformly). Define the set A as the collection of all points that have their orbits contained in their ω-limit sets (that is, A is the union of all minimal sets of Φ).*

Then $\mu(X \setminus A) = 0$.

Proof: The proof is divided into five parts presented below as separate lemmas. To show that the set $X \setminus A$ has measure zero, we will decompose it into smaller parts. Each such part will, in turn, be expressed as a countable union of even smaller sets, and so on, until we prove that the smallest division consists entirely of sets of measure zero.

The structure of the proof: Lemmas 2.6 to 2.9 give conditions that (if proven) imply the previous statement. That is, Lemma 2.6 gives a condition that needs to be proven to finish the proof of the main theorem; Lemma 2.7 gives a condition that implies the one from Lemma 2.6, and so on. Finally, Lemma 2.10 proves the condition from Lemma 2.9.

Let $\forall!$ denote "for almost all points except for a set of measure zero."

Lemma 2.6. *It is enough to prove*

$$\forall\varepsilon > 0\ \forall! x \in X\ \exists t_1, t_2, \ldots \to \infty\ \textit{s.t.}\ \forall i\ d(\Phi(t_i, x), x) < \varepsilon.$$

Proof: Indeed, if $A_{\frac{1}{k}}$ is the set of all $x \in X$ satisfying the above for $\varepsilon = \frac{1}{k}$, then $A = \bigcap_{k=1}^{\infty} A_{\frac{1}{k}}$. Therefore, $\mu(X \backslash A) \leq \sum_{k=1}^{\infty} \mu(X \backslash A_{\frac{1}{k}}) = 0$. □

Lemma 2.7. *To prove the condition in Lemma 2.6, it suffices to show that*
$\forall\varepsilon > 0\ \forall x \in X\ \forall! y \in B(x, \varepsilon) \exists t_1, t_2, \ldots \to \infty\ \textit{s.t.}\ \forall i\ \Phi(t_i, y) \in B(x, \varepsilon)$.

Proof: For a given ε we choose a countable cover of X with such balls (X is separable metric, so it is Lindelöf). Note that inside every ball from the covering each point that does satisfy the right part of the above property belongs to $A_{2\varepsilon}$. Inside every ball from the covering there is only a set of measure zero of points that do not satisfy the right part of the above property, so $X \backslash A_{2\varepsilon}$ is contained in a countable union of sets of measure zero. □

Lemma 2.8. *For a given ε and x, define*

$$B_n = \{y \in B(x, \varepsilon)\ :\ \Phi([n, \infty), y) \cap B(x, \varepsilon) = \emptyset\}.$$

Then B_n are measurable, and it is enough to show that $\mu(B_n) = 0$.

Proof: To prove that the sets B_n are measurable, we will prove that they are closed. Suppose $y \notin B_n$. In this case, for some $t \geq n$ we have $\Phi(t, y) \in B(x, \varepsilon)$. By continuity of Φ, there is a neighborhood of y that also enters $B(x,\varepsilon)$ after time t, proving that the complement of B_n is open.

Let B denote the set of points y in the ball $B(x,\varepsilon)$ for which the right part of the condition in Lemma 2.7 is not satisfied. Then $B = \bigcup_{n=1}^{\infty} B_n$, and we are done. □

Lemma 2.9. *For a fixed ε, x, and n, define*

$$B_n^j = \{y \in B_n : \Phi(\mathbb{R}, y) \subset \overline{B(x,j)}\}.$$

Then B_n^j are measurable, and it is enough to show that $\mu(B_n^j) = 0$.

Proof: To prove that the sets B_n^j are measurable, we will prove that they are closed. The proof is analogous to the one from Lemma 2.8: If $y \notin B_n^j$, then either $y \notin B_n$ (and the same is true for some neighborhood of y), or for some t we have $\Phi(t,y) \notin \overline{B(x,j)}$. By continuity of Φ, we get a neighborhood of y that also leaves $\overline{B(x,j)}$. This proves that B_n^j are closed.

By our assumption that every orbit is bounded, we have $B_n = \bigcup_{j=1}^{\infty} B_n^j$. This finishes the proof of this lemma. □

Lemma 2.10. *The measure of every set B_n^j is zero.*

Proof: Consider the sets $\Phi(p\cdot(n+1), B_n^j)$ where p ranges over the natural numbers. All of these sets are subsets of $\overline{B(x,j)}$. We claim that they are pairwise disjoint (so essentially, we are proving that the set B_n^j is a weakly wandering set).

It is easy to see that the first set ($p = 0$) is disjoint from any other set in the collection. Indeed, B_n^j is a subset of $B(x,\varepsilon)$, while $\Phi(p\cdot(n+1), B_n^j)$ does not intersect that set when $p > 0$.

Now if y belonged to both $\Phi(p\cdot(n+1), B_n^j)$ and $\Phi(q\cdot(n+1), B_n^j)$ where $p < q$, then $\Phi(-p\cdot(n+1), y)$ would be an element of both B_n^j and $\Phi((q-p)\cdot(n+1), B_n^j)$, contradicting the above.

At the same time, each of these sets has the measure equal to $\mu(B_n^j)$. Since the measure of $\overline{B(x,j)}$ is finite, it follows that $\mu(B_n^j) = 0$. □

With the chain of all five lemmas proven, so is our main theorem. □

Remark 2.11. The assumption that Φ is measure preserving is too strong and can be replaced with the following technicality: If Y is a measurable subset of X then for any $t > 0$, the set $\Phi(t, Y)$ contains some measurable set Y' such that $\mu(Y) \leq \mu(Y')$.

Remark 2.12. Theorem 2.5 is presented here in terms of continuous time dynamical systems because of its connections with the results mentioned in the Introduction. The same proof is also valid for the discrete case.

Remark 2.13 Another way of proving Theorem 2.5 is to use the following proposition:

If a dynamical system Φ on a separable metric space X with an invariant complete Borel measure μ has no minimal sets, then the measure of X is either zero or infinity.

Proof: (Sketch). Using the concepts from the proof of Theorem 2.5, we obtain around every point in X an open ball that is mapped out of itself for all times greater than certain time T_0. If the measure of X is finite, then each such ball has to be of measure zero. Then by the Lindelöf property the measure of the whole space is also zero.

To obtain Theorem 2.5, we first remove the set A, and then we divide whatever is left into countably many balls of radius 1. Each such ball in turn can be expressed as a union of collections of points that stay within distance n of it. Each such collection is contained in some ball, so its measure is finite. Using the above proposition for every collection separately, we obtain the desired result. □

Acknowledgment. The author would like to thank Professor Krystyna Kuperberg for her advice and the referee for comments that helped to improve the structure of the proof.

References

[1] G. Kuperberg, *A volume-preserving counterexample to the Seifert Conjecture*, Comment. Math. Helv., **71** (1996), 70–97.

[2] K. Kuperberg and C. Reed, *A rest point free dynamical system on R^3 with uniformly bounded trajectories*, Fund. Math., **114** (1981), 229–234.

[3] D. Mauldin, Ed., *The Scottish Book: Mathematics from the Scottish Café*, Boston: Birkhäuser, 1981.

Department of Mathematics, Auburn University, Auburn University, Alabama 36849 USA

Current address: Institute of Mathematics, Jagiellonian University, ul. Reymonta 4, 30-059 Kraków, Poland

E-mail address: kulczmc@auburn.edu