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FANS WHOSE HYPERSPACES ARE CONES

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Abstract. Answering a question by S. B. Nadler, Jr. and the author, it is proved that if F is a fan, which is homeomorphic to the cone over a compactum, then each of its hyperspaces is homeomorphic to the cone over a compactum.

1. Introduction

Let $\mathcal{C}(X)$ denote a hyperspace of subcontinua of a continuum X with the Hausdorff metric [6]. Let us note that there are some similarities between the hyperspace $\mathcal{C}(X)$ and the cone over X (the similarities are discussed in [6, p. 59–60]). Thus, the study of when $\mathcal{C}(X)$ is actually homeomorphic to its cone is natural. It is also natural to ask if given a continuum X, is $\mathcal{C}(X)$ homeomorphic to the cone over a continuum Y [15, (8.25)]? Recently, it has been an interest to study the n–fold symmetric products and n–fold hyperspaces of continua ([7], [8], [3], [9], [10], [11]–[13]). R. Schori showed that the 2–fold hyperspace of $[0, 1]$ is homeomorphic to $[0, 1]^4$ [5, Lemma 1]. Hence, geometric models for these hyperspaces would be nice to have. Having this in mind, S. B. Nadler, Jr. and the author asked [13, 3.8], "Does there exist a hereditarily decomposable continuum X that is not an arc such that $\mathcal{C}_n(X)$ is homeomorphic to the cone over a finite−dimensional continuum for some integer $n \geq 2$?" We give an affirmative answer to this question (see 3.2). In fact, we show that if F is a fan, which is homeomorphic to the cone

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over a compactum, then all its hyperspaces are homeomorphic to the cone over a compactum. In this way, we present geometric models for the $n-$fold symmetric products and the n–fold hyperspaces of fans which are cones.

2. Notation and Terminology

If Z is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}_\varepsilon(A)$.

We denote the unit interval $[0,1]$ by I. The set of positive integers is denoted by $\mathbb{N}$.

A *continuum* is a nonempty compact connected metric space.

Hyperspaces. Given a continuum X, we define its *hyperspaces* as the following sets:

$$2^X = \{A \subset X \mid A \text{ is closed and nonempty}\}$$

$$\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\},\ n \in \mathbb{N}$$

$$\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\},\ n \in \mathbb{N}.$$

It is known that 2^X is a metric space with the Hausdorff metric, $\mathcal{H}$, defined as follows:

$$\mathcal{H}(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}_\varepsilon(B) \text{ and } B \subset \mathcal{V}_\varepsilon(A)\},$$

(see [16, (0.1)]). In fact, 2^X is an arcwise connected continuum [16, (1.13)]; for each $n \in \mathbb{N}$, $\mathcal{C}_n(X)$ is an arcwise connected continuum [7, 3.1], and $\mathcal{F}_n(X)$ is a continuum [1, p. 877].

Fans. A *dendroid* is an arcwise connected and hereditarily unicoherent continuum. A *fan* is a dendroid with exactly one ramification point (i.e., with only one point which is the common part of three otherwise disjoint arcs) [2]. The unique ramification point of a fan F is called the *top of* F; τ always denotes the top of a fan. By an *end point of a fan* F, we mean an end point in the classical sense, which means a point e of F that is a nonseparating point of any arc in F that contains e; $E(F)$ denotes the set of all end points of a fan F. A *leg of a fan* F is the unique arc in F from τ to some end point of F. Given two points x and y of a fan F, xy denotes the unique arc in F joining x and y.

Given an $m \in \mathbb{N}$, an $m-od$ is a fan for which $E(F)$ has exactly m elements.

A fan F is said to be *smooth* provided that whenever $\{x_i\}_{i=1}^{\infty}$ is a sequence in F converging to a point x of F, then the sequence of arcs $\{\tau x_i\}_{i=1}^{\infty}$ converges to the arc τx.

Cones. The *cone over a compactum* Y, denoted by $\text{Cone}(Y)$, is the quotient space $(Y \times I)/(Y \times \{1\})$ obtained from the Cartesian product $Y \times I$ by shrinking $Y \times \{1\}$ to a point v called the *vertex of the cone* [16, p. 41]; v always denotes the vertex of a cone. The *base of* $\text{Cone}(Y)$ is $\{(y,0) \mid y \in Y\}$, which we denote by $\mathcal{B}(Y)$.

3. Main Theorem

Given a fan F, let $\mathcal{G}(F)$ denote either of the hyperspaces 2^F, or $\mathcal{C}_n(F)$, or $\mathcal{F}_n(F)$, for $n \geq 2$. Let us note that the following result is already known for $\mathcal{C}_1(F)$ [12, 4.3].

Theorem 3.1. *If F is a fan which is homeomorphic to the cone over a compact metric space, then $\mathcal{G}(F)$ is homeomorphic to the cone over a continuum.*

Proof: Let F be a fan which is a cone. Then, F is smooth. We assume by ([4, Corollary 4, p. 90] and [2, Theorem 9, p. 27]) that F is embedded in $\mathbb{R}^2$, $\tau = (0,0)$ is the top of F, and the legs of F are convex arcs of length one [12, 4.2]. Given two points a and b of $\mathbb{R}^2$, $[a,b]$ denotes the convex arc in $\mathbb{R}^2$ whose end points are a and b, and $||a||$ denotes the norm of a in $\mathbb{R}^2$. Given an element A of $\mathcal{G}(F)$ and $r \geq 0$, $rA = \{ra \mid a \in A\}$. Note that for $r = 0$, $rA = \{(0,0)\} = \{\tau\}$.

Let $E(F) = \{e_\lambda\}_{\lambda \in \Lambda}$. Then, $F = \text{Cone}(E(F))$ by [12, 4.2]. Note that this equality implies that $E(F)$ is closed in F; hence, $E(F)$ is a compactum.

Let $\mathcal{B} = \bigcup\{\{A \in \mathcal{G}(F) \mid e_\lambda \in A\} \mid \lambda \in \Lambda\}$.

Let $\varphi \colon \mathcal{B} \times I \to \mathcal{G}(F)$ be given by

$$\varphi((A,t)) = (1-t)A.$$

Clearly, φ is well defined. Observe that if $t \in [0,1)$ and $A \in \mathcal{B}$, then $\tau \in \varphi((A,t))$ if and only if $\tau \in A$. We show that φ is continuous. Let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{2}$. Let $A, B \in \mathcal{G}(F)$ and $t, s \in [0,1]$

be such that $\mathcal{H}(A,B) < \delta$ and $|t-s| < \delta$. Let $a \in A$; then there exists $b \in B$ such that $||a-b|| < \delta$. Note that

$$||(1-t)a-(1-s)b|| \leq ||(1-t)a-(1-t)b|| + ||(1-t)b-(1-s)b|| \leq$$

$$(1-t)||a-b|| + |s-t|||b|| \leq ||a-b|| + |s-t| < 2\delta = \varepsilon.$$

So, $\varphi((A,t)) \subset \mathcal{V}_\varepsilon(\varphi((B,s)))$. Similarly, $\varphi((B,s)) \subset \mathcal{V}_\varepsilon(\varphi((A,t)))$. Therefore, $\mathcal{H}(\varphi((A,t)),\varphi((B,s))) < \varepsilon$ and φ is continuous.

We show that φ is one−to−one on $\mathcal{B}\times[0,1)$. Let $t,s \in [0,1)$, and let $A,B \in \mathcal{B}$. Suppose that $\varphi((A,t)) = \varphi((B,s))$. Since $A,B \in \mathcal{B}$, there exist $e_\lambda, e_{\lambda'} \in E(F)$ such that $e_\lambda \in A$ and $e_{\lambda'} \in B$.

Case (1). $\tau \notin A$. Then $\tau \notin B$. Let $[a,e_\lambda]$ be the component of A containing e_λ. Let $[b,e_{\lambda'}]$ be the component of B containing $e_{\lambda'}$. Since $\varphi((A,t)) = \varphi((B,s))$, there exist $[b_{\lambda'},c_{\lambda'}] \subset A$ and $[b_\lambda,c_\lambda] \subset B$ such that

$$[(1-t)a,(1-t)e_\lambda] = [(1-s)b_\lambda,(1-s)c_\lambda]$$

and

$$[(1-s)b,(1-s)e_{\lambda'}] = [(1-t)b_{\lambda'},(1-t)c_{\lambda'}].$$

From the first equality we obtain that $(1-t)e_\lambda = (1-s)c_\lambda$, which implies that $1-t = (1-s)||c_\lambda|| \leq 1-s$. From the second inequality we obtain that $(1-s)e_{\lambda'} = (1-t)c_{\lambda'}$, which implies that $1-s = (1-t)||c_{\lambda'}|| \leq 1-t$. Therefore, $t=s$. Hence, $A=B$.

Case (2). $\tau \in A$. Then, $\tau \in B$. Let us observe that either $[\tau,e_\lambda] \subset A$ or there exists $a \in A$ such that $[a,e_\lambda] \subset A$. In either case, as in Case (1), we conclude that $(1-t)e_\lambda = (1-s)c_\lambda$, for some $c_\lambda \in E(B)$, and that $(1-s)e_{\lambda'} = (1-t)c_{\lambda'}$, for some $c_{\lambda'} \in E(A)$. These two equalities imply that $t=s$. Hence, $A=B$.

We show that φ is onto. Let $B \in \mathcal{G}(F)$. If $B = \{\tau\}$, then $\varphi((A,1)) = \{\tau\}$ for any $A \in \mathcal{B}$. Thus, assume $B \neq \{\tau\}$. If $B \cap E(F) \neq \emptyset$, then $\varphi((B,0)) = B$.

Suppose $B \cap E(F) = \emptyset$. Let $t = \inf\{||b-e_\lambda|| \mid b \in B \text{ and } e_\lambda \in E(F)\}$. Since $B \neq \{\tau\}$, $t \neq 1$. Then there exists $\lambda_0 \in \Lambda$ such that $||b_{\lambda_0} - e_{\lambda_0}|| = t$, where $b_{\lambda_0} \in B \cap [\tau,e_{\lambda_0}]$.

Let $A = \frac{1}{1-t}B$. Note that for λ_0, $\frac{1}{1-t}b_{\lambda_0} \in A \cap [\tau,e_{\lambda_0}]$. Since $\frac{1}{1-t}b_{\lambda_0} = e_{\lambda_0}$, we have that $e_{\lambda_0} \in A$. Hence, $A \in \mathcal{B}$, and $\varphi((A,t)) = \frac{1-t}{1-t}B = B$.

By the Transgression Lemma [16, 3.22], the hyperspace $\mathcal{G}(F)$ is homeomorphic to Cone($\mathcal{B}$). Since no point of $\mathcal{G}(F)$ arcwise disconnects $\mathcal{G}(F)$ [15, (11.5)], we have that $\mathcal{B}$ is a continuum. □

Since, clearly, an $m-$od is a fan homeomorphic to the cone over a finite set, the following result answers [13, 3.8].

Corollary 3.2. *Let m and n be positive integers. If F is an m – od, then $\mathcal{G}(F) \in \{\mathcal{F}_n(F), \mathcal{C}_n(F)\}$ is homeomorphic to the cone over a finite – dimensional continuum.*

Question 3.3. Does there exist a hereditarily decomposable continuum X that is neither an arc nor an $m-$od such that $\mathcal{C}_n(X)$ is homeomorphic to the cone over a finite−dimensional continuum for some integer $n \geq 2$?

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