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CLASSIFICATION OF FINITE ALEXANDER QUANDLES

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ABSTRACT. Two finite Alexander quandles with the same number of elements are isomorphic iff their $\mathbb{Z}[t^{\pm 1}]$ -submodules $\text{Im}(1-t)$ are isomorphic as modules. This yields specific conditions on when Alexander quandles of the form $\mathbb{Z}_n[t^{\pm 1}]/(t-a)$ where $\gcd(n, a) = 1$ (called *linear* quandles) are isomorphic, as well as specific conditions on when two linear quandles are dual and which linear quandles are connected. We apply this result, obtaining a procedure for classifying Alexander quandles of any finite order, and as an application, we list the numbers of distinct and connected Alexander quandles with up to fifteen elements.

1. INTRODUCTION

In [4], D. Joyce defines the fundamental quandle, an algebraic invariant of knots which classifies classical knots. The set of quandles forms a category whose axioms are algebraic versions of the three Reidemeister moves. Quandles are useful both for defining new knot invariants (as in [1]) and for improving our understanding of old ones (see [2], for example).

The ability to distinguish quandles would allow us to distinguish knots. While there is not yet a complete classification theorem for general quandles, there are classification results for quandles of prime order [6] and for indecomposable quandles of prime squared order [3]. In this paper, we classify finite Alexander quandles by

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reducing the problem of comparing finite Alexander quandles to comparing certain $\mathbb{Z}[t^{\pm 1}]$ -submodules.

Definition 1.1. A *quandle* is a set X with a binary operation written as exponentiation satisfying

- (i) For every $a, b \in X$, there exists a unique $c \in X$ such that $a = c^b$;
- (ii) for every $a, b, c \in X$, we have $a^{bc} = a^{cb^c}$; and
- (iii) for every $a \in X$, we have $a^a = a$.

Any module over $\Lambda = \mathbb{Z}[t^{\pm 1}]$ is a quandle under the operation $a^b = ta + (1-t)b$. Quandles of this form are called *Alexander quandles*. To obtain finite Alexander quandles, we typically consider $\Lambda_n/(h)$ where $\Lambda_n = \mathbb{Z}_n[t^{\pm 1}]$ and h is a monic polynomial in t . In an earlier version of [6], the questions, when are two Alexander quandles of the form $\Lambda_n/(t-a)$ with $\gcd(n, a) = 1$ (we call Alexander quandles of this form *linear*) isomorphic, and when are two linear quandles dual, were posed.

To answer these questions, we first consider the general case of when two arbitrary Alexander quandles of finite cardinality are isomorphic. We obtain a result which reduces the problem of comparing Alexander quandles to comparing certain Λ -submodules. We then apply this result, obtaining a pair of simple conditions on a and b which are necessary and sufficient for two linear Alexander quandles $\Lambda_n/(t-a)$ and $\Lambda_n/(t-b)$ to be isomorphic.

In the course of answering the question of classifying linear quandles, we also answer the question of when are linear quandles dual, and we obtain results on when Alexander quandles are connected.

2. ALEXANDER QUANDLES AND Λ -MODULES

Since the quandle structure of an Alexander quandle is determined by its Λ -module structure, any isomorphism of Λ -modules is also an isomorphism of Alexander quandles. The converse is not true, however; $\Lambda_9/(t-4)$ is isomorphic to $\Lambda_9/(t-7)$ as an Alexander quandle but not as a Λ -module.

Nonetheless, an isomorphism of Alexander quandles is, in a sense, almost an isomorphism of Λ -modules; in fact (after applying a shift, if necessary), the restriction of a quandle isomorphism $f : M \rightarrow N$ to the submodule $(1-t)M$ is a Λ -module isomorphism onto the

image of the restriction. Theorem 2.1 says that the converse is true as well; that is, we can determine whether two Alexander quandles of the same finite cardinality are isomorphic simply by comparing these Λ -submodules. This reduces the problem of classifying finite Alexander quandles to comparing Λ -modules of the form $(1 - t)M$.

Theorem 2.1. *Two finite Alexander quandles M and N of the same cardinality are isomorphic as quandles iff there is an isomorphism of Λ -modules $h : (1 - t)M \rightarrow (1 - t)N$.*

Proof: Let M and N be finite Alexander quandles and $f : M \rightarrow N$ a quandle isomorphism. We may assume without loss of generality that $f(0) = 0$ since $f' : M \rightarrow N$ defined by $f'(x) = f(x) + c$ is also an isomorphism of Alexander quandles for any $c \in N$. Then $f(tx + (1 - t)y) = tf(x) + (1 - t)f(y)$ implies

$$f(tx) = f(tx + (1 - t)0) = tf(x) + (1 - t)f(0) = tf(x)$$

and

$$f((1 - t)y) = f(t0 + (1 - t)y) = tf(0) + (1 - t)f(y) = (1 - t)f(y)$$

so that

$$(2.1) \quad f(tx + (1 - t)y) = f(tx) + f((1 - t)y).$$

Denote $M' = (1 - t)M$ and $N' = (1 - t)N$. Since $t^{-1} \in \Lambda$, every element of M is tx for some $x \in M$, and since $f(0) = 0$, f takes the coset $0 + M'$ of M' in $\bar{M} = M/M'$ to the coset $0 + N'$ of N' in $\bar{N} = N/N'$, so we have that $h = f|_{M'} : M' \rightarrow N'$ is a homomorphism of Λ -modules. Since f is injective, its restriction h is a bijection onto its image $0 + N' = N'$, and hence, h is an isomorphism of Λ -modules.

Conversely, suppose $h : M' \rightarrow N'$ is an isomorphism of finite Λ -modules with $|M| = |N|$. Let $A \subset M$ be a set of representatives of cosets of M' in M . Then every $m \in M$ has the form $m = \alpha + \omega$ for a unique $\alpha \in A$ and $\omega \in M'$. We will show that there exists a bijection $k : A \rightarrow B$ onto a set B of representatives of cosets of N' in \bar{N} such that the map $f : M \rightarrow N$ defined by

$$f(\alpha + \omega) = k(\alpha) + h(\omega)$$

is an isomorphism of Alexander quandles (though typically not of Λ -modules).

Let $\alpha_1, \alpha_2 \in A$ and $\omega_1, \omega_2 \in (1-t)M$. For any $\alpha_1 \in A$, we have $t\alpha_1 = \alpha_1 - (1-t)\alpha_1$, so that

$$\begin{aligned} & f(t(\alpha_1 + \omega_1) + (1-t)(\alpha_2 + \omega_2)) \\ &= f(\alpha_1 + t\omega_1 + (1-t)(\alpha_2 - \alpha_1 + \omega_2)) \\ &= k(\alpha_1) + h(t\omega_1 + (1-t)(\alpha_2 - \alpha_1 + \omega_2)) \\ &= k(\alpha_1) + th(\omega_1) + h((1-t)\alpha_2) \\ &\quad - h((1-t)\alpha_1) + (1-t)h(\omega_2). \end{aligned}$$

On the other hand,

$$\begin{aligned} & tf(\alpha_1 + \omega_1) + (1-t)f(\alpha_2 + \omega_2) \\ &= t(k(\alpha_1) + h(\omega_1)) + (1-t)(k(\alpha_2) + h(\omega_2)) \\ &= tk(\alpha_1) + th(\omega_1) + (1-t)k(\alpha_2) + (1-t)h(\omega_2). \end{aligned}$$

So for f to be a homomorphism of quandles, it is sufficient that

$$(2.2) \quad (1-t)k(\alpha_1) - h((1-t)\alpha_1) = (1-t)k(\alpha_2) - h((1-t)\alpha_2)$$

for all $\alpha_1, \alpha_2 \in A$. We will show that, given a set of coset representatives $A \subset M$, we can choose a set $B \subset N$ of coset representatives and a bijection $k : A \rightarrow B$ so that $(1-t)k(\alpha) = h((1-t)\alpha)$ for all $\alpha \in A$, which satisfies (2.2) and thus yields a homomorphism $f : M \rightarrow N$ of Alexander quandles. Since this f is set-wise the Cartesian product $k \times h$ of the bijections $k : A \rightarrow B$ and $h : M' \rightarrow N'$, f is bijective and, hence, an isomorphism of quandles.

Denote $M'' = (1-t)^2M$, $\bar{M} = M'/M''$ and similarly for N . The isomorphism $h : M' \rightarrow N'$ induces an isomorphism $\bar{h} : \bar{M} \rightarrow \bar{N}$. There are surjective maps $\psi : \bar{M} \rightarrow \bar{M}$ and $\phi : \bar{N} \rightarrow \bar{N}$ induced by multiplication by $(1-t)$. Then $|M'| = |N'|$ and $|M| = |N|$ imply that $|\bar{M}| = |\bar{N}|$, and, in turn, $|\bar{M}| = |\bar{N}|$.

Since $\psi(y) = \psi(0 + y) = \psi(0) + \psi(y)$ implies that $|\psi^{-1}(y)| \geq |\psi^{-1}(0)|$, since for each element of $\psi^{-1}(0)$ there is an element of $\psi^{-1}(y)$. Similarly, $\psi(0) = \psi(y) + \psi(-y)$ implies that $|\psi^{-1}(0)| \geq |\psi^{-1}(y)|$. Hence, $|\psi^{-1}(y)| = |\psi^{-1}(0)| = |\psi^{-1}(y')|$ for all $y, y' \in \bar{M}$, $|\psi^{-1}(y)| = |\bar{M}|/|\bar{M}|$ for all $y \in \bar{M}$. Likewise, $|\phi^{-1}(h(y))| = |\bar{N}|/|\bar{N}| = |\bar{M}|/|\bar{M}|$ for all $y \in \bar{M}$. Thus, there is a bijection of sets

$g : \bar{M} \rightarrow \bar{N}$ such that the diagram

$$\begin{array}{ccc} \bar{M} & \xrightarrow{g} & \bar{N} \\ \psi \downarrow & & \downarrow \phi \\ \bar{\bar{M}} & \xrightarrow{\bar{h}} & \bar{\bar{N}} \end{array}$$

commutes.

Let B be a set of coset representatives for \bar{N} . Then there is a unique bijection $k : A \rightarrow B$ such that

$$\begin{array}{ccc} A & \xrightarrow{k} & B \\ \downarrow & & \downarrow \\ \bar{M} & \xrightarrow{g} & \bar{N} \\ \psi \downarrow & & \downarrow \phi \\ \bar{\bar{M}} & \xrightarrow{\bar{h}} & \bar{\bar{N}} \end{array}$$

commutes. In particular, $\bar{h}\psi(\alpha) = \phi k(\alpha)$, that is,

$$(2.3) \quad \bar{h}((1-t)\alpha + (1-t)^2M) = (1-t)k(\alpha) + (1-t)^2N.$$

Define $\gamma : M' \rightarrow \bar{\bar{M}}$ and $\epsilon : N' \rightarrow \bar{\bar{N}}$ by $\gamma((1-t)m) = (1-t)m + (1-t)^2M \in \bar{\bar{M}}$ and $\epsilon((1-t)n) = (1-t)n + (1-t)^2N \in \bar{\bar{N}}$, the classes of $(1-t)m$ and $(1-t)n$ in $\bar{\bar{M}}$ and $\bar{\bar{N}}$, respectively. We then have commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{(1-t)} & M' \\ \downarrow & & \downarrow \gamma \\ \bar{M} & \xrightarrow{\psi} & \bar{\bar{M}} \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{(1-t)} & N' \\ \downarrow & & \downarrow \epsilon \\ \bar{N} & \xrightarrow{\phi} & \bar{\bar{N}}. \end{array}$$

Equation (2.3) then says that outside rectangle of the diagram

$$\begin{array}{ccc} A & \xrightarrow{k} & B \\ (1-t) \downarrow & & \downarrow (1-t) \\ M' & \xrightarrow{h} & N' \\ \gamma \downarrow & & \downarrow \epsilon \\ \bar{\bar{M}} & \xrightarrow{\bar{h}} & \bar{\bar{N}} \end{array}$$

commutes. The bottom square commutes by definition of \bar{h} , and thus we have $\epsilon(h((1-t)\alpha)) = \epsilon((1-t)k(\alpha))$, that is,

$$h((1-t)\alpha) + (1-t)^2N = (1-t)k(\alpha) + (1-t)^2N.$$

In particular, there is a $\xi \in N$ so that

$$h((1-t)\alpha) = (1-t)k(\alpha) + (1-t)^2\xi = (1-t)(k(\alpha) + (1-t)\xi).$$

Then for each $\alpha \in A$ with $\xi \neq 0$ we may replace $k(\alpha)$ with the coset representative $k'(\alpha) = k(\alpha) + (1-t)\xi$ to obtain a new set B' of coset representatives for \bar{N} and a bijection $k' : A \rightarrow B'$ with $(1-t)k'(\alpha) = h((1-t)\alpha)$ so that (2.2) is satisfied. Then $f : M \rightarrow N$ by $f(\alpha + \omega) = k'(\alpha) + h(\omega)$ for all $\alpha \in A$ is an isomorphism of Alexander quandles, as required. \square

As a consequence, we obtain Corollary 2.2, which gives specific conditions on a and b for $\Lambda_n/(t-a)$ and $\Lambda_n/(t-b)$ to be isomorphic Alexander quandles when a and b are coprime to n .

Denote $N(n, a) = \frac{n}{\gcd(n, 1-a)}$ for any $a \in \mathbb{Z}_n$. We will use the symbol \cong to denote an isomorphism of quandles and \approx to denote an isomorphism of Λ -modules.

Corollary 2.2. *Let a and b be coprime to n . Then the Alexander quandles $\Lambda_n/(t-a)$ and $\Lambda_n/(t-b)$ are isomorphic iff $N(n, a) = N(n, b)$ and $a \equiv b \pmod{N(n, a)}$.*

Proof: By Theorem 2.1,

$$\Lambda_n/(t-a) \cong \Lambda_n/(t-b) \iff (1-t)[\Lambda_n/(t-a)] \approx (1-t)[\Lambda_n/(t-b)].$$

As \mathbb{Z} -modules, $(1-t)[\Lambda_n/(t-a)]$ is $(1-a)\mathbb{Z}_n$ and $(1-t)[\Lambda_n/(t-b)]$ is $(1-b)\mathbb{Z}_n$, with the action of t given by multiplication by a in $(1-a)\mathbb{Z}_n$ and by b in $(1-b)\mathbb{Z}_n$.

The \mathbb{Z} -module $(1-a)\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_n/\text{Ann}(1-a)$, so

$$\begin{aligned} \Lambda_n/(t-a) \cong \Lambda_n/(t-b) &\iff \mathbb{Z}_n/\text{Ann}(1-a) \approx \mathbb{Z}_n/\text{Ann}(1-b) \\ &\iff \text{Ann}(1-a) = \text{Ann}(1-b) \\ &\iff \text{Ord}_{\mathbb{Z}_n}(1-a) = \text{Ord}_{\mathbb{Z}_n}(1-b) \\ &\iff \frac{n}{\gcd(n, 1-a)} = \frac{n}{\gcd(n, 1-b)} \\ &\iff N(n, a) = N(n, b). \end{aligned}$$

Denote $n' = N(n, a) = N(n, b)$. Then $(1-t)[\Lambda_n/(t-a)]$ is $\mathbb{Z}_{n'}$ with t acting by multiplication by a , and if $N(n, a) = N(n, b) = n'$ then $(1-t)[\Lambda_n/(t-b)]$ is $\mathbb{Z}_{n'}$ with t acting by multiplication by b .

Multiplication by a agrees with multiplication by b on $\mathbb{Z}_{n'}$ iff $a \equiv b \pmod{n'}$, so the Λ -module structures on $\mathbb{Z}_{n'}$ determined by a and b agree iff $a \equiv b \pmod{n'}$. \square

Definition 2.3. A quandle M is *connected* if it has only one orbit, i.e., if the set $\{a^b : b \in M\} = M$ for all $a \in M$. In particular, an Alexander quandle is connected if $(1 - t)M = M$.

Corollary 2.4. *Two finite connected Alexander quandles are isomorphic iff they are isomorphic as Λ -modules.*

Proof: This follows from the proof of Theorem 2.1. Specifically, if M and N are connected and $f : M \rightarrow N$ is an isomorphism of quandles with $f(0) = 0$, then f is an isomorphism of Λ -modules. \square

Corollary 2.5. *A linear Alexander quandle $\Lambda_n/(t - a)$ is connected iff $\gcd(n, 1 - a) = 1$.*

Proof: An Alexander quandle is connected iff $M = (1 - t)M$. Since $(1 - t)[\Lambda_n/(t - a)]$ is $\mathbb{Z}_{N(n,a)}$; with t acting by multiplication by a , we have $\Lambda_n/(t - a)$ is connected iff $N(n, a) = n$, that is, iff $\gcd(n, 1 - a) = 1$. \square

Corollary 2.6. *No linear Alexander quandle $\Lambda_n/(t - a)$ with n even is connected.*

Proof: To have a linear quandle $\Lambda_n/(t - a)$ with n elements, we must have $\gcd(n, a) = 1$, so if n is even, a must be odd. But then $1 - a$ is even and $\gcd(n, 1 - a) \neq 1$, and $\Lambda_n/(t - a)$ is not connected. \square

For each $y \in X$ we can define a map of sets $f_y : X \rightarrow X$ by $f_y(x) = x^y$. Quandle axiom (i) then says that f_y is a bijection for each $y \in X$. We may then define a new quandle structure on X by $x^{\bar{y}} = f_y^{-1}(x)$; this is the *dual* quandle of X .

Lemma 2.7. *The dual of an Alexander quandle X is the set X with quandle operation given by $x^{\bar{y}} = t^{-1}x + (1 - t^{-1})y$.*

Proof: If $f_y(x) = c = tx + (1 - t)y$ then $t^{-1}c = x + (t^{-1} - 1)y \Rightarrow x = t^{-1}c + (1 - t^{-1})y$; thus, $f_y^{-1}(x) = t^{-1}x + (1 - t^{-1})y$. \square

Corollary 2.8. *Let a, b be coprime to n . Then $\Lambda_n/(t - a)$ is dual to $\Lambda_n/(t - b)$ iff $N(n, a) = N(n, b)$ and $ab \equiv 1 \pmod{N(n, a)}$. In particular, a linear Alexander quandle $\Lambda_n/(t - a)$ is self-dual iff a is a square mod $N(n, a)$.*

Proof: If n and a are coprime, then a is invertible in \mathbb{Z}_n and the dual of $\Lambda_n/(t - a)$ is given by $\Lambda_n/(t - a^{-1})$ by Lemma 2.7. Then

Corollary 2.2 says that $\Lambda_n/(t-b)$ is isomorphic to $\Lambda_n/(t-a^{-1})$ iff $N(n,b) = N(n,a)^{-1}$ and $b \equiv a^{-1} \pmod{N(n,b)}$.

Since $\gcd N(n,a) = 1$, we have $\gcd(n, 1-a) = \gcd N(n, -a(1-a^{-1})) = \gcd N(n, 1-a^{-1})$, so that $N(n,a) = N(n,a^{-1})$, as required. \square

3. \mathbb{Z} -AUTOMORPHISMS AND COMPUTATIONS

Let X be a finite Alexander quandle and let X_A denote X regarded as an Abelian group, called the *underlying Abelian group* of X . The map $\phi : X_A \rightarrow X_A$ defined by $\phi(x) = tx$ is a homomorphism of \mathbb{Z} -modules. Since $t^{-1} \in \Lambda$, the map $\psi : X_A \rightarrow X_A$ defined by $\psi(x) = t^{-1}x$ is a two-sided inverse for ϕ as $\psi(\phi(x)) = t^{-1}tx = x$ and $\phi(\psi(x)) = tt^{-1}x = x$, and ϕ is in fact a \mathbb{Z} -automorphism.

Conversely, if A is a finite Abelian group and $\phi : A \rightarrow A$ is a \mathbb{Z} -module automorphism, we can give A the structure of an Alexander quandle by defining $tx = \phi(x)$. This yields a general strategy for listing all finite Alexander quandles of a given size n : first, list all Abelian groups A of order n ; then, for each element of $\text{Aut}_{\mathbb{Z}}(A)$ find $(1-t)A = \text{Im}(1-\phi)$ and compare these as Λ -modules. In practice, for low order (i.e., $|A| \leq 15$) Alexander quandles, this procedure in its full generality is necessary only for one case, namely Alexander quandles with underlying Abelian group isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. We shall see that Alexander quandles with $X_A \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$ are isomorphic to linear Alexander quandles (in six cases) or to Alexander quandles with underlying group $(\mathbb{Z}_2)^3$ (in two cases).

We first obtain a few simplifying results:

Lemma 3.1. *If the underlying Abelian group X_A of X is cyclic, then X is linear.*

Proof: Suppose $X_A = \mathbb{Z}_n$. Then for any $x \in \mathbb{Z}_n$ and any $\phi \in \text{Aut}_{\mathbb{Z}}(\mathbb{Z}_n)$, we must have $\phi(x) = \phi(x \cdot 1) = x\phi(1)$, so the action of t agrees with multiplication by $a = \phi(1)$ on \mathbb{Z}_n . Further, we must have $\gcd(n, a) = 1$ since ϕ is surjective. Hence, X is \mathbb{Z}_n with t acting by multiplication by a , that is, $X \cong \Lambda_n/(t-a)$. \square

Remark 3.2. Lemma 3.1 is also noted in [6].

Corollary 3.3. *For any prime p , there are exactly $p-1$ distinct Alexander quandles with p elements, namely $\Lambda_p/(t-a)$ for $a = 1, \dots, p-1$. Further, every Alexander quandle of prime order is*

either trivial ($\Lambda_p/(t-1) \cong T_p$, the trivial quandle of p elements) or connected.

Proof: If p is prime, $N(n, a) = \frac{n}{\gcd(p, 1-a)} = 1$ for each $a \in 1, \dots, p-1$. Then by Corollary 2.2, these are all distinct. By Lemma 3.1, every quandle of order p is linear, so these are all of the Alexander quandles of order p .

Since $\gcd(p, 1-a) = 1$ for $a = 2, \dots, p-1$, Corollary 2.5 gives us that $\Lambda_p/(t-a)$ is connected. \square

Corollary 3.4. *Let $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ be a product of powers of distinct primes. Then there are exactly $N_{p_1} N_{p_2} \dots N_{p_k}$ distinct Alexander quandles of order n , where N_{p_i} is the number of distinct Alexander quandles of order $p_i^{e_i}$.*

Proof: Since any \mathbb{Z} -automorphism must respect order, any Alexander quandle structure on a direct sum of Abelian groups $A_{p_1^{e_1}} \oplus \dots \oplus A_{p_k^{e_k}}$ with order $p_1^{e_1}, \dots, p_k^{e_k}$ must respect this direct sum structure. Hence, we may obtain a complete list of Alexander quandles of order n by listing all direct sums of Alexander quandles of orders $p_1^{e_1}, \dots, p_k^{e_k}$. \square

Corollary 3.5. *If the order of an Alexander quandle $n \equiv 2 \pmod{4}$, the quandle is not connected.*

Proof: If $n \equiv 2 \pmod{4}$, then the underlying Abelian group of the quandle has a summand of \mathbb{Z}_2 . Hence, the quandle has a summand isomorphic to $\Lambda_2/(t+1) \cong T_2$ and, therefore, is not connected. \square

In light of Corollary 3.4, to classify finite Alexander quandles, it is sufficient to consider Alexander quandles of prime power order. Alexander quandles with prime order are cyclic as Abelian groups and, hence, are linear quandles and are classified by Corollary 3.3. Alexander quandles whose orders are products of distinct primes are classified by Corollary 3.4.

If the underlying Abelian group of X is $(\mathbb{Z}_p)^n$, then X is not only a Λ -module but also a Λ_p -module, so we may use the classification theorem for finitely generated modules over a PID. Thus, any Alexander quandle X with $X_A = (\mathbb{Z}_p)^n$ must be of the form $\Lambda_p/(h_1) \oplus \dots \oplus \Lambda_p/(h_k)$ with $h_1|h_2|\dots|h_k$, $h_i \in \Lambda_p$ and $\sum \deg(h_i) = n$. We may further assume without loss of generality that each $h_i \in \mathbb{Z}_p[t]$ is monic and has nonzero constant term.

Proposition 3.6. *An Alexander quandle $M = \Lambda_p/(h)$, p a prime, is connected iff $(1 - t) \nmid h$.*

Proof: Since M is finite, $(1 - t)M = M$ iff $(1 - t) : M \rightarrow M$ is bijective. If $(1 - t) \mid h$ then $h = (1 - t)g$ for some nonzero $g \in M$, and, hence, $\ker(1 - t) \neq \{0\}$, so $(1 - t)$ fails to be injective.

Conversely, $(1 - t)$ is prime in Λ , so $(1 - t)$ coprime to h implies that every $l \in \Lambda$ may be written as $a(1 - t) + bh$ for some $a, b \in \Lambda$. Hence, every $m \in M$ is $a(1 - t)$ for some $a \in M$. \square

Proposition 3.7. *The Alexander quandle $\Lambda_{p^n} / (t^n + \sum_{i=0}^{n-1} a_i t^i)$ is connected iff $\sum_{i=0}^{n-1} a_i = -1$.*

Proof: By 3.6, $\Lambda_{p^n} / (t^n + \sum_{i=0}^{n-1} a_i t^i)$ is connected iff $(t - 1) \mid t^n + \sum_{i=0}^{n-1} a_i t^i$. That is, $\Lambda_{p^n} / (t^n + \sum_{i=0}^{n-1} a_i t^i)$ is connected iff there are $b_i \in \Lambda_p$, $0 \leq i \leq n - 2$ such that

$$(t - 1) \left(t^{n-1} + \sum_{i=0}^{n-2} b_i t^i \right) = t^n + \sum_{i=0}^{n-1} a_i t^i.$$

Comparing coefficients, we must have that $a_{n-1} + b_{n-2} = -1$, $b_i = a_i + b_{i-1}$ for all $1 \leq i \leq n - 2$, and $b_0 = a_0$. Then, $\sum_{i=0}^{n-1} a_i = -1$. Conversely, if $\sum_{i=0}^{n-1} a_i = -1$, define $b_0 = a_0$, $b_i = a_i + b_{i-1}$ for all $1 \leq i \leq n - 2$, and $a_{n-1} + b_{n-2} = -1$. \square

Proposition 3.8. *There are $2p^2 - 3p - 1$ connected Alexander quandles of order p^2 where p is prime.¹*

Proof: Every Alexander quandle of order p^2 has underlying Abelian group \mathbb{Z}_{p^2} or $\mathbb{Z}_p \oplus \mathbb{Z}_p$. A linear quandle $\Lambda_{p^2} / (t - a)$ of order p^2 is connected iff $\gcd(1 - a, p) = 1$ and there are $p(p - 2)$ such quandles.

An Alexander quandle M with underlying Abelian group $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is a module over the PID Λ_p , so we have either $M \cong \Lambda_p / (t - a) \oplus \Lambda_p / (t - a)$ or $M \cong \Lambda_p / (t^2 + at + b)$ where $b \neq 0$. There are $p - 2$ connected quandles of the first type and $(p - 1)^2$ of the second type, so in total there are $2p^2 - 3p - 1$ connected Alexander quandles of order p^2 . \square

¹This agrees with the result of M. Graña in [3].

| X_A | Module | $\text{Im}(1-t)$ |
|--------------------|--|-------------------------|
| $(\mathbb{Z}_2)^2$ | $(\Lambda_2/(t+1))^2$ | 0 |
| | $\Lambda_2/(t^2+1)$ | $\Lambda_2/(t+1)$ |
| | $\Lambda_2/(t^2+t+1)$ | $\Lambda_2/(t^2+t+1)$ |
| $(\mathbb{Z}_2)^3$ | $(\Lambda_2/(t+1))^3$ | 0 |
| | $\Lambda_2/(t+1) \oplus \Lambda_2/(t^2+1)$ | $\Lambda_2/(t+1)$ |
| | $\Lambda_2/(t^3+1)$ | $\Lambda_2/(t^2+t+1)$ |
| | $\Lambda_2/(t^3+t+1)$ | $\Lambda_2/(t^3+t+1)$ |
| | $\Lambda_2/(t^3+t^2+1)$ | $\Lambda_2/(t^3+t^2+1)$ |
| | $\Lambda_2/(t^3+t^2+t+1)$ | $\Lambda_2/(t^2+1)$ |
| $(\mathbb{Z}_3)^2$ | $(\Lambda_3/(t+2))^2$ | 0 |
| | $(\Lambda_3/(t+1))^2$ | $(\Lambda_3/(t+1))^2$ |
| | $\Lambda_3/(t^2+2)$ | $\Lambda_3/(t+1)$ |
| | $\Lambda_3/(t^2+1)$ | $\Lambda_3/(t^2+1)$ |
| | $\Lambda_3/(t^2+2t+2)$ | $\Lambda_3/(t^2+2t+2)$ |
| | $\Lambda_3/(t^2+2t+1)$ | $\Lambda_3/(t^2+2t+1)$ |
| | $\Lambda_3/(t^2+t+2)$ | $\Lambda_3/(t^2+t+2)$ |
| | $\Lambda_3/(t^2+t+1)$ | $\Lambda_3/(t+2)$ |

TABLE 1. Computations of $\text{Im}(1-t)$ for $(\mathbb{Z}_2)^2$, $(\mathbb{Z}_2)^3$ and $(\mathbb{Z}_3)^2$.

For arbitrary values of n and p we may classify Alexander quandles with underlying Abelian group $(\mathbb{Z}_p)^n$ listing all possible Λ -modules with underlying group $(\mathbb{Z}_p)^n$ and comparing the submodules $\text{Im}(1-t)$.

Results of applying this procedure to Alexander quandles with underlying Abelian group $(\mathbb{Z}_2)^2$, $(\mathbb{Z}_2)^3$ and $(\mathbb{Z}_3)^2$ are collected in Table 1. As we expected, these results agree with Proposition 3.8.

Note that by Theorem 2.1 and Corollary 2.2, the results in Table 1 show that $\Lambda_2/(t^2+1) \cong \Lambda_4/(t-3)$ and $(\Lambda_2/(t+1))^2 \cong \Lambda_4/(t-1) \cong T_4$, the trivial quandle of order 4, while $\Lambda_2/(t^2+t+1)$ is the only connected Alexander quandle of order 4.

Alexander quandles with underlying Abelian group $(\mathbb{Z}_2)^3$ include $\Lambda_2/(t+1) \oplus \Lambda_2/(t^2+1) \cong \Lambda_8/(t-5)$ and $(\Lambda_2/(t+1))^3 \cong T_8$. Also, Theorem 2.1 yields an isomorphism $\Lambda_8/(t-3) \cong \Lambda_8/(t-7)$; otherwise, the order eight quandles listed are all distinct. Of these,

only $\Lambda_2/(t^3 + t^2 + 1)$ and $\Lambda_2/(t^3 + t + 1)$ are connected. Note that none of the linear Alexander quandles of order eight are connected.

Among Alexander quandles with Abelian group $(\mathbb{Z}_3)^2$, we have $\Lambda_9/(t - 4) \cong \Lambda_9/(t - 7) \cong \Lambda_9/(t^2 + t + 1)$ (the first isomorphism was noted in [1] and the second also follows from Proposition 4.1 of [5]); otherwise, the linear quandles of order nine and the quandles listed in Table 1 are all distinct. Note that five of the eight listed quandles of order nine are connected; of the linear quandles of order nine, $\Lambda_9/(t - 2)$, $\Lambda_9/(t - 5)$ and $\Lambda_9/(t - 8)$ are connected.

To count distinct Alexander quandles whose underlying Abelian group is neither cyclic nor a direct sum of n copies of \mathbb{Z}_p , the following observation is useful.

Lemma 3.9. *The number of conjugacy classes in $\text{Aut}_{\mathbb{Z}}(X_A)$ is an upper bound on the number of distinct Alexander quandles X with underlying Abelian group X_A .*

Proof: Let $\phi_1, \phi_2 \in \text{Aut}_{\mathbb{Z}}X_A$. Then if $t_1 = \phi_1(1)$ and $t_2 = \phi_2(1)$, we have $\phi_2^{-1}\phi_1\phi_2$ acting by multiplication by $t_2^{-1}t_1t_2 = t_1$, since multiplication in Λ is commutative. Thus, any two conjugate automorphisms define the same Alexander quandle structure. \square

To complete the classification of Alexander quandles with up to fifteen elements, we now only need to consider the case $X_A = \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

Proposition 3.10. *There are three distinct Alexander quandle structures definable on the Abelian group $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, given by \mathbb{Z} -automorphisms $\phi_1 = \text{id}$, $\phi_2((1, 0)) = (1, 1)$, $\phi_2((0, 1)) = (0, 1)$, $\phi_3((1, 0)) = (1, 1)$, and $\phi_3((0, 1)) = (2, 1)$. Further, these quandles are isomorphic to previously listed quandles, namely $(\mathbb{Z}_4 \oplus \mathbb{Z}_2, \phi_1) \cong T_8$, $(\mathbb{Z}_4 \oplus \mathbb{Z}_2, \phi_2) \cong \Lambda_2/(t + 1) \oplus \Lambda_2/(t^2 + 1)$, and $(\mathbb{Z}_4 \oplus \mathbb{Z}_2, \phi_3) \cong \Lambda_2/(t^3 + t^2 + t + 1)$.*

Proof: Direct calculation shows that $\text{Aut}_{\mathbb{Z}}(\mathbb{Z}_4 \oplus \mathbb{Z}_2) \cong D_8$, the dihedral group of order eight, so by Lemma 3.9, there are at most five Alexander quandle structures on $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. Of the eight \mathbb{Z} -automorphisms of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, one is the identity, yielding the trivial quandle structure; five have $\text{Im}(1 - t) \cong \Lambda_2/(t + 1)$ (including ϕ_2) and, hence, yield quandles isomorphic to $\Lambda_2/(t+1) \oplus \Lambda_2/(t^2+1)$; and two have $\text{Im}(1 - t) \cong \Lambda_2/(t^2 + 1)$ (including ϕ_3), yielding quandles isomorphic to $\Lambda_2/(t^3 + t^2 + t + 1)$. \square

| n | # of Alexander quandles | # connected |
|-----|----------------------------|----------------|
| 2 | 1 | 0 |
| 3 | 2 | 1 |
| 4 | 3 | 1 |
| 5 | 4 | 3 |
| 6 | 2 | 0 |
| 7 | 6 | 5 |
| 8 | 7 | 2 |
| 9 | 11 | 8 |
| 10 | 4 | 0 |
| 11 | 10 | 9 |
| 12 | 6 | 1 |
| 13 | 12 | 11 |
| 14 | 6 | 0 |
| 15 | 8 | 3 |

TABLE 2. The number of Alexander quandles and connected Alexander quandles of size $n \leq 15$.

We now have enough information to determine all Alexander quandles with up to fifteen elements. In light of corollaries 3.3 and 3.4, we list in Table 2 only the numbers of distinct and connected Alexander quandles of each order.

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