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DENSE $\mathbb{R}^n$'S IN n-MANIFOLDS

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Abstract. We prove that every separable connected n-manifold has a dense subspace homeomorphic to $\mathbb{R}^n$, answering a question asked by P. Nyikos during the 1998 Spring Topology Conference in Fairfax, Virginia. We also prove a stronger version of this fact for the case of compact manifolds.

In 1998, P. Nyikos asked whether every compact connected n-manifold contained a dense subspace homeomorphic to $\mathbb{R}^n$. The purpose of this short note is to prove that this is indeed the case even in the more general class of separable connected manifolds. We also consider the question of whether it is possible to construct this imbedding in such a way that it could be extended to the boundary of the unit n-ball (where $B^n \cong \mathbb{R}^n$ is a natural subspace) in the case of compact connected manifolds. The latter question was suggested by Jeff Norden.

We use standard topological notation and facts (see [4]). B^n will denote the open unit ball $\{ x \in \mathbb{R}^n : \|x\| < 1 \}$. If X is a topological space, $X^{(n)}$ will stand for the set of points of X with euclidean neighborhoods of dimension n; $\overline{A}$ will denote the closure of A. All manifolds are topological (i.e., carry no extra structure), possibly with boundary; for a manifold M, the submanifold of its boundary points is denoted by ∂M. We do not assume any separation axioms in section 1. Elsewhere all spaces are metrizable.

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1. Noncompact case

In this section we provide a simple answer to Nyikos's question. The following simple lemma is part of mathematical folklore, but we give its proof here for the reader's convenience.

Lemma 1. *Let M be an n-manifold, let $V \subseteq M$ be homeomorphic to B^n, and let $x, y \in V$. Then there exists a homeomorphism $s_{x,y} : M \to M$ such that $s_{x,y}(y) = x$, and $s_{x,y}|_{M \setminus V} = \mathrm{id}|_{M \setminus V}$.*

Proof: By the invariance of domain (see [4, Theorem 4.8.16]), V is an open subset of M. By shrinking V we may assume that $\overline{V}$ is homeomorphic to the closed unit ball. Now use the geometrically obvious fact that any two points in the interior of the closed unit ball can be swapped by a homeomorphism which does not move the boundary. □

Lemma 2. *Let M be a connected n-manifold without boundary, and let $i : B^n \to M$ be an imbedding. Then for any $0 < \varepsilon < 1$, any $x \in M$ there exists an imbedding $j_{i,\varepsilon,x} : B^n \to M$ such that $x \in j_{i,\varepsilon,x}(B^n)$ and $j_{i,\varepsilon,x}|_{B_\varepsilon} = i|_{B_\varepsilon}$, where $B_\varepsilon = \{ x \in B^n : \|x\| \leq 1 - \varepsilon \}$.*

Proof: Consider the set

$$S = \{ y \in M : \text{there exists a } j_{i,\varepsilon,y} \text{ as above} \}.$$

Obviously, S is open. We will show that S is closed, thus finishing the proof. Indeed, let $x \in \overline{S}$. If $x \in i(B_\varepsilon)$, then there is nothing to prove; otherwise, choose a $x \in V \subseteq M$ homeomorphic to B^n such that $V \cap i(B_\varepsilon) = \emptyset$. Pick a $y \in \operatorname{Int} V \cap S$ and consider $j_{i,\varepsilon,x} = s_{x,y} \circ j_{i,\varepsilon,y}$, where $s_{x,y}$ is as in Lemma 1. It is easy to check that $j_{i,\varepsilon,x}$ is as desired. □

Theorem 1. *Let X be a topological space such that $X^{(n)}$ is dense in X, and is separable and connected. Then there exists a dense open $D \subseteq X$ such that D is homeomorphic to $\mathbb{R}^n$.*

Proof: By passing to $X^{(n)}$, it is enough to prove the theorem in the case of X being a separable connected n-manifold without boundary. Let $\langle r_n : n \in \omega \rangle$ be a countable dense subset of X. By induction, one can build sequences $\langle j_n : n \in \omega \rangle$ and $\varepsilon_n \to 0$ such that the following properties hold (we use the notation introduced in Lemma 2):

(1) $j_0 : B^n \to X$ is an imbedding such that $r_0 \in j_0(B_{\varepsilon_0})$;
(2) $j_n = j_{j_{n-1},\varepsilon_n,r_n}$; and
(3) $r_n \in j_n(B_{\varepsilon_{n+1}})$.

Now define $i(x) = j_n(x)$ provided $x \in B_{\varepsilon_{n+1}}$. The properties of $j_{j,\varepsilon,x}$ and (2) imply that i is 1–1 and continuous. Now (3) implies that $i : B^n \to X$ is a dense imbedding. The fact that $i(B^n)$ is open follows from the invariance of domain. □

Corollary 1. *A connected n-manifold M is separable iff it contains a dense subspace homeomorphic to* $\mathbb{R}^n$*.*

2. Compact case

Let us now consider the following stronger version of the question discussed in the previous section.

Question 1. Let M be a compact connected n-manifold. Is it possible to find a mapping $f : \overline{B^n} \to M$ such that f is onto, and $f \mid B^n$ is 1–1?

First, one should note that the construction in the previous section does not necessarily lead to such a mapping. Indeed, the subspace of S^2 obtained by throwing out a space homeomorphic to the graph of $\sin 1/x$, $x \in (0, 1]$, together with the "limit segment" $\{0\} \times [0, 1]$, is homeomorphic to B^2. But no such homeomorphism can be extended to $\overline{B^2}$, since otherwise the boundary of $\overline{B^2}$ $(= S^1)$ would be mapped on a space which is not pathwise connected.

Below, we present a proof that the answer is yes. The proof remains somewhat unsatisfactory in that it relies on several high-powered results in manifold theory and lacks the simplicity of the construction in section 1.

The first simple observation one can make is that the answer to Question 1 is positive if M is triangulable. We will generalize this observation in Lemma 3 below. Before we prove the lemma, let us recall the definition of handlebody decomposition (see [2] for a more general definition).

Definition 1. Let M be an n-manifold, M_0 be a clean submanifold of M. A *handlebody decomposition* of M on M_0 is a representation of M as a finite union

$$M_0 \subset M_1 \subset \cdots \subset M_m = M$$

by clean submanifolds such that, for each i, $H_i = \overline{M_i \setminus M_{i-1}}$ is a clean compact submanifold of M_i, and $(H_i, H_i \cap M_{i-1}) \cong (B^k, \partial B^k) \times B^{m-k}$ for some k, $0 \leq k \leq m$. H_i is called a *handle.*

In the definition above, *clean* means that the manifold in question is bi-collared in the larger manifold (i.e., an imbedding of the smaller manifold can be extended to an imbedding of its product with the reals).

Lemma 3. *Let M be an n-manifold such that there exists a handlebody decomposition of M on M_0, where M_0 is homeomorphic to $\overline{B^n}$. Then there exists an f that satisfies the requirements of Question 1.*

Proof: We prove the lemma by induction on the number of handles. Suppose that whenever there is a handlebody decomposition

$$M_0 \subset M_1 \subset \cdots \subset M_{m-1} = M'$$

we can find an $f : \overline{B^n} \to M'$ as required by Question 1. As will be seen from the proof below, we can require that f be 1–1 on any given small $T \subset S^{n-1} = \overline{B^n} \setminus B^n$, where T is homeomorphic to $\overline{B^{n-1}}$ and its boundary is bi-collared (the latter can always be achieved by shrinking T). Now let $M = M' \cup H$, where H is a handle. By Lemma 4, we can assume that f maps the cone $\mathrm{C}\,\overline{B^{n-1}}$ onto M', that f is 1–1 on the "base" ($\cong \overline{B^{n-1}}$) of the cone, and that the image of the base is in ∂H. Again, by shrinking T we can assume that the boundary of T is bi-collared in ∂H; therefore, Lemma 4 allows us to assume that there exists a 1–1 mapping $g : \mathrm{C}\,\overline{B^{n-1}} \to H$ such that the image of the base under g is exactly T. Since any homeomorphism of the base of the cone onto itself can be extended to the whole cone, we can also assume, by changing g, that f and g are the same on the base. Now "gluing together" f and g provides the desired map $f \cup g : \mathrm{S}\,\overline{B^{n-1}} \to M$, where $\mathrm{S}\,\overline{B^{n-1}} \cong \overline{B^n}$ denotes the suspension of $\overline{B^{n-1}}$. □

The proof above uses the following simple lemma, which shows that a "hole in the skin of $\overline{B^n}$" can be "straightened out" if the hole is bi-collared in the skin.

Lemma 4. *Let T be a subset of $S^{n-1} = \overline{B^n} \setminus B^n$ homeomorphic to $\overline{B^{n-1}}$ whose boundary is bi-collared in S^{n-1}. Then there exists a* *homeomorphism* $h : \mathrm{C}\,\overline{B^{n-1}} \to \overline{B^n}$ *such that the image of the base of the cone under* h *is* T.

Proof: It is easy to see that it is enough to prove the existence of a homeomorphism $h : \overline{B^n} \to \overline{B^n}$ such that the image of the "upper hemisphere," $\{\, (x_1, \ldots, x_n) \in S^{n-1} : x_i \geq 0 \,\}$ is T. But the existence of such h on S^{n-1} follows from the generalized Schoenflies Theorem (see [1, Theorem 19.11]), and the desired homeomorphism can be obtained by extending h radially to the whole $\overline{B^n}$. □

Lemma 3 will be used to handle the majority of cases for which Question 1 has an affirmative answer. In the case of dimension 4, however, we need a different approach (since the general handlebody decomposition results of [2] fail in dimension 4). The following proposition will be used to handle this case.

Lemma 5. *Let* X *be a compact space and* $S \subset X$ *be a closed subspace of* X *such that* S *is countable, and* $X \setminus S$ *is a connected* n*-manifold,* $n \geq 3$ *which is also a simplicial complex. Then there is an* f *as in Question 1.*

Proof: We will sketch a proof only for the case of S being a single point. The general proof is similar but it requires an induction on the Cantor-Bendixson index of S and is not used in the sequel.

From now on assume that $S = \{z\}$. Then it is easy to see that all the conditions of Lemma 6 are satisfied for $M = X \setminus \{z\}$ so there exists an $f : \overline{B^n} \times [0,1) \to X \setminus \{z\}$ such that f is onto and $f \mid B^n \times (0,1)$ is 1–1. Since the "tubes" of Lemma 6 form a locally finite collection, it is possible to extend f to a one point compactification of its domain. □

Lemma 6. *Let a triangulation* $K = \{\, \sigma_i : i \in \omega \,\}$ *of a connected* n*-manifold* M*,* $n \geq 3$*, have the following property: for any* $k \in \omega$ *there exists a connected open set* $U \subset M$ *such that all but finitely many simplices of* K *are in* U *and* $\bigcup\{\, \sigma_i : i \leq k \,\} \subset M \setminus U$*. Then there exists an* $f : \overline{B^n} \times [0,1) \to M$ *such that* f *is onto and* $f \mid B^n \times (0,1)$ *is 1–1.*

Proof: It will be more convenient to build an $f : \overline{B^n} \times [0,\infty) \to M$ with similar properties. The construction below also ensures that the family $\{\, f(\overline{B^n} \times [k, k+1] \,\}_{k=0}^{\infty}$ is locally finite (this is used in

Lemma 5). The idea is to build the "centerline" of the cylinder first.

Begin by constructing a path $p : [0,\infty) \to M$ with the property that $p([k+1,\infty)) \cap \sigma_k = \emptyset$ and $p(m_k) =$ "the baricenter of σ_k" for some $m_k \in [0,\infty)$. This can be done by induction on k, using the property of K stated in the lemma and path-connectedness of M. Now one can easily find a piecewise linear $q : [0,\infty) \to M$ with the same properties as p. (It is enough to observe that for any $k \in \omega$ there exists a piecewise linear $q_k : [k,k+1] \to M$ which lies inside the simplices hit by $p([k,k+1])$, hits the baricenter of every simplex hit by $p([k,k+1])$, and starts and ends at $p(k)$ and $p(k+1)$, respectively.) Using the fact that M is a manifold, we can modify q in an arbitrarily small way to assume that q does not intersect the $(n-2)$-skeleton of K. A similar modification, together with the fact that $\dim M \geq 3$, will ensure that no whole "segment" of q lies inside the $(n-1)$-skeleton of K and that q is 1-1 and goes through (the interior of) every n-simplex of K at least once. Finally, we can assume that for every $\sigma \in K$ the set $q^{-1}(\sigma)$ is a collection of finitely many nondegenerate intervals. (Finiteness follows from the piecewise linearity of q and the fact that K forms a locally finite cover; nondegeneracy can be achieved by "tweaking" q.)

The construction outlined above leads to a piecewise linear path $q : [0,\infty) \to M$ and a strictly increasing sequence $r_k \in [0,\infty)$ such that for any $\sigma \in K$ the set $q^{-1}(\sigma)$ is a disjoint union of finitely many intervals of the form $[r_i, r_{i+1}]$. Note also that every $q(r_i)$ lies in the interior of an $(n-1)$-simplex of K and that $q(r_i) \neq q(r_j)$ for $i \neq j$.

Now fix an arbitrary n-simplex $\sigma \in K$ and let $r_{k_1}, \ldots, r_{k_m}$ list all the points r_i such that $q((r_i, r_{i+1})) \subset \sigma$. Let $f_{k_j} : \overline{B^n} \to \sigma$ denote the imbeddings built in Lemma 7 where we put $q_i^0 = q(r_{k_i})$, $q_i^1 = q(r_{k_i+1})$. The collection of f_k's has the following properties:

1. each $f_k : \overline{B^n} \to M$ is an imbedding and $\bigcup_{k=0}^{\infty} f_k(\overline{B^n}) = M$;
2. $f_i(B^n) \cap f_j(B^n) = \emptyset$ for $i \neq j$;
3. for any $k \in \omega$ there exists a $T \subset M$ homeomorphic to $\overline{B^{n-1}}$ which is bicollared in both $f_k(S^{n-1})$ and $f_{k+1}(S^{n-1})$; the existence of such T follows from the fact that $f_k(S^{n-1}) \cap f_{k+1}(S^{n-1})$ contains a neighborhood of $q(r_{k+1})$, which is open in the $(n-1)$-skeleton of K.

Finally, using a construction identical to that of Lemma 3, we can build the required $f : \overline{B^n} \times [0, \infty) \to M$ by induction so that $f|\overline{B^n} \times [0, k]$ is obtained by "gluing" the appropriate modifications of $f_0, \dots, f_{k-1}$.

Note that the images of f_k's form a locally finite decomposition of M into n-balls. Such decomposition is also, of course, provided by K. The reason for the somewhat tedious construction above lies in the need for an infinite "string" of touching n-balls (see property (3) above) which K may not contain (see also Question 2). □

The next lemma shows that every n-simplex can be represented as a finite union of tubes with prescribed "entrance" and "exit" points.

Lemma 7. *Let σ be an n-dimensional simplex, $n \geq 3$, q_i^ν, $i = 1, \dots, k$, and $\nu = 0, 1$ be disjoint points in $\partial\sigma$. Then there exist imbeddings $f_i : \overline{B^n} \to \sigma$, $i = 1, \dots, k$ such that $\bigcup_i f_i(\overline{B^n}) = \sigma$, $f_i(B^n) \cap f_j(B^n) = \emptyset$ for $i \neq j$, and $f_i(\overline{B^n})$ contains an open (in σ) neighborhood of $\{q_i^0, q_i^1\}$ for every $i = 1, \dots, k$.*

Proof: It is not difficult to come up with a homeomorphism of σ onto an n-dimensional cube in such a way that q_i^0 lies "directly beneath" q_i^1. Cutting the cube into "prisms" and taking their preimages under the homeomorphism finishes the construction. □

Theorem 2. *Let M be a compact connected n-manifold. Then there exists a mapping $f : \overline{B^n} \to M$ such that f is onto and $f \mid B^n$ is 1–1.*

Proof: It follows from [2, Theorem III.2.1] that in case of $\dim M \leq 3$ or $\dim M \geq 6$, M has a handlebody decomposition. The results of [3] imply the existence of such a decomposition in dimension 5. Lemma 3 then finishes the proof. When $\dim M = 4$, Theorem 2.2.3 of [3] implies that $M \setminus \{z\}$ is smoothable (and thus triangulable). Now an application of Lemma 5 completes the proof. □

It is natural to ask for a more combinatorial proof of Lemma 6 which leads to the following question. Let X be a traingulated connected n-manifold. For the question below, define the *graph* of the triangulation by taking the n-simplices of the triangulation as vertices and adding an edge connecting every pair of n-simplices that have a common $(n-1)$-face.

Question 2. In the situation of Lemma 6, does there exist a subdivision K' of K whose graph is Hamiltonian, i.e., it admits a path that goes through every vertex (of the graph) exactly once?

The author does not know the answer to the question above for dimensions higher than 3.

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