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ON THE NON-NORMALITY OF $\beta X \setminus \{p\}$ FOR NON-DISCRETE SPACES X

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ABSTRACT. For a non-compact metrizable space X without isolated points and a point $p \in \beta X \setminus X$, $\beta X \setminus \{p\}$ is shown to be non-normal in the following cases: (1) $\dim X = 0$; (2) p is a remote point.

As is pointed out in [4, Question 13, p. 105], it is difficult to determine whether $\beta\omega \setminus \{p\}$ is normal or not for a point $p \in \beta\omega \setminus \omega = \omega^*$ (obviously, $\beta\omega \setminus \{p\}$ is normal for any $p \in \omega^*$ iff $\omega^* \setminus \{p\}$ is normal for any $p \in \omega^*$). Under CH, it is known to be non-normal [9], [12], but, without CH or under the negation of CH, we seem to know only very little.

Here, deviating a bit from this central interest, we study $\beta X \setminus \{p\}$ for non-discrete spaces X in ZFC and show the following two theorems. These extend S. Logunov's recent results [5], [6], which have an additional assumption that X is separable.

Theorem 1. *If X is a non-compact, metrizable, strongly zero-dimensional space without isolated points and $p \in \beta X \setminus X$ is an arbitrary point, then $\beta X \setminus \{p\}$ is not normal.*

Theorem 2. *If X is a non-compact metrizable space without isolated points and $p \in \beta X \setminus X$ is an arbitrary remote point, then $\beta X \setminus \{p\}$ is not normal.*

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A point $p \in \beta X \setminus X$ is called a *remote point* of X if it does not belong to the closure of any nowhere dense subset of X. It is known [2] that every non-compact metrizable space has at least 2^c remote points.

B. E. Šapirovskiĭ [10, p. 1060] called a point $x \in X$ a *butterfly point* (or *b-point*) of X if there are two closed sets of X such that (a) each of them accumulates at x, and (b) they meet at exactly the point x.

Obviously, if X is not compact and $x \in \beta X \setminus X$ is a butterfly point of βX, then $\beta X \setminus \{x\}$ is not normal. Conversely, if Z is normal and $Z \setminus \{x\}$ is not normal, then x is a butterfly point of Z. Therefore, the point p in each of our theorems is a butterfly point of βX. Actually, we will show that p is also a butterfly point of $\beta X \setminus X$. However, we do not know whether $\beta X \setminus (X \cup \{p\})$ is non-normal or not, even under the additional assumption that X is locally compact.

1. Proof of Theorem 1

Throughout, we will use the symbol $\mathcal{A}^* = \bigcup\{S : S \in \mathcal{A}\}$ for a family $\mathcal{A}$ of subsets of X. Also the closure operation Cl will be always taken in βX unless otherwise specified.

Let X be any non-compact, metrizable, strongly-zero-dimensional space without isolated points. Then we note that X has a base $\mathcal{B} = \{B, C, \dots\}$ consisting of nonempty clopen sets, each B being associated with three sets $B^{(i)} \in \mathcal{B}$, $i = 1, 2, 3$, such that

(1) $B \supset B^{(i)}$ and $B^{(i)} \cap B^{(j)} = \emptyset$ for $i \neq j$;
(2) if $B \subsetneq C$, then either $B \subset C^{(i)}$ for some i or $B \cap \bigcup_i C^{(i)} = \emptyset$;
(3) for each B, C, either $B \cap C = \emptyset$, $B \subset C$, or $B \supset C$;
(4) each B is contained in only finitely many members of $\mathcal{B}$.

In fact, first take any disjoint open cover $\mathcal{B}_1$ of diameter ≤ 1. Since X has no isolated points, each $B \in \mathcal{B}_1$ is covered by infinitely many disjoint open sets of diameter $\leq 1/2$. They form, with B running over $\mathcal{B}_1$, a cover $\mathcal{B}_2$ of X. The sets $B^{(i)}$ are taken arbitrarily from the cover of B. And each $B \in \mathcal{B}_2$ is covered by infinitely many disjoint open sets of diameter $\leq 1/3$. Proceeding in this way, we obtain a base $\mathcal{B} = \bigcup_n \mathcal{B}_n$ which satisfies the conditions (1) through (4).

Let us take and fix any point $p \in \beta X \setminus X$. We will show that p is a butterfly point of βX.

Now consider the collection $\mathcal{U}$ of all disjoint open covers $\sigma \subset \mathcal{B}$ of X.

We may suppose that $\mathrm{Cl}\, B \not\ni p$ for any $B \in \mathcal{B}$, since $\mathcal{B} \setminus \{B \in \mathcal{B} : \mathrm{Cl}\, B \ni p\}$ is also a base of X and satisfies (1) through (4).

The point p helps us introduce a linear order in $\mathcal{U}$ (see [5], [6]). That is, two elements σ and τ are considered equivalent, written $\sigma \sim \tau$, if $p \in \mathrm{Cl}(\sigma \cap \tau)^*$. And define

$$\sigma \leq \tau \text{ iff } p \in \mathrm{Cl}(\sigma \,|\, \tau)^*,$$

where we let $\sigma \,|\, \tau = \{B \in \sigma : B \subset C \text{ for some } C \in \tau\}$. The fact that this order is well-defined on equivalence classes follows from the facts that each $\sigma \in \mathcal{U}$ is a disjoint open cover of X, and that $\mathrm{Cl}\, \delta^* \cap \mathrm{Cl}(\sigma \setminus \delta)^* = \emptyset$ for any $\delta \subset \sigma$. The linearity of the order (that is, either $\sigma \leq \tau$ or $\sigma \geq \tau$ occurs for any σ, τ) follows from condition (3). The details are as follows.

Lemma 1. (i) $\sigma \sim \tau \wedge \tau \sim \pi \implies \sigma \sim \pi$,
(ii) $\sigma \leq \tau \wedge \tau \sim \pi \implies \sigma \leq \pi$,
(iii) $\sigma \sim \upsilon \wedge \sigma \leq \tau \implies \upsilon \leq \tau$,
(iv) $\sigma \leq \tau \wedge \tau \leq \pi \implies \sigma \leq \pi$,
(v) $\sigma \leq \tau \wedge \tau \leq \sigma \implies \sigma \sim \tau$,
(vi) $\sigma \not\leq \tau \implies \tau \leq \sigma$.

Proof: (i). Since $p \in \mathrm{Cl}(\sigma \cap \tau)^*$ and $p \in \mathrm{Cl}(\tau \cap \pi)^*$, it follows that $p \in \mathrm{Cl}\big((\sigma \cap \tau)^* \cap (\tau \cap \pi)^*\big)$. Note that $(\sigma \cap \tau)^* \cap (\tau \cap \pi)^* = (\sigma \cap \tau \cap \pi)^*$ because τ is a disjoint cover. Now $(\sigma \cap \tau \cap \pi)^* \subset (\sigma \cap \pi)^*$ implies $p \in \mathrm{Cl}(\sigma \cap \pi)^*$.

(ii). Let $\tau' = \{C \in \tau : C \text{ contains some } B \in \sigma \,|\, \tau\}$. Then $(\tau')^* \supset (\sigma \,|\, \tau)^*$ implies $p \in \mathrm{Cl}(\tau')^*$. This further implies $p \in \mathrm{Cl}\big((\tau')^* \cap (\tau \cap \pi)^*\big) = \mathrm{Cl}(\tau' \cap \tau \cap \pi)^* = \mathrm{Cl}(\tau' \cap \pi)^*$ and $p \in \mathrm{Cl}\big((\sigma \,|\, \tau)^* \cap (\tau' \cap \pi)^*\big)$. Since τ is disjoint, it follows that $(\sigma \,|\, \tau)^* \cap (\tau' \cap \pi)^* \subset \{B \in \sigma \,|\, \tau : B \text{ is contained in some } C \in \tau' \cap \pi\}^*$. The latter set is included in $\{B \in \sigma : B \text{ is contained in some } C \in \tau \cap \pi\}^* \subset \{B \in \sigma : B \text{ is contained in some } C \in \pi\}^* = (\sigma \,|\, \pi)^*$.

(iii). Since $p \in \mathrm{Cl}(\sigma \cap \upsilon)^*$, it follows that $p \in \mathrm{Cl}\big((\sigma \,|\, \tau)^* \cap (\sigma \cap \upsilon)^*\big) = \mathrm{Cl}((\sigma \,|\, \tau) \cap \upsilon)^*$. It is obvious that $(\sigma \,|\, \tau) \cap \upsilon \subset \upsilon \,|\, \tau$.

(iv). Let $\tau' = \{C \in \tau : C \text{ contains some } B \in \sigma \,|\, \tau\}$. Then since $p \in \mathrm{Cl}(\sigma \,|\, \tau)^*$ and $p \in \mathrm{Cl}(\tau \,|\, \pi)^*$, it follows that $p \in \mathrm{Cl}\big((\sigma \,|\, \tau)^* \cap$

$(\tau')^* \cap (\tau \,|\, \pi)^*\big) = \mathrm{Cl}\big((\sigma \,|\, \tau)^* \cap (\tau' \cap (\tau \,|\, \pi))^*\big)$. Since τ is disjoint, the latter set is included in $\{B \in \sigma \,|\, \tau : B$ is contained in some $C \in \tau' \cap (\tau \,|\, \pi)\}^* \subset \{B \in \sigma : B$ is contained in some $C \in \tau \,|\, \pi\}^* \subset \{B \in \sigma : B$ is contained in some $D \in \pi\}^* = (\sigma \,|\, \pi)^*$.

(v). This is seen by noting $(\sigma \,|\, \tau)^* \cap (\tau \,|\, \sigma)^* \subset (\sigma \cap \tau)^*$.

(vi). Since $p \notin \mathrm{Cl}(\sigma \,|\, \tau)^*$, it follows that $p \in \mathrm{Cl}(\sigma \setminus \sigma \,|\, \tau)^*$. Take any neighborhood O of p and suppose that $B \in \sigma \setminus \sigma \,|\, \tau$ meets O. The open set $B \cap O$ meets some $C \in \tau$. This means C meets B. Since the inclusion $B \subset C$ cannot happen, condition (3) implies $C \subset B$ and hence, $C \in \tau \,|\, \sigma$. Thus, $p \in \mathrm{Cl}(\tau \,|\, \sigma)^*$. □

In the sequel, we will let symbols $\sigma, \tau, \ldots$ denote both covers and their equivalence classes. There won't be any confusion.

Note that smaller σ's are more important in the present argument. Obviously, if σ refines τ, then $\sigma \leq \tau$. Conversely, if $\sigma \leq \tau$, then there are $\rho, \varphi \in \mathcal{U}$ such that $\rho \sim \sigma$, $\tau \sim \varphi$, ρ refines τ, and σ refines φ.

For each $\sigma \in \mathcal{U}$, take a closed subset

$$H_\sigma = \bigcap \{\mathrm{Cl}\,\delta^* : \delta \subset \sigma \text{ and } p \in \mathrm{Cl}\,\delta^*\}$$

of $\beta X \setminus X$. Note that H_σ is well-defined on the equivalence class of σ; more precisely,

Lemma 2. *If $\sigma \leq \tau$, then $H_\sigma \subset H_\tau$.*

Proof: Suppose $\sigma \leq \tau$, let $x \in H_\sigma$ and take any $\varepsilon \subset \tau$ such that $p \in \mathrm{Cl}\,\varepsilon^*$. Let τ' be the set of $C \in \tau$ which contains some $B \in \sigma \,|\, \tau$, and δ be the set of $B \in \sigma \,|\, \tau$ which is contained in some $C \in \varepsilon \cap \tau'$. Note that $p \in \mathrm{Cl}\big((\varepsilon \cap \tau')^* \cap (\sigma \,|\, \tau)^*\big)$. Since τ is disjoint, it follows that $(\varepsilon \cap \tau')^* \cap (\sigma \,|\, \tau)^* \subset \delta^*$, and hence, that $p \in \mathrm{Cl}\,\delta^*$. This means $x \in \mathrm{Cl}\,\delta^*$ because $x \in H_\sigma$, and $x \in \mathrm{Cl}\,\varepsilon^*$ because $\varepsilon^* \supset \delta^*$, and finally $x \in H_\tau$ because $\varepsilon \subset \tau$ is arbitrary. □

Lemma 3. *Any neighborhood O of p in βX contains some H_σ.*

Proof: Since $\dim \beta X = 0$, we may suppose that the neighborhood O is clopen. Then for any point $x \in X$, choose a neighborhood $B(x) \in \mathcal{B}$ such that either $B(x) \subset O$ or $B(x) \cap O = \emptyset$, depending on whether x belongs to $O \cap X$ or not. By conditions (3) and (4), the family $\{B(x) : x \in X\}$ contains a disjoint open cover. Let σ denote it. Then $p \in \mathrm{Cl}(\{B(x) : x \in O\})^* = \mathrm{Cl}(\{B \in \sigma : B \subset O\})^*$. Let $\delta = \{B : B \subset O\}$. Obviously, $H_\sigma \subset \mathrm{Cl}\,\delta^* \subset O$. □

Lemma 4. *For each equivalence class σ and $i = 1, 2, 3$, there is a point $r_{\sigma,i} \in H_\sigma$ such that $r_{\sigma,i} \in \mathrm{Cl}\{B^{(i)} : B \in \tau\}^*$ for any $\tau \le \sigma$.*

Proof: It suffices to see that the collection

$$\begin{array}{ll} \delta^* & \delta \subset \sigma \text{ and } p \in \mathrm{Cl}\,\delta^* \\ \{B^{(i)} : B \in \tau\}^* & \tau \le \sigma \end{array}$$

has the finite intersection property, that is, for each $\delta \subset \sigma$ satisfying $p \in \mathrm{Cl}\,\delta^*$ and $\tau_1, \dots, \tau_n \le \sigma$,

$$\delta^* \cap \bigcap_{j=1}^{n} \{B^{(i)} : B \in \tau_j\}^* \ne \emptyset.$$

Take any δ and $\tau_1, \dots, \tau_n$, and, for each j, let σ_j be the subfamily of σ consisting of sets each of which contains a member of $\tau_j \,|\, \sigma$. Since $p \in \mathrm{Cl}(\tau_j \,|\, \sigma)^*$, it follows that $p \in \mathrm{Cl}(\sigma_j)^*$ and that $p \in \mathrm{Cl}\bigl(\delta^* \cap \bigcap_j (\sigma_j)^*\bigr) = \mathrm{Cl}\bigl(\delta \cap \bigcap_j \sigma_j\bigr)^*$.

Take any $S \in \delta \cap \bigcap_j \sigma_j$. For each j, S contains a member of τ_j, and hence, by (3), is expressed as the union of some members of τ_j.

If $S \in \tau_j$ for all j, then there is nothing more to do because $S^{(i)} \subset \delta^* \cap \bigcap_{j=1}^{n} \{B^{(i)} : B \in \tau_j\}^*$. Otherwise, for each j with $S \notin \tau_j$, since $S^{(i)} \subset S$, τ_j contains a member which meets $S^{(i)}$. By (4), find a maximal $B_1 \in \bigcup\{\tau_j : S \notin \tau_j\}$ that meets $S^{(i)}$ ("maximal" in the sense of set-inclusion). Since $B_1 \subset S$, it follows from condition (2) that $B_1 \subset S^{(i)}$.

Next, if S or $B_1 \in \tau_j$ for all j, then there is nothing more to do. Otherwise, find a maximal $B_2 \in \bigcup\{\tau_j : S, B_1 \notin \tau_j\}$ which meets $B_1^{(i)}$. Since $B_2 \cap B_1 \ne \emptyset$, it follows from condition (3) that $B_2 \subset B_1$. Hence, by (2), $B_2 \subset B_1^{(i)}$.

Next consider B_2 and proceed similarly. Then eventually this process terminates because we are dealing with only finitely many τ_j's. □

Now we are ready to complete the proof of Theorem 1.

For each $i = 1, 2, 3$, let

$$K_i = \{r_{\sigma,i} : \sigma \in \mathcal{U}\}.$$

Then, by our lemmas 3 and 4, $p \in \mathrm{Cl}\,K_i$.

Take any σ. Then $K_i \setminus H_\sigma \subset \{r_{\tau,i} : \tau > \sigma\}$ because $r_{\tau,i} \in H_\tau \subset H_\sigma$ for $\tau \le \sigma$. Since $r_{\tau,i} \in \mathrm{Cl}\{B^{(i)} : B \in \sigma\}^*$ for $\sigma < \tau$, it follows that $K_i \subset H_\sigma \cup \mathrm{Cl}\{B^{(i)} : B \in \sigma\}^*$. This implies $\mathrm{Cl}\, K_i \cap \mathrm{Cl}\, K_j \subset H_\sigma$ for $i \neq j$ because $\mathrm{Cl}\{B^{(i)} : B \in \sigma\}^* \cap \mathrm{Cl}\{B^{(i)} : B \in \sigma\}^* = \emptyset$ for $i \neq j$ by (2). Thus, $\mathrm{Cl}\, K_i \cap \mathrm{Cl}\, K_j = \bigcap_\sigma H_\sigma = \{p\}$ by Lemma 3, and p is a butterfly point of both βX and $\beta X \setminus X$.

It might happen that, for some i, $r_{\sigma,i} = p$ for sufficiently small σ such that p is not an accumulation point of K_i. But, by Lemma 4, there is at most one such i, so it does not matter; in fact, this is the reason why we have constructed three sequences $\{r_{\sigma,i} : \sigma\}$, $i = 1, 2, 3$.

2. Proof of Theorem 2

We prove Theorem 2 by modifiying the proof of the previous section. But first we must clarify what the family $\mathcal{B}$ is in this case.

Let us begin with Arhangel'skii's regular base for metrizable spaces (see [3, 5.4.6]). A base $\mathcal{G}$ of a space X is called *regular* if, for any point $x \in X$ and any of its neighborhood U, there is a neighborhood $V \subset U$ of x such that only finitely many members of $\mathcal{G}$ meet both V and $X \setminus U$. If X is metrizable, then it has a regular base, defined as the union of locally finite refinements of the open covers of X consisting of all $1/(n+1)$-neighborhoods, $n = 0, 1, 2, \dots$, of points. Arhangel'skii shows that the converse to this is true, that is, every T_1-space having a regular base is metrizable, but we do not need it here.

The following seems well-known (see [3, Lemma 5.4.3]).

Proposition 1. *A metrizable space X has a σ-locally finite base $\mathcal{G} = \bigcup \mathcal{G}_i$, called a regular base, such that every cover $\mathcal{W} \subset \mathcal{G}$ has a locally finite subcover.*

Another of our devices is the following (see [6]). Here a family $\mathcal{A}$ of subsets of a set E is said to *densely cover* E when $\mathrm{Cl}_X \mathcal{A}^* = \mathrm{Cl}_X E$. Also note that we will call here the family $\mathcal{A}$ locally finite if it is locally finite in the whole space X (that is, not just in the union $\bigcup \mathcal{A}$).

Proposition 2. *For every locally finite family $\mathcal{W}$ of nonempty open sets, there is a locally finite family $\mu(\mathcal{W})$ of disjoint nonempty open sets such that*

(a) *for every $W \in \mathcal{W}$ and $U \in \mu(\mathcal{W})$, either $W \cap U = \emptyset$ or $W \supset U$ holds;*

(b) *every $W \in \mathcal{W}$ is densely covered by a subfamily of $\mu(\mathcal{W})$.*

Proof: For each finite subfamily $\varphi \subset \mathcal{W}$, consider

$$K_\varphi = \left(\bigcap \varphi\right) \setminus \mathrm{Cl}_X \left(\bigcup (\mathcal{W} \setminus \varphi)\right),$$

and let $\mu(\mathcal{W})$ be the family of all nonempty K_φ. $\mu(\mathcal{W})$ consists of disjoint sets, because, for $W \in \varphi \setminus \varphi'$, $K_\varphi \subset W$ and $K_{\varphi'} \cap W = \emptyset$. Also $\mu(\mathcal{W})$ is locally finite, because an open set which meets K_φ meets every $W \in \varphi$.

To see that $\mu(\mathcal{W})$ densely covers each member of $\mathcal{W}$, take any $U \in \mathcal{W}$ and $x \in U$. Further, take any open neighborhood (in X) $O \subset U$ of the point x. We may suppose that $\{W \in \mathcal{W} : O \cap W \neq \emptyset\}$ is a finite set. Let φ denote it. Obviously, $\varphi \ni U$. Note that $O \cap \mathrm{Cl}_X W = \emptyset$ for every $W \notin \varphi$. If $O \cap \bigcap \varphi \neq \emptyset$, then $O \cap K_\varphi \neq \emptyset$. If $O \cap \bigcap \varphi = \emptyset$, then find a set $\varphi' \subset \varphi$ such that $U \in \varphi'$ and $O \cap \bigcap \varphi' \neq \emptyset$, and $O \cap \bigcap \varphi' \cap W = \emptyset$ for any $W \in \varphi \setminus \varphi'$. Then $K_{\varphi'} \subset U$ and $O \cap K_{\varphi'} \neq \emptyset$. □

Now we are ready to define the family $\mathcal{B}$.

$\mathcal{B}$ is going to be a π-base of X and satisfy the conditions (1) through (4), although now the following condition (1′) replaces (1):

(1′) $B \supset \mathrm{Cl}_X B^{(i)}$ and $\mathrm{Cl}_X B^{(i)} \cap \mathrm{Cl}_X B^{(j)} = \emptyset$ for $i \neq j$.

First of all, fix a regular base $\mathcal{G} = \bigcup_{n=0}^{\infty} \mathcal{G}_n$ of X as in Proposition 1, and let $\mathcal{B}_1 = \mu(\mathcal{G}_0)$.

Since X contains no isolated points, each $B \in \mathcal{B}_1$ contains three nonempty open sets $\tilde{B}^{(1)}, \tilde{B}^{(2)}, \tilde{B}^{(3)}$ such that $B \supset \mathrm{Cl}_X \tilde{B}^{(i)}$ for each $i = 1, 2, 3$, and $\mathrm{Cl}_X \tilde{B}^{(i)} \cap \mathrm{Cl}_X \tilde{B}^{(j)} = \emptyset$ for $i \neq j$. Let $\mathcal{C}_1$ be the family of all $\tilde{B}^{(i)}$. Then $\mathcal{C}_1$ is locally finite in X.

Now let $\mathcal{B}_2 = \mu(\mathcal{B}_1 \cup \mathcal{C}_1 \cup \mathcal{G}_1)$. For each $B \in \mathcal{B}_1$, there obviously are $B^{(i)} \in \mathcal{B}_2$ such that the condition (1′) is satisfied.

Next, each $B \in \mathcal{B}_2$ contains three nonempty open sets $\tilde{B}^{(1)}$, $\tilde{B}^{(2)}$, $\tilde{B}^{(3)}$ such that $B \supset \mathrm{Cl}_X \tilde{B}^{(i)}$ for each $i = 1, 2, 3$, and $\mathrm{Cl}_X \tilde{B}^{(i)} \cap \mathrm{Cl}_X \tilde{B}^{(j)} = \emptyset$ for $i \neq j$. Let $\mathcal{C}_2$ be the family of all $\tilde{B}^{(i)}$, and $\mathcal{B}_3 = \mu(\mathcal{B}_2 \cup \mathcal{C}_2 \cup \mathcal{G}_2)$.

Proceeding in this way, we obtain a family $\mathcal{B} = \bigcup_n \mathcal{B}_n$ which is a π-base of X, and satisfies (1′), (2), (3) and (4).

Once we have taken a remote point p, the rest of the proof of Theorem 2 is similar to that of Theorem 1. As before we remove all B from $\mathcal{B}$, if any, such that $\mathrm{Cl}\, B \ni p$.

The collection $\mathcal{U}$ now consists of all *locally finite* disjoint families $\sigma \subset \mathcal{B}$ which cover X densely, i.e., $\mathrm{Cl}_X\, \sigma^* = X$. The equivalence relation and the order are defined in exactly the same way. Although it is no longer true that $\mathrm{Cl}\, \delta^* \cap \mathrm{Cl}(\sigma \setminus \delta)^* = \emptyset$ for $\delta \subset \sigma$, we have instead that if U and V are open sets of X and $p \in \mathrm{Cl}\, U \cap \mathrm{Cl}\, V$, then $p \in \mathrm{Cl}(U \cap V)$ because p is a remote point. This is enough to ensure the validity of the equivalence and the order relation.

Other modifications are equally minor. We mention only the modified proof of Lemma 3.

Take open neighborhoods P, Q of the point p such that $\mathrm{Cl}_X\, Q \subset P \subset \mathrm{Cl}_X\, P \subset O$. Then for any point $x \in X$ choose a neighborhood $G(x) \in \mathcal{G}$ such that either $G(x) \subset P$ or $G(x) \cap \mathrm{Cl}_X\, Q = \emptyset$, depending on whether x belongs to $\mathrm{Cl}_X\, Q$ or not. We can assume $\mathrm{Cl}\, G(x) \not\ni p$. The cover $\{G(x) : x \in X\}$ has a locally finite subcover π by Proposition 1. By conditions (3) and (4) and our construction, a disjoint locally finite family $\sigma \subset \mathcal{B}$ is found such that σ refines π and covers X densely (take $\sigma(G) \subset \mathcal{B}$, for each $G \in \pi$, which covers G densely, and consider maximal members of the family $\bigcup\{\sigma(G) : G\}$). Then $p \in \mathrm{Cl}(\{G(x) : x \in \mathrm{Cl}_X\, Q\})^* \subset \mathrm{Cl}(\{B \in \sigma : B \subset P\})^*$. Let $\delta = \{B : B \subset P\}$. Obviously, $H_\sigma \subset \mathrm{Cl}\, \delta^* \subset O$.

3. Remarks

Remark 1. The referee pointed out the following to the author.

A. Błaszczyk and A. Szymański [1] show that every *near* point p of a non-compact metrizable space X without isolated points is a butterfly point of βX. Here p is called a near point of X if p belongs to the closure of a closed discrete set $\subset X$.

According to the referee, suppose that $p \in \mathrm{Cl}\, D$ for a closed discrete subset $D \subset X$, and pick a discrete (in X) family $\{U(d) : d \in D\}$ of closed neighborhoods $U(d)$ of d. Then, for a neighborhood base $\mathcal{N}$ at p,

$$H = \bigcap_{O \in \mathcal{N}} \mathrm{Cl} \left[\bigcup_{d \in O} U(d) \right]$$

is a closed set of βX and accumulates at p, and moreover $H \cap \mathrm{Cl}\, D = \{p\}$.

A non-near point is called a *far* point. The referee's point is it would be interesting to know whether every far-but-not-remote point is a butterfly point.

Remark 2. Obviously, a minor modification of the foregoing establishes in ZFC that if $p \in \operatorname{Cl} D \setminus D$ for a countable discrete set $D \subset \omega^*$, then p is a butterfly point of ω^*. And the proof yields also that $\omega^* \setminus \{p\}$ is not normal. This was probably what was meant by "the best result so far" in [4, Question 13, p. 105].

Alan Dow told the author in a private communication that he also knows of a ZFC point p with a similar property.

Remark 3. A really easy proof of the non-normality of $\beta\omega \setminus \{p\}$ under CH is actually given in [11, 3.30]; it shows, without recourse to the P-points, that every p is a butterfly point of ω^*.

Remark 4. The readers should note that Logunov has published two other papers in related topics (see [7], [8]).

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