Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY
PROCEEDINGS
Volume 27, No. 1, 2003
Pages 345–349

UNIVERSAL ULTRAMETRIC SPACE OF WEIGHT τ^ω

JERRY E. VAUGHAN

Abstract. We give an affirmative solution to a problem raised by A. J. Lemin by showing that any ultrametric space of weight at most τ^ω can be isometrically embedded into the ultrametric space (LW_τ, Δ) as constructed by A. J. Lemin and V. A. Lemin.

1. Introduction

A. J. Lemin and V. A. Lemin constructed for every cardinal $\tau \geq 2$ an ultrametric space called (LW_τ, Δ) into which every ultrametric space of weight at most τ can be isometrically embedded [2, Main Theorem] (such a space is called τ-universal). The weight of (LW_τ, Δ) is τ^ω, and this fact raises the natural question [1, Problem 2]: Is (LW_τ, Δ) a τ^ω-universal space, i.e., can every ultrametric space of weight at most τ^ω be isometrically embedded into (LW_τ, Δ)?

In this paper we answer this question in the affirmative by proving that (LW_τ, Δ) is indeed τ^ω-universal (Theorem 1.2).

Recall that a metric space (X, d) is called an *ultrametric space* provided the metric d satisfies the Strong Triangle Inequality: For all $x, y, z \in X$

$$d(x, z) \leq \max\{d(x, y), d(y, z)\},$$

2000 *Mathematics Subject Classification.* 03E10, 54A25, 54C25, 54E35.
Key words and phrases. isometric embedding, isosceles property, ultrametric, weight.

or, equivalently, satisfies the following Isosceles Property: For all $x, y, z \in X$

$\triangle_{IP}(x, y, z)$: Among the numbers $\{d(x, y), d(y, z), d(x, z)\}$ two are equal and greater than or equal to the third.

Definition 1.1 ([2]). The underlying set of (LW_τ, Δ) is the set of all functions f from the set $\mathbb{Q}_+$ of positive rational numbers into τ, which are eventually zero (i.e.,there exists a real number r such that $f(x) = 0$ for all $x > r$). The ultrametric Δ is defined by $\Delta(f, g) = \sup\{x \in \mathbb{Q}_+ : f(x) \neq g(x)\}$.

Theorem 1.2. *Let $\tau \geq 2$. Then LW_τ is τ^ω-universal.*

We give a complete proof of Theorem 1.2 in this paper. Our proof of Theorem 1.2 follows the two cases used by the Lemins in their proof [2, Main Theorem] and differs from the Lemins' proof mainly in Case 1 where we use a new construction of certain functions in LW_τ (see Lemma 1.3 below). Our proof also differs from the Lemins' proof in that we define the isometry directly on all of X as in [3], rather than first defining it on a dense subset as in [2], and we employ a different presentation of the use of the Isosceles Property.

Lemma 1.3. *If $b \geq 0, h \in LW_\tau$ and $Y \subset LW_\tau$ with $|Y| < \tau^\omega$, then there exists $f \in LW_\tau$ such that $\Delta(f, h) = b$, and for all $y \in Y$, $\Delta(h, y) \geq b$.*

Proof: If $b = 0$ then $f = h$ works; so we assume $b > 0$. Put $d_0 = 0$ and for $n \geq 1$ pick an increasing sequence of positive irrational numbers (d_n) converging up to b. Let I_n denote the interval of rational numbers $(d_n, d_n{+}1) \cap \mathbb{Q}_+$. Since there are τ^ω functions with domain I_n and range τ, and $|Y| < \tau^\omega$, we may pick a function g_n with domain I_n and range τ such that $g_n \neq y \restriction I_n$ for all $y \in Y \cup \{h\}$. We define

$$f = \cup\{g_n : n \in \omega\} \cup h \restriction [b, \infty).$$

Clearly, $f \in LW_\tau$, and $\Delta(f, h) \leq b$. Moreover, $\Delta(f, y) \geq b$ for all $y \in Y \cup \{h\}$ because f and y differ below b arbitrarily close to b. In particular, $\Delta(f, h) = b$. □

2. Proof of the Theorem

Let (X, d) be an ultrametric space with weight not more that τ^ω. It follows that the cardinality of X is no more that τ^ω; so we may assume $|X| = \tau^\omega$ and in a one-one fashion, well-order $X = \{x_\alpha : \alpha < \tau^\omega\}$. We want to isometrically embed X into (LW_τ, Δ).

Assume we have defined $f_\alpha \in LW_\tau$ for $\alpha < \gamma$ such that the map $x_\alpha \mapsto f_\alpha$ satisfies $d(x_\alpha, x_\beta) = \Delta(f_\alpha, f_\beta)$ for $\alpha < \beta < \gamma$. We define f_γ and show that the assignment $x_\gamma \mapsto f_\gamma$ extends the isometry. First, (as in [2]), define

$$d_\gamma = \inf\{d(x_\alpha, x_\gamma) : \alpha < \gamma\}.$$

Case 1: $d(x_\gamma, x_\beta) = d_\gamma$ for some $\beta < \gamma$. Let $Y = \{f_\alpha : \alpha < \gamma\}$. Thus, $|Y| < \tau^\omega$. So by Lemma 1.3, there exists $f_\gamma \in LW_\tau$ such that $\Delta(f_\gamma, f_\beta) = d_\gamma$, and $\Delta(f_\gamma, f_\alpha) \geq d_\gamma$ for all $\alpha < \gamma$. To see that the assignment $x_\gamma \mapsto f_\gamma$ extends the isometry, we fix $\alpha < \gamma$ and show that $d(x_\gamma, x_\alpha) = \Delta(f_\gamma, f_\alpha)$. We may rename the following two sets of distances as indicated

$$\{d(x_\gamma, x_\beta), d(x_\beta, x_\alpha), d(x_\gamma, x_\alpha)\} = \{d_\gamma, a, b\}$$

$$\{\Delta(f_\gamma, f_\beta), \Delta(f_\beta, f_\alpha), \Delta(f_\gamma, f_\alpha)\} = \{d_\gamma, a, b'\}$$

because the first number in each set equals d_γ and the two middle numbers are equal by the Induction Hypothesis. We want to show that $b = b'$. If $a < d_\gamma$ then by $\triangle_{IP}(x_\gamma, x_\beta, x_\alpha)$, $b = d_\gamma$, and by $\triangle_{IP}(f_\gamma, f_\beta, f_\alpha)$, $b' = d_\gamma$; so $b = b'$. Similarly, if $d_\gamma < a$ then $a = b$ and $a = b'$; so $b = b'$. Finally, if $d_\gamma = a$, it does not follow in general that $b = b'$, but in our case we also know that $d_\gamma \leq b$ by definition of d_γ, and $d_\gamma \leq b'$ by Lemma 1.3. Thus, by the isosceles property, we have $d_\gamma = a = b$ and $d_\gamma = a = b'$; so $b = b'$. Hence, $d(x_\gamma, x_\alpha) = \Delta(f_\gamma, f_\alpha)$.

Case 2: $d_\gamma < d(x_\gamma, x_\alpha)$ for all $\alpha < \gamma$. Thus, for every $\epsilon > 0$ there exists $\alpha < \gamma$ such that $0 < d(x_\gamma, x_\alpha) - d_\gamma < \epsilon$. Pick $\alpha_1 < \gamma$ such that $d(x_\gamma, x_{\alpha_1}) < d_\gamma + 1$. Continue by induction to pick for each $i > 1$ an ordinal $\alpha_i < \gamma$ such that

$$d(x_\gamma, x_{\alpha_i}) < \min\{\frac{1}{i}(d_\gamma + 1), \min\{d(x_\gamma, x_{\alpha_j}) : j < i\}\}.$$

Put $b_i = d(x_\gamma, x_{\alpha_i})$ for all i. Then (b_i) is a strictly decreasing sequence of real numbers converging to d_γ. From $x_{\alpha_i} \mapsto f_{\alpha_i}$, define

$$f_\gamma = \cup\{f_{\alpha_i}\restriction(b_i, \infty) : i \geq 1\} \cup \{(x, 0) : x \in \mathbb{Q}_+ \text{ and } x \leq d_\gamma\}.$$

To see that f_γ is a well defined function and an element of LW_τ, it suffices to show that $i < k$ implies $f_{\alpha_i}(x) = f_{\alpha_k}(x)$ for all $x > b_i$, and hence to show $\Delta(f_{\alpha_i}, f_{\alpha_k}) \leq b_i$. To see this, note that $\triangle_{IP}(x_{\alpha_i}, x_{\alpha_k}, x_\gamma)$ implies $d(x_{\alpha_i}, x_{\alpha_k}) = b_i$; hence, by the induction hypothesis we have, in fact, $\Delta(f_{\alpha_i}, f_{\alpha_k}) = d(x_{\alpha_i}, x_{\alpha_k}) = b_i$. Thus, f_γ is well defined, and clearly by definition, $\Delta(f_\gamma, f_{\alpha_i}) \leq b_i$ for all i.

To see that the assignment $x_\gamma \mapsto f_\gamma$ extends the isometry, we fix $\alpha < \gamma$ and show that $d(x_\gamma, x_\alpha) = \Delta(f_\gamma, f_\alpha)$. Choose i so large that $b_i = d(x_\gamma, x_{\alpha_i}) < d(x_\gamma, x_\alpha)$. We may rename the following two sets of distances as indicated

$$\{d(x_\gamma, x_\alpha), d(x_\alpha, x_{\alpha_i}), d(x_\gamma, x_{\alpha_i})\} = \{c, a, b_i\}$$

$$\{\Delta(f_\gamma, f_\alpha), \Delta(f_\alpha, f_{\alpha_i}), \Delta(f_\gamma, f_{\alpha_i})\} = \{f, a, g\}$$

because the last number in the first set equals b_i, and the two middle numbers are equal by the induction hypothesis. We chose i so that $b_i < c$ and hence, by $\triangle_{IP}(x_\gamma, x_\alpha, x_{\alpha_i})$, $c = a$. We also know $g \leq b_i < a$; hence, by $\triangle_{IP}(f_\gamma, f_\alpha, f_{\alpha_i})$, $f = a$. Thus, $c = f$, i.e., $d(x_\gamma, x_\alpha) = \Delta(f_\gamma, f_\alpha)$, and this completes the proof.

Remark 2.1. Our result that (LW_τ, Δ) is τ^ω-universal renews interest in a problem raised by the Lemins [2, Problem 1]: does there exist a τ-universal ultrametric space of weight τ? In [3], we gave a consistent, affirmative solution to this problem (we assumed the singular cardinal hypothesis), and in ZFC an affirmative solution for a proper class of cardinals τ of countable cofinality. The problem, however, is not completely solved in ZFC.

References

[1] A. J. Lemin, *Imbedding of ultrametric spaces in Banach spaces, Lebesgue spaces, two-point products and τ-universal spaces*. Preprint.

[2] A. J. Lemin and V. A. Lemin, *On a universal ultrametric space*, Topology Appl. **103** (2000), 339–345.

[3] J. E. Vaughan, *Universal ultrametric spaces of smallest weight*, Topology Proc. **24** (1999), 611–619.

Department of Mathematics, University of North Carolina at Greensboro, Greensboro, NC 27402
E-mail address: vaughanj@uncg.edu