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AN EXPANSIVE HOMEOMORPHISM ON A TWO-DIMENSIONAL PLANAR CONTINUUM

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Abstract. A homeomorphism $h : X \to X$ is called *expansive* provided that for some fixed $c > 0$ and every $x, y \in X$ there exists an integer n, dependent only on x and y, such that $d(h^n(x), h^n(y)) > c$. A two-dimensional planar continuum that admits an expansive homeomorphism is constructed.

1. Introduction

A *continuum* is a nondegenerate compact connected metric space. A homeomorphism $h : X \to X$ is called *expansive* provided that for some fixed $c > 0$ and every distinct $x, y \in X$ there exists an integer n, dependent only on x and y, such that $d_X(h^n(x), h^n(y)) > c$. Expansive homeomorphisms exhibit chaotic behavior in that no matter how close two points are either their forward or reverse images will eventually be a certain distance apart. The Plykin attractor [4] and the dyadic solenoid [5] are examples of continua that admit expansive homeomorphisms.

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A continuum is *1-dimensional* if for every $\epsilon > 0$ there exists a finite open cover $\mathcal{U}$ whose mesh is less than ϵ such that for every $y \in X$, y is in at most 2 elements of $\mathcal{U}$. If X is a plane continuum, then X is 2-dimesional if X contains an open disk and 1-dimensional otherwise. A planar continuum X is a *non-separating plane* continuum provided that $\mathbb{R}^2 - X$ is connected. The Plykin attractor is a Lakes of Wada continuum that separates the plane into 4 components and is the most widely known 1-dimensional planar continuum that admits an expansive homeomorphism. It has been shown by the author that 1-dimensional non-separating plane continua do not admit expansive homeomorphisms. (In fact, tree-like continua do not admit expansive homeomorphisms [3].) This paper gives an example of a 2-dimensional planar continuum that admits an expansive homeomorphism and separates the plane. However, the following question remains open: *Does there exist a 2-dimensional non-separating plane continuum that admits expansive homeomorphism?*

2. Inverse limits and the Plykin attractor

A useful method of constructing continua is through *inverse limits*. Let $\{G_i\}_{i=1}^{\infty}$ be a sequence of topological spaces. For each $i < j$, let $f_i^j : G_j \longrightarrow G_i$ be a continuous function called a *bonding map*. If $f_k^j = f_k^i \circ f_i^j$ for each $k < i < j$, then the collection $\{G_i, f_i^j\}_{i=1}^{\infty}$ is called an *inverse system*. Each of the spaces G_i is called a *factor space* of the inverse system.

Since each bonding map f_k^j is determined by the collection of one step bonding maps $f_i^{i+1} = f_i$ for $k \leq i < j$, it is sufficient to consider only these maps. The inverse system $\{G_i, f_i\}_{i=1}^{\infty}$ is sometimes written as

$$G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} G_3 \ \ldots \ G_{i-1} \xleftarrow{f_{i-1}} G_i \xleftarrow{f_i} \ \ \ldots .$$

Every inverse system $\{G_i, f_i\}_{i=1}^{\infty}$ determines a topological space X called the *inverse limit* of the system and is written $\varprojlim\{G_i, f_i\}_{i=1}^{\infty}$. The space X is the subspace of the Cartesian product $\prod_{i=1}^{\infty} G_i$ given by

$$X = \varprojlim\{G_i, f_i\}_{i=1}^{\infty} = \{(x_i)_{i=1}^{\infty} \in \textstyle\prod_{i=1}^{\infty} G_i | f_i(x_{i+1}) = x_i \text{ for each } i\}.$$

X has the subspace topology induced on it by $\prod_{i=1}^{\infty} G_i$. If $\mathbf{x} = (x_i)_{i=1}^{\infty}$ and $\mathbf{y} = (y_i)_{i=1}^{\infty}$ are two points of the inverse limit, we define distance to be

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i},$$

where d_i is the metric on G_i. If each of the factor spaces $G = G_i$ and each of the bonding maps $f = f_i$ are the same, then there is natural homeomorphism on the inverse limit $\widehat{f} : X \longrightarrow X$ defined by

$$\widehat{f}(x_1, x_2, x_3 ...) = (f(x_1), f(x_2), f(x_3), ...) = (f(x_1), x_1, x_2, ...).$$

We call $\widehat{f}$ the *shift homeomorphism* on X. Notice that

$$\widehat{f}^{-1}(x_1, x_2, x_3 ...) = (x_2, x_3, x_4, ...).$$

The construction of the Plykin attractor will be through the inverse limit of the common factor space P with common bonding map f. Let $P = [0, 4]/\{A, B\}$ where A is the identification of 0,1, and 2 and B is the identification of 3 and 4 in the interval $[0, 4]$ (see Figure 1). Define

$$d_a(x) = \begin{cases} \min\{|x|, |x-1|, |x-2| & x \in [0, 3]\} \\ \min\{|x-3|+1, |x-4|+1, & x \in (3, 4]\} \end{cases}$$

and

$$d_b(x) = \begin{cases} \min\{|x|+1, |x-1|+1, |x-2|+1 & x \in [0, 2]\} \\ \min\{|x-3|, |x-4|, & x \in (2, 4]\} \end{cases} .$$

Define distance on P by

$$d_P(x, y) = \min\{|x - y|, d_a(x) + d_a(y), d_b(x) + d_b(y)\}.$$

Let $f : P \longrightarrow P$ be the bonding map defined by

$$f(x) = \begin{cases} 2.5 - 2x & 0 \le x < .25 \\ 1 + 2(x - .25) & .25 \le x < .75 \\ 2.5 - 2(1 - x) & .75 \le x < 1 \\ 2.5 - 4(x - 1) & 1 \le x < 1.125 \\ 1 + 8(x - 1.125) & 1.125 \le x < 1.375 \\ 3 + 4(x - 1.375) & 1.375 \le x < 1.625 \\ 1 + 8(1.875 - x) & 1.625 \le x < 1.875 \\ 2.5 - 4(2 - x) & 1.875 \le x < 2 \\ 2.5 + 2(x - 2) & 2 \le x < 2.75 \\ 3 - 2(x - 2.75) & 2.75 \le x < 3 \\ 2.5 - 2(x - 3) & 3 \le x < 3.25 \\ 2(x - 3.25) & 3.25 \le x < 3.75 \\ 2.5 - 2(4 - x) & 3.75 \le x < 4 \end{cases}.$$

Motivation for $f(x)$ comes from Figure 5.6 on page 210 of [2]. For a pictorial representation of f see Figure 2. Let $\mathcal{P} = \varprojlim\{P, f_i\}_{i=1}^{\infty}$ where $f_i = f$ for each i. Then $\mathbf{x} = \{x_1, x_2, ...\}$ is an element of $\mathcal{P}$ provided that $f(x_{i+1}) = f_i(x_{i+1}) = x_i$ for each i. For $\mathbf{x}, \mathbf{y} \in \mathcal{P}$, define distance as $\mathrm{d}_{\mathcal{P}}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} 2^{-i}(\mathrm{d}_P(x_i, y_i))$. Let $\widehat{f} : \mathcal{P} \longrightarrow \mathcal{P}$ be the shift homeomorphism induced by f.

Proposition 1. *Suppose $\mathbf{x} = (x_1, x_2, ...) \in \mathcal{P}$, then the following are true:*
(1) If $x_i \in (0, 1)$ then $x_{i+1} \in (3.25, 3.75)$.
(2) If $x_i \in (1, 2)$ then $x_{i+1} \in (.25, .75) \cup (1.125, 1.25) \cup (1.75, 1.875)$.
(3) If $x_i \in (2, 2.5)$ then $x_{i+1} \in (0, .25) \cup (1, 1.125) \cup (1.875, 2) \cup (1.25, 1.3125) \cup (1.6875, 1.75) \cup (3, 3.25) \cup (3.75, 4)$.
(4) If $x_i \in (2.5, 3)$ then $x_{i+1} \in (1.3125, 1.375) \cup (1.625, 1.6875) \cup (2, 2.25) \cup (2.75, 3)$.
(5) If $x_i \in (3, 4)$ then $x_{i+1} \in (1.375, 1.625) \cup (2.25, 2.75)$.

The next two lemmas show how f expands the distances between points.

Lemma 2. *Suppose $x_{i-1} = y_{i-1}$ and $x_i \neq y_i$, then either $d_P(x_i, y_i) \geq .0625$ or $d_P(x_{i+1}, y_{i+1}) \geq .0625$.*

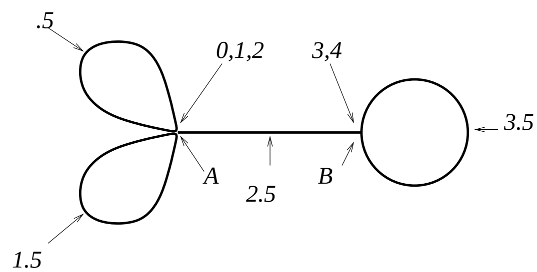

Figure 1

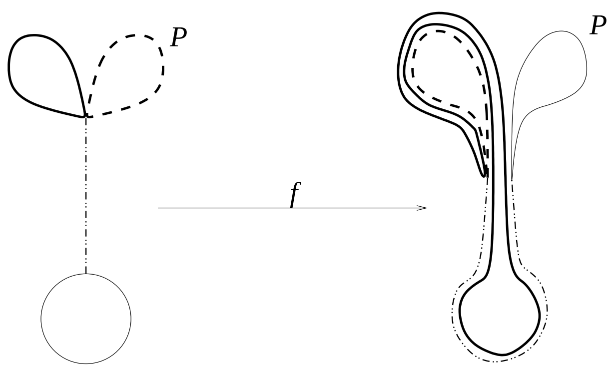

Figure 2

Proof. The proof of this lemma contains a large number of cases of which 3 are shown. The proof of the other cases are similar.

Case 1. Suppose $x_i, y_i \in (0,1)$. Since f maps the interval $(0,.75)$ one-to-one and onto $(1,2.5)/A$, f maps $(.25,1)$ one-to-one and onto $(1,2.5)/A$ and $x_{i-1} = y_{i-1}$, we may take $x_i \in (0,.25)$ and $y_i \in (.75,1)$. Then $x_{i+1} \in (3.25, 3.375)$ and $y_{i+1} \in (3.625, 3.75)$. Hence,

$\mathrm{d}_P(x_{i+1}, y_{i+1}) \geq .25 > .0625$.

Case 2. Suppose $x_i \in (0,1)$ and $y_i \in (1,2)$. Then $x_{i+1} \in (3.25, 3.75)$ and $y_{i+1} \in (.25,.75) \cup (1.125, 1.25) \cup (1.75, 1.875)$. Hence,

$$d_P(x_{i+1}, y_{i+1}) \geq .375 > .0625.$$

Case 3. Suppose $x_i \in (0,1)$ and $y_i \in (2, 2.5)$. Then $x_{i-1} \in (1, 2.5)/A$ and $y_{i-1} \in (2.5, 3.5)$. Thus, $x_{i-1} \neq y_{i-1}$ which is a contradiction. □

Lemma 3. *Suppose $d_P(x_i, y_i) < .0625$ for all $i \geq 1$ then*

$$d_P(f(x_1), f(y_1)) \geq 2\, d_P(x_1, y_1).$$

Proof. The proof of this lemma also contains a large number of cases of which 2 are shown. Again, the proof of the other cases are similar. From Lemma 2 we may assume that $x_1 \neq y_1$.

Case A. Suppose $x_1, y_1 \in (3,4)$. There are 8 subcases to consider:

Case A.1. Suppose $x_1, y_1 \in (3, 3.25)$, then $d_P(x_1, y_1) = |y_1 - x_1|$. Here, $f(x_1), f(y_1) \in (2, 2.5)$. So, $d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2.5 - 2(y_1 - 3) - (2.5 - 2(x_1 - 3))| = 2|y_1 - x_1| = 2d_P(x_1, y_1)$.

Case A.2. Suppose $x_1 \in (3, 3.25)$, $y_1 \in (3.25, 3.5)$, then $d_P(x_1, y_1) = y_1 - x_1$. Here, $f(x_1) \in (2, 2.5)$, $f(y_1) \in (0, .5)$. So, $d_P(f(x_1), f(y_1)) = f(x_1) - 2 + f(y_1) = 2.5 - 2(x_1 - 3) - 2 + 2(y_1 - 3.25) = 2(y_1 - x_1) = 2d_P(x_1, y_1)$.

Case A.3. Suppose $x_1, y_1 \in (3.25, 3.5)$, then $d_P(x_1, y_1) = |y_1 - x_1|$. Here, $f(x_1), f(y_1) \in (0, .5)$. So, $d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2(y_1 - 3.25) - (2(x_1 - 3.25))| = 2|y_1 - x_1| = 2d_P(x_1, y_1)$.

Case A.4. Suppose $x_1 \in (3.375, 3.5)$, $y_1 \in (3.5, 3.625)$ then $d_P(x_1, y_1) = y_1 - x_1$. Here, $f(x_1) \in (.25, .5)$, $f(y_1) \in (.5, .75)$. So, $d_P(f(x_1), f(y_1)) = f(y_1) - f(x_1) = 2(y_1 - 3.25) - (2(x_1 - 3.25)) = 2(y_1 - x_1) = 2d_P(x_1, y_1)$.

Case A.5. Suppose $x_1 \in (3.5, 3.75)$, $y_1 \in (3.5, 3.75)$ then $d_P(x_1, y_1) = |y_1 - x_1|$. Here, $f(x_1), f(y_1) \in (.5, 1)$. So, $d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2(y_1 - 3.25) - (2(x_1 - 3.25))| = 2|y_1 - x_1| = 2d_P(x_1, y_1)$.

Case A.6. Suppose $x_1 \in (3.625, 3.75)$, $y_1 \in (3.75, 3.875)$ then $d_P(x_1, y_1) = y_1 - x_1$. Here, $f(x_1) \in (.5, 1)$, $f(y_1) \in (2, 2.25)$. So, $d_P(f(x_1), f(y_1)) = f(y_1) - 2 + 1 - f(x_1) = 2.5 - 2(4 - y_1) - 2 + 1 - (2(x_1 - 3.25)) = 2(y_1 - x_1) = 2d_P(x_1, y_1)$.

Case A.7. Suppose $x_1, y_1 \in (3.75, 4)$ then $d_P(x_1, y_1) = |y_1 - x_1|$. Here, $f(x_1), f(x_2) \in (2, 2.5)$. So, $d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2.5 - 2(4 - y_1) - (2.5 - 2(4 - x_1))| = 2|y_1 - x_1| = 2d_P(x_1, y_1)$.

Case A.8. Suppose $x_1 \in (3, 3.25)$, $y_1 \in (3.75, 4)$ then $x_2 \in (2.25, 2.375) \cup (1.375, 1.4375)$ and $y_2 \in (2.625, 2.75) \cup (1.5625, 1.625)$. So, $d_P(x_2, y_2) \geq .125 > .0625$ which is a contradiction.

Case B. Suppose $x_1 \in (2, 3)$ and $y_1 \in (3, 4)$. Since $\mathrm{d}_P(x_1, y_1) < .0625$, there are 2 subcases to consider:

Case B.1. Suppose $x_1 \in (2.9375, 3)$ and $y_1 \in (3, 3.0625)$ then $d_P(x_1, y_1) = |y_1 - x_1|$. Here, $f(x_1) \in (2.5, 2.625)$ and $f(y_1) \in (2.375, 2.5)$. So, $d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2.5 - 2(y_1 - 3) - (3 - 2(x_1 - 2.75))| = 2|y_1 - x_1|$.

Case B.2. Suppose $x_1 \in (2.9375, 3)$ and $y_1 \in (3.9375, 4)$ then $d_P(x_1, y_1) = |7 - x_1 - y_1|$. Here, $f(x_1) \in (2.5, 2.625)$ and $f(y_1) \in (2.375, 2.5)$. So, plane $d_P(f(x_1), f(y_1)) = |f(y_1) - f(x_1)| = |2.5 - 2(4 - y_1) - (3 - 2(x_1 - 2.75))| = 2|7 - x_1 - y_1|$. $\square$

The next theorem states that the shift homeomorphism on the Plykin attractor is expansive.

Theorem 4. *$\widehat{f} : \mathcal{P} \longrightarrow \mathcal{P}$ is an expansive homeomorphism with expansive constant .0625.*

Proof. Suppose $\mathbf{x}=\{x_1, x_2, ...\}$, $\mathbf{y}= \{y_1, y_2, ...\}$ are distinct points of $\mathcal{P}$. Let i be the smallest index such that $x_i \neq y_i$.

Case 1. Suppose $i = 1$ and $\mathrm{d}_P(x_k, y_k) \geq .0625$ for some $k \geq 1$. Then $\widehat{f}^{1-k}(\mathbf{x})= \{x_k, x_{k+1}, ...\}$ and $\widehat{f}^{1-k}(\mathbf{y})= \{y_k, y_{k+1}, ...\}$. Hence,

$$\mathrm{d}_{\mathcal{P}}(\widehat{f}^{1-k}(\mathbf{x}), \widehat{f}^{1-k}(\mathbf{y})) \geq \mathrm{d}_P(x_k, y_k) \geq .0625.$$

Case 2. Suppose $i = 1$ and $\mathrm{d}_P(x_k, y_k) < .0625$ for all positive integers k. Let n be an integer such that

$$log_2(.0625/\mathrm{d}_P(x_1, y_1)) \leq n < log_2(.0625/\mathrm{d}_P(x_1, y_1)) + 1.$$

Then it follows from Lemma 3 that

$$\mathrm{d}_{\mathcal{P}}(\widehat{f}^n(\mathbf{x}), \widehat{f}^n(\mathbf{y})) \geq \mathrm{d}_P(f^n(x_1), f^n(y_1)) \geq 2^n \mathrm{d}_P(x_1, y_1)) \geq .0625.$$

Case 3. Suppose $i > 1$. Then by Lemma 2, either $\mathrm{d}_P(x_i, y_i) \geq .0625$ or $\mathrm{d}_P(x_{i+1}, y_{i+1}) \geq .0625$. This is similar to Case 1) by letting $k = i$ or $k = i + 1$. □

3. Construction of the 2-dimensional plane continuum X that admits an expansive homeomorphism.

The 2-dimensional planar continuum X that admits an expansive homeomorphism that is constructed is the compactification of a disk minus two points, $D - \{a, b\}$, whose boundary contains $\mathcal{P}_1$ and $\mathcal{P}_2$ (see figure 5). Let $\widehat{f_1}$ and $\widehat{f_2}$ be shift homeomorphisms on Plykin attractors $\mathcal{P}_1$ and $\mathcal{P}_2$. The expansive homeomorphisms F on X uses the shift homeomorphism $\widehat{f_1}$ when restricted to $\mathcal{P}_1$ and the inverse of the shift homeomorphism $\widehat{f_2}^{-1}$ when restricted to $\mathcal{P}_2$. If two distinct points $\mathbf{x}, \mathbf{y} \in D - \{a, b\}$ have different 1st coordinates, then it is shown that there is a positive integer n such that $\mathrm{d}_X(F^n(\mathbf{x}), F^n(\mathbf{y})) > c$. On the other hand if $\mathbf{x}, \mathbf{y}$ have different 2nd coordinates, then it is shown that there is a positive integer n such that $\mathrm{d}_X(F^{-n}(\mathbf{x}), F^{-n}(\mathbf{y})) > c$, where c is the expansive constant.

The construction of X begins by creating 2 homeomorphic continua Y_1, Y_2 with inverse limits that use single bonding maps h_1, h_2 on factor spaces G_1, G_2. To construct the factor space, define

$$T = \{(x, y) | x \geq 0, y \geq 0, x + y \leq 2\}$$

$$L_1 = \{(x, 0) | 2 \leq x \leq 2.5\},\ L_2 = \{(0, y) | 2 \leq y \leq 2.5\}$$

$$P_1 = \{(x, p_1) | x \in P\},\ P_2 = \{(p_2, y) | y \in P\}.$$

Then let $G_1 = T \cup L_1 \cup P_1$, where the point $(2.5, 0)$ in L_1 is identified to the point (A, p_1) in P_1 and let $G_2 = T \cup L_2 \cup P_2$, where the point $(0, 2.5)$ in L_2 is identified to the point (p_2, A) in P_2 (see Figures 3 and 4).

To define distance on G_1 and G_2, let $\pi^k_{G_1}, \pi^k_{G_2}$ be the projection maps on the kth coordinate for G_1 and G_2, where $k \in \{1, 2\}$. Also for $k \in \{1, 2\}$ and $x, y \in P_k$ define

$$\mathrm{d}_{G_k}(x, y) = \mathrm{d}_{P_k}(x, y) = \mathrm{d}_P(\pi^k_{G_k}(x), \pi^k_{G_k}(y)).$$

If $x, y \in T \cup L_k$, define

$$\mathrm{d}_{G_k}(x,y) = \mathrm{d}_{T\cup L_k}(x,y) = |\pi^1_{G_k}(x) - \pi^1_{G_k}(y)| + |\pi^2_{G_k}(x) - \pi^2_{G_k}(y)|.$$

If $x \in T \cup L_1$ and $y \in P_1$, define

$$\mathrm{d}_{G_1}(x,y) = \mathrm{d}_{T\cup L_1}(x,(2.5,0)) + \mathrm{d}_{P_1}((A,p_1),y).$$

If $x \in T \cup L_2$ and $y \in P_2$, define

$$\mathrm{d}_{G_2}(x,y) = \mathrm{d}_{T\cup L_2}(x,(0,2.5)) + \mathrm{d}_{P_2}((p_2,A),y).$$

To define the bonding maps h_1, h_2, let $\phi, \psi : [0,2] \longrightarrow [0,2]$ by

$$\phi(z) = \begin{cases} 2z & 0 \le z < .5 \\ z + .5 & .5 \le z < 1 \\ 1 + .5z & 1 \le z \le 2 \end{cases},$$

$$\psi(z) = \begin{cases} .5z & 0 \le z < 1 \\ z - .5 & 1 \le z < 1.5 \\ 2(z-1) & 1.5 \le z \le 2 \end{cases}.$$

Define $g_1 : T \cup L_1 \longrightarrow G_1$ by

$$g_1(x,y) = \begin{cases} (\phi(x), \psi(y)) & x \le 1.5 \\ (x + .25, (1.75 - x)\psi(y)) & 1.5 < x \le 1.75 \\ (2(x - 1.75) + 2, 0) & 1.75 < x \le 2 \\ (x, p_1) & 2 < x \le 2.5 \end{cases}.$$

Define $g_2 : T \cup L_2 \longrightarrow G_2$ by

$$g_2(x,y) = \begin{cases} (\psi(x), \phi(y)) & y \le 1.5 \\ ((1.75 - y)\psi(x), y + .25) & 1.5 < y \le 1.75 \\ (0, 2(x - 1.75) + 2) & 1.75 < y \le 2 \\ (p_2, y) & 2 < y \le 2.5 \end{cases}.$$

Let $h_1 : G_1 \longrightarrow G_1$ be defined by

$$h_1(w) = \begin{cases} g_1(\pi^1_{G_1}(w), \pi^2_{G_1}(w)) & w \in T \cup L_1 \\ (f(\pi^1_{G_1}(w)), p_1) & w \in P_1 \end{cases}.$$

Let $h_2 : G_2 \longrightarrow G_2$ be defined by

$$h_2(w) = \begin{cases} g_2(\pi^1_{G_2}(w), \pi^2_{G_2}(w)) & w \in T \cup L_2 \\ (p_2, f(\pi^2_{G_2}(w))) & w \in P_2 \end{cases}.$$

The following 2 lemmas examines the movement of points under the bonding maps h_1 and h_2.

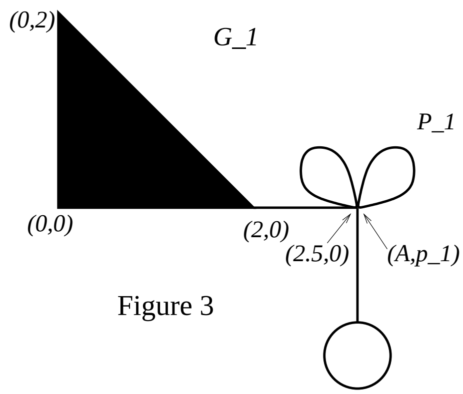

Figure 3

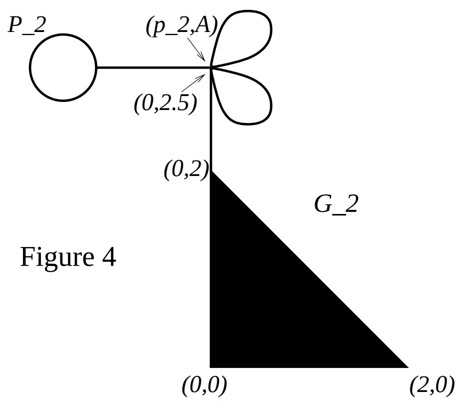

Figure 4

Lemma 5. *Suppose that* $k \in \{1, 2\}$ *and that* $(r, s) \in T \cup L_k$ *such that* $r \neq 0$. *Then there exists a positive integer* n *such that* $h_k^{n-1}(r, s) \in T \cup L_k$ *and* $h_k^n(r, s) \in P_k$

Proof. The proof is shown for $k = 1$. The proof is similar for $k = 2$. Here $h_1((r, s)) = g_1((r, s))$. So it suffices to find an n such that $g_1^n(r, s) \in P_1$.

Case 1. Suppose $(r,s) \in L_1$. Then $g_1((r,s)) = (r,p_1) \in P_1$. So take $n = 1$.

Case 2. Suppose $(r,s) \in T$ such that $1.75 < r \leq 2$. Then $g_1((r,s)) \in L_1$. Now Case 1 applies. Here take $n = 2$.

Case 3. Suppose $(r,s) \in T$ such that $1.5 < r \leq 1.75$. Then $1.75 < \pi^1_{G_1}(g_1((r,s))) \leq 2$, so Case 2 applies. Now take $n = 3$.

Case 4. Suppose $(r,s) \in T$ such that $1 < r \leq 1.5$. Then $\pi^1_{G_1}(g_1((r,s))) = \phi(r) = 1 + .5r$. So, $1.5 < \pi^1_{G_1}(g_1((r,s))) = 1 + .5r \leq 1.75$ and hence, Case 3 applies. Here, let $n = 4$.

Case 5. Suppose $(r,s) \in T$ such that $.5 < r \leq 1$. Then $\pi^1_{G_1}(g_1((r,s))) = \phi(r) = r + .5$. So, $1 < \pi^1_{G_1}(g_1((r,s))) = r + .5 \leq 1.5$. Thus, Case 4 applies, so let $n = 5$.

Case 6. Suppose $(r,s) \in T$ such that $0 < r \leq .5$. Then $\pi^1_{G_1}(g_1((r,s))) = \phi(r) = 2r$. Let m be an integer such that $.5 < 2^m r \leq 1$. Then, $.5 < \pi^1_{G_1}(g_1^m((r,s))) = 2^m r \leq 1$. Hence, let $n = m + 5$. □

Lemma 6. *Suppose that $k \in \{1,2\}$ and $(r,s) \in T \cup L_k$. Then for every $\epsilon > 0$, there exists a negative integer n such that $\pi^1_{G_k}(h^n_k(r,s)) < \epsilon$.*

Proof. Proof follows from doing proof of Lemma 5 in reverse. □

Lemmas 7 and 8 show how h_1 and h_2 expands the distances between points.

Lemma 7. *Suppose $(r,s), (x,y) \in T \cup L_1$ such that $r \neq x$. Then there exists a positive integer n such that $d_{G_1}(h^n_1(r,s), h^n_1(x,y)) \geq .0625$.*

Proof. Assume $r > x$.

Case 1. Suppose $x = 0$. We may assume that $r < .0625$. Let n be a positive integer such that $2^{n-1}r < .0625 \leq 2^n r$. Then $\mathrm{d}_{G_1}(h^n_1(r,s), h^n_1(0,y)) \geq |\pi^1_{G_1}(h^n_1(r,s)) - \pi^1_{G_1}(h^n_1(0,y))| = |(\phi^n(r) - \phi^n(0))| = 2^n r \geq .0625$.

Case 2. Suppose $0 < x < r$. Then by Lemma 5, we may choose n to be a positive integer such that $h_1^{n-1}(r,s) \in T \cup L_1$ and $h_1^n(r,s) \in P_1$.

Case 2a. Suppose $h_1^n(x,y) \in P_1$. Let $r_n = \pi_{G_1}^1(h_1^n(r,s))$ and $x_n = \pi_{G_1}^1(h_1^n(x,y))$. Then r_n and x_n are distinct elements of $[2,2.5]_P$. Since $f : [2,2.5]_{P_1} \longrightarrow [2.5,3.5]_{P_1}$ is one-to-one, $f(r_n)$ and $f(x_n)$ are distinct elements of $[2.5,3.5]_P$. Also, since $f : [2.5,3.5]_P \longrightarrow [0,.5]_{P_1} \cup [2,3]_P \cup [3.5,4_{P_1}]$ is one-to-one, $f^2(r_n)$ and $f^2(x_n)$ are distinct elements of $[0,.5]_{P_1} \cup [2,3]_P \cup [3.5,4]_P$.

It follows from Lemma 3 that there exists a $k \geq 0$ such that $\mathrm{d}_P(f^k(x_n), f^k(r_n)) \geq .0625$. Hence $\mathrm{d}_{G_1}(h^{n+k}(r,s), h^{n+k}(x,y)) = \mathrm{d}_{P_1}((f^k(r_n),p_1),(f^k(x_n,p_1)) = \mathrm{d}_P(f^k(x_n), f^k(r_n)) \geq .0625$.

Case 2b. Suppose $h_1^n(x,y) \notin P_1$. We may assume that

$$\mathrm{d}_{G_1}(\pi_{G_1}^1(h_1^n(r,s)), \pi_{G_1}^1(h_1^n(x,y)) \leq .0625.$$

Hence, $\pi_{G_1}^1(h_1^n(x,y)) \in [2.4375,2.5]_{L_1}$ and $\pi_{G_1}^1(h_1^{n+1}(r,s)) \in [2,2.0625]_P$. Then $x_{n+1} = \pi_{G_1}^1(h_1^{n+1}(x,y)) \in [2.4375,2.5]_P$ and $r_{n+1} = \pi_{G_1}^1(h_1^{n+1}(r,s)) \in [2.5,2.625]_P$. Then the proof is similar to Case 2a. □

Lemma 8. *Suppose $(r,s),(x,y) \in T \cup L_2$ such that $s \neq y$. Then there exists a positive integer n such that $d_{G_2}(h_2^n(r,s), h_2^n(x,y)) \geq .0625$.*

Proof. Proof is similar to proof of Lemma 7. □

Let $Y_1 = \varprojlim\{G_1, h_1\}_{i=1}^{\infty}$, and $Y_2 = \varprojlim\{G_2, h_2\}_{i=1}^{\infty}$. Let $k \in \{1,2\}$. Each element $\widehat{\mathbf{w}} \in Y_k$ is an infinite sequence of ordered pairs $\widehat{\mathbf{w}} = ((x_1,y_1),(x_2,y_2),...)$ where $h_k(x_i,y_i) = (x_{i-1},y_{i-1})$. Let $H_1 : Y_1 \longrightarrow Y_1$ and $H_2 : Y_2 \longrightarrow Y_2$ be shift homeomorphisms induced from the inverse limit constructions.

Define projection maps $\{\pi_{Y_k}^i, \pi_{Y_k}^{i,1}, \pi_{Y_k}^{i,2}\}$ such that $\pi_{Y_k}^i(\widehat{\mathbf{w}}) = (x_i,y_i)$, $\pi_{Y_k}^{i,1}(\widehat{\mathbf{w}}) = x_i$ and $\pi_{Y_k}^{i,2}(\widehat{\mathbf{w}}) = y_i$. If $\widehat{\mathbf{w}}, \widehat{\mathbf{z}} \in Y_k$, then $\mathrm{d}_{Y_k}(\widehat{\mathbf{w}}, \widehat{\mathbf{z}}) = \sum_{i=1}^{\infty} 2^{-i}\mathrm{d}_{G_k}(\pi_{Y_k}^i(\widehat{\mathbf{w}}), \pi_{Y_k}^i(\widehat{\mathbf{z}}))$.

Lemma 9. *Let $k \in \{1,2\}$. If $\widehat{\mathbf{w}}, \widehat{\mathbf{z}} \in Y_k$ and $\pi^{i,k}_{Y_k}(\widehat{\mathbf{w}}) \neq \pi^{i,k}_{Y_k}(\widehat{\mathbf{z}})$ for some i, then there exists an integer n such that $d_{Y_k}(H^n_k(\widehat{\mathbf{w}}), H^n_k(\widehat{\mathbf{z}})) \geq$ .0625*

Proof. Proof is for $k = 1$. Suppose that i is the smallest positive integer such that $\pi^{i,1}_{Y_1}(\widehat{\mathbf{w}}) \neq \pi^{i,1}_{Y_1}(\widehat{\mathbf{z}})$. Then for all $m \geq i$, $\pi^{m,1}_{Y_1}(\widehat{\mathbf{w}}) \neq \pi^{m,1}_{Y_1}(\widehat{\mathbf{z}})$. Let $w_1 = \pi^{i,1}_{Y_1}(\widehat{\mathbf{w}})$, $w_2 = \pi^{i,2}_{Y_1}(\widehat{\mathbf{w}})$, $z_1 = \pi^{i,1}_{Y_1}(\widehat{\mathbf{z}})$, and $z_2 = \pi^{i,2}_{Y_1}(\widehat{\mathbf{z}})$. It will be shown that there exists an integer n such that $\mathrm{d}_{G_1}(h^n_1(w_1,w_2), h^n_1(z_1,z_2)) \geq .0625$.

Case 1. Suppose that $h^j_1(w_1,w_2), h^j_1(z_1,z_2) \in P_1$ for every j. Then $h^j_1(w_1,w_2) = (f^j(w_1), p_1)$ and $h^j_1(z_1,z_2) = (f^j(z_1), p_1)$. Hence , by Lemmas 2 and 3, there exists an integer n such that

$$\mathrm{d}_{G_1}(h^n_1(w_1,w_2), h^n_1(z_1,z_2)) = \mathrm{d}_P(f^n(w_1), f^n(z_1)) \geq .0625.$$

Case 2. Suppose that $h^j_1(w_1,w_2) \in P_1$ for every j, but there exists an α such that $h^\alpha_1(z_1,z_2) \notin P_1$. Then, by Lemma 6, there exists an integer n such that $\pi^1_{G_1}(h^n_1(z_1,z_2)) < 1$. Then,

$$\mathrm{d}_{G_1}(h^n_1(w_1,w_2), h^n_1(z_1,z_2)) \geq 2.5 - 1 = 1.5.$$

Case 3. Suppose that there exists an j such that $h^j_1(w_1,w_2)$, $h^j_1(z_1,z_2) \notin P_1$. Then by Lemma 7, there exists an n such that

$$\mathrm{d}_{G_1}(h^n_1(w_1,w_2), h^n_1(z_1,z_2)) \geq .0625.$$

It follows that $\mathrm{d}_{Y_1}(H^{n+i}_1(\widehat{\mathbf{w}}), H^{n+i}_1(\widehat{\mathbf{z}})) = \mathrm{d}_{Y_1}(H^n_1 \circ H^i_1(\widehat{\mathbf{w}}), H^n_1 \circ H^i_1(\widehat{\mathbf{z}})) \geq \mathrm{d}_{G_1}(h^n_1(w_1,w_2), h^n_1(z_1,z_2)) \geq .0625.$ □

Let $X = Y_1 \cup Y_2/T$ such that if $\widehat{\mathbf{w}} \in Y_1$ and $\widehat{\mathbf{z}} \in Y_2$, then $\widehat{\mathbf{w}}$ is identified to $\widehat{\mathbf{z}}$ if and only if $\pi^1_{Y_1}(\widehat{\mathbf{w}}), \pi^1_{Y_1}(\widehat{\mathbf{z}}) \in T$, $\pi^{1,1}_{Y_1}(\widehat{\mathbf{w}}) = \pi^{1,1}_{Y_1}(\widehat{\mathbf{z}})$, and $\pi^{1,2}_{Y_1}(\widehat{\mathbf{w}}) = \pi^{1,2}_{Y_1}(\widehat{\mathbf{z}})$. The projection maps are defined as $\pi_X(\widehat{\mathbf{w}}) = \pi_{Y_1}(\widehat{\mathbf{w}})$ if $\widehat{\mathbf{w}} \in Y_1$ and $\pi_X(\widehat{\mathbf{w}}) = \pi_{Y_2}(\widehat{\mathbf{w}})$ if $\widehat{\mathbf{w}} \in Y_2$. Let

$$d_X(\widehat{\mathbf{w}}, \widehat{\mathbf{z}}) = \begin{cases} d_{Y_1}(\widehat{\mathbf{w}}, \widehat{\mathbf{z}}) & \widehat{\mathbf{w}}, \widehat{\mathbf{z}} \in Y_1 \\ d_{Y_2}(\widehat{\mathbf{w}}, \widehat{\mathbf{z}}) & \widehat{\mathbf{w}}, \widehat{\mathbf{z}} \in Y_2 \\ \inf(\{d_{Y_1}(\widehat{\mathbf{w}}, \widehat{\mathbf{q}}) + d_{Y_2}(\widehat{\mathbf{q}}, \widehat{\mathbf{z}}) | \widehat{\mathbf{q}} \in Y_1 \cap Y_2\}) & \widehat{\mathbf{w}} \notin Y_2, \widehat{\mathbf{z}} \notin Y_1 \end{cases}$$

Let $F : X \longrightarrow X$ be defined by

$$F(\widehat{\mathbf{w}}) = \begin{cases} H_1(\widehat{\mathbf{w}}) & 0 \leq \pi_X^{1,2}(\widehat{\mathbf{w}}) \leq 1.5 \\ H_1(\widehat{\mathbf{w}}) & \pi_X^{1,2}(\widehat{\mathbf{w}}) = p_1 \\ H_2^{-1}(\widehat{\mathbf{w}}) & 0 \leq \pi_X^{1,1}(\widehat{\mathbf{w}}) \leq 1.5 \\ H_2^{-1}(\widehat{\mathbf{w}}) & \pi_X^{1,1}(\widehat{\mathbf{w}}) = p_2 \end{cases}.$$

Notice that if $0 \leq \pi_X^{1,2}(\widehat{\mathbf{w}}) \leq 1.5$ and $0 \leq \pi_X^{1,1}(\widehat{\mathbf{w}}) \leq 1.5$, then $H_1(\widehat{\mathbf{w}}) = H_2^{-1}(\widehat{\mathbf{w}})$. Hence, F is a homeomorphism.

Theorem 10. $F : X \longrightarrow X$ *is an expansive homeomorphism.*

Proof. Suppose $\widehat{\mathbf{w}}, \widehat{\mathbf{z}} \in X$ such that $\widehat{\mathbf{w}} \neq \widehat{\mathbf{z}}$. Then there exists an integer i such that $\pi_X^i(\widehat{\mathbf{w}}) \neq \pi_X^i(\widehat{\mathbf{z}})$.

Case 1. Suppose $\pi_X^{i,1}(\widehat{\mathbf{w}}) \neq \pi_X^{i,1}(\widehat{\mathbf{z}})$. Then $\widehat{\mathbf{w}}, \widehat{\mathbf{z}} \in Y_1$. Hence, by Lemma 9, there exists an n such that

$$\mathrm{d}(F^n(\widehat{\mathbf{w}}), F^n(\widehat{\mathbf{z}})) = \mathrm{d}_{Y_1}(H_1^n(\widehat{\mathbf{w}}), H_1^n(\widehat{\mathbf{z}})) \geq .0625.$$

Case 2. Suppose $\pi_X^{i,2}(\widehat{\mathbf{w}}) \neq \pi_X^{i,2}(\widehat{\mathbf{z}})$. Then $\widehat{\mathbf{w}}, \widehat{\mathbf{z}} \in Y_2$. Hence, by Lemma 9, there exists an n such that

$$\mathrm{d}(F^n(\widehat{\mathbf{w}}), F^n(\widehat{\mathbf{z}})) = \mathrm{d}_{Y_2}(H_2^n(\widehat{\mathbf{w}}), H_2^n(\widehat{\mathbf{z}})) \geq .0625.$$

□

The fact that X is planar follows from an application of the Anderson-Choquet Embedding Theorem [1] and although not difficult to show, it is tedious and will be left out.

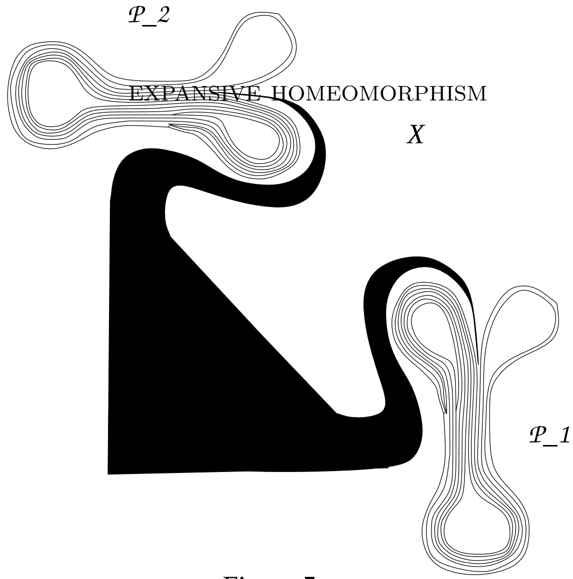

Figure 5

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