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**INVERSE LIMITS OF TENT MAPS WITHOUT THE
PSEUDO-ORBIT SHADOWING PROPERTY**

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ABSTRACT. In this paper we examine the topological structure of inverse limits generated by tent maps without the pseudo-orbit shadowing property. We demonstrate strong connections between the kneading sequence of the map and the topology of the inverse limit.

1. INTRODUCTION AND OUTLINE

Inverse limits of unimodal maps of the interval form a wide class of continua that have received much attention. A useful model for all of the combinatorially dynamics present in other families of unimodal maps is the tent map. The tent map is a piecewise-linear, unimodal map of the interval, $[0, 1]$, that fixes the origin. It is defined by

$$T_a(x) = \begin{cases} ax & \text{if } x \leq 1/2 \\ a(1-x) & \text{if } x \geq 1/2, \end{cases}$$

where $a \in [\sqrt{2}, 2]$.

We will restrict this map to the interval $[T_a^2(1/2), T_a(1/2)]$, which is called the *core* of the map. The reason that we restrict our attention to this interval is because the interval $[T_a(1/2), 1]$ contributes nothing to the inverse limit space and the interval $[0, T_a^2(1/2)]$ gives rise only to a ray that limits onto the subcontinuum generated by the map restricted to the core. For these reasons, we will consider

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the tent map to be given by

$$f_m(x) = \begin{cases} mx + 2 - m & \text{if } x \leq 1 - 1/m \\ m(1 - x) & \text{if } x \geq 1 - 1/m, \end{cases}$$

$m \in [\sqrt{2}, 2]$, which is the core of the tent map rescaled to the interval. This is a subset of the family \mathcal{G} of $g_{b,c}$ maps, which have been studied extensively, [6], [7], [9].

These tent map cores form a one parameter family of spaces many of which have the property that they are (except at some exceptional points) locally the product of a Cantor set and an arc. However Barge, Brucks and Diamond showed that most (in a measure-theoretic sense) of these spaces have a bizarre form of self-similarity: not only is it the case that every neighborhood contains a homeomorph of the entire space, but every neighborhood contains a homeomorph of *every* space occurring in this family of spaces, [1]. They accomplished this by examining tent maps with a dense postcritical orbit. It is well known that inverse limits of tent maps with a finite postcritical orbit have the property that, except at finitely many points, the space is locally the product of a Cantor set and an arc. This leaves out the case of an inverse limit of a tent map with an infinite but sparse postcritical orbit. Very little has been done to understand the topological structure of the inverse limit in this case, a notable paper addressing this case is due to Brucks and Bruin, [3].

We examine a family of tent map cores described by Coven, Kan, and Yorke in [5] that do not have the pseudo-orbit shadowing property. We identify a countable set of points in the inverse limit that are invariant under any autohomeomorphism and show that one point, the point in the inverse limit corresponding to the orientation reversing fixed point, is fixed under any autohomeomorphism.

2. CHAININGS OF INVERSE LIMITS, TURNLINKS, AND ESSENTIAL TURNLINKS

A *chain*, $\{L_1, L_2, \dots, L_n\}$, is a finite collection of sets called *links* with the property that $L_i \cap L_j \neq \emptyset$ if, and only if, $|i - j| < 2$. If a chain, \mathcal{C} , covers a continuum, X we call it a *chaining* of X and if the largest diameter of each of its links is less than some positive number ϵ then we call it an ϵ -*chaining* of X . If X can be ϵ -chained

for all positive numbers ϵ then we call X *chainable* and it is well known that the inverse limit of a chainable continuum is itself a chainable continuum.

Let \mathcal{C} be a chaining of a continuum X . Let $L \in \mathcal{C}$. Call the link L a *turnlink* provided there exists an adjacent link M and a refinement, \mathcal{D} , of \mathcal{C} with subchain $\{D_i\}_{i=a}^b$ such that

- $\bigcup_{i=a}^b D_i \subset L \cup M$;
- $D_a, D_b \subset M \setminus L$;
- $D_i \subset L$ for some $a < i < b$.

In this case we say that \mathcal{D} turns in L . If there exists a positive number ϵ such that all ϵ -chains of X have a turnlink in L then we call L an *essential turnlink*. The definition of turnlink and essential turnlink is due to Bruin, and he proved that the homeomorphic image of an essential turnlink is again an essential turnlink, [4].

Let A be a set. By \bar{A} we will mean the closure of A and by A' we will mean the collection of limit points of A .

3. TENT MAPS WITHOUT THE PSEUDO-ORBIT SHADOWING PROPERTY

Let $f : X \rightarrow X$ be a mapping of a continuum, X . Let δ be a positive number. A δ -pseudo-orbit is a sequence $\{x_0, x_1, x_2 \dots\}$ such that $d(f(x_i), x_{i+1}) \leq \delta$ for all $i \geq 0$. Due to considerations regarding machine-aided approximations of dynamical systems, it is quite important to know when pseudo-orbits can be approximated by actual orbits. A sequence, $\{x_0, x_1, x_2 \dots\}$, is said to be ϵ -shadowed by another sequence, $\{y_0, y_1, y_2, \dots\}$, if $d(x_i, y_i) < \epsilon$ for all $i \geq 0$. f has the *pseudo-orbit shadowing property* if for every $\epsilon > 0$, there is a $\delta > 0$ such that every δ -pseudo-orbit can be ϵ -shadowed by an actual orbit.

Coven, Kan and Yorke proved the following theorems:

Theorem 3.1. [5, Theorem 6.1] *For almost every $a \in [\sqrt{2}, 2]$, the tent map with parameter a , T_a , has the pseudo-orbit shadowing property.*

Theorem 3.2. [5, Theorem 7.4] *The set of parameters for which T_a does not have the pseudo-orbit shadowing property is locally uncountable, i.e. its intersection with any open set is uncountable.*

The inverse limits we are considering in this section are a subset of those without the pseudo-orbit shadowing property. So it is quite a large collection of maps but clearly overshadowed by its complement.

The ω -limit set of a point x under a mapping f is

$$\omega_f(x) = \bigcap_{i \in \mathbb{N}} \overline{\{f^j(x) : j \geq i\}}.$$

We will denote it by $\omega(x)$ when no confusion will arise.

The *itinerary* of a point x under a unimodal map f with critical point c , denoted by $I_f(x)$ is a sequence of symbols, t_0, t_1, t_2, \dots from $\{R, L, C\}$ such that

- $t_i = R$ if $f^i(x) > c$;
- $t_i = L$ if $f^i(x) < c$;
- $t_i = C$ if $f^i(x) = c$.

The convention is that we stop recording the itinerary of a point as soon as the first C occurs, if it ever does. The *kneading sequence* for a unimodal map f with critical point c is given by $K(f) = I_f[f(c)]$.

A point x of the domain of a map is called *prefixed* if there exists a positive integer i and a fixed point p of the map f such that $f^i(x) = p$. A tent map, T_a , has a unique orientation-reversing fixed point which we will denote by p_a or simply by p when no confusion will arise. The set of parameters, $a \in [\sqrt{2}, 2]$, for which the critical point of T_a is prefixed is dense, [5], and all of these parameter values in $[\sqrt{2}, 2)$ have the property that the critical point is mapped to p_a . Such parameter values are called *prefixed parameters*. The kneading sequence of a tent map with a prefixed parameter is some finite word of symbols from $\{R, L\}$, call it W_a , followed by an infinite string of R 's, which we will denote by R^∞ . So if a is a prefixed parameter in $[\sqrt{2}, 2)$, then $K(T_a) = W_a \hat{\ } R^\infty$, where $\hat{\ }$ denotes concatenation of words.

Coven, Kan and Yorke proved Theorem 3.2 by showing that for every increasing sequence of positive integers, $\{n_i\}_{i=1}^\infty$, and every prefixed parameter, a , in $[\sqrt{2}, 2)$, there is a parameter b close to a such that

$$K(T_b) = W_a \hat{\ } R^{n_1} \hat{\ } W_a \hat{\ } R^{n_2} \hat{\ } \dots$$

and that this tent map does not have the pseudo-orbit shadowing property, [5]. It is clear that these tent maps have an infinite postcritical orbit that is not dense.

Let f be the core of a tent map with critical point c , orientation reversing fixed point p , and slope m , such that there exists a prefixed word, W_a , and an increasing sequence of positive integers $\{n_i\}_{i=1}^\infty$ such that

$$K(f) = W_a \wedge R^{n_1} \wedge W_a \wedge R^{n_2} \wedge \dots$$

Let α be the length of W_a .

Notice that since the kneading sequence of f contains subwords that only contain the symbol R of ever increasing length, the critical point must get mapped closer and closer to the fixed point, p . This is because p is repelling and with every application of f intervals in $[c, 1]$ are stretched by a factor of m . So the interval $\langle f^{\alpha+1}(c), p \rangle$, which is a subset of $[c, 1]$ must have diameter less than $\frac{1}{m^{n_1}}$, where by $\langle a, b \rangle$ we mean $[a, b]$ if $a < b$ and we mean $[b, a]$ otherwise. It follows similarly that the diameter of $\langle f^{(\sum_{i < k} n_i) + k \cdot \alpha + 1}(c), p \rangle$ is less than $\frac{1}{m^{n_k}}$.

Let B be a word from the alphabet $\{R, L, C\}$. Denote by A_B the subinterval of $[0, 1]$ of points, x , with the property that $I_f(x)$ begins with the word B . If B is an initial segment of another word C then we will write $B \leq C$.

Lemma 3.3. *If B and D are words from $K(f)$ with $B \leq D$ then $A_B \supseteq A_D$. But if $B \not\leq D$ and $D \not\leq B$ then $A_B \cap A_D = \emptyset$.*

Proof. The proof is immediate because every point, x , with $D \leq I_f(x)$ also has $B \leq I_f(x)$ in the first case, and in the second it is clear that there are no points, x , with $I_f(x)$ having two distinct initial segments. \square

Hence

$$A_{R^{n_1}} \supset A_{R^{n_2}} \supset A_{R^{n_3}} \supset \dots \supset A_{R^\infty} = \{p\}.$$

Also

$$A_{R^{n_1}} \supset A_{R^{n_1} \wedge W_a} \supset A_{R^{n_1} \wedge W_a \wedge R^{n_2}}$$

and

$$A_{R^{n_j} \wedge W_a} \cap A_{R^{n_k} \wedge W_a} = \emptyset$$

for all positive integers $j \neq k$. But notice that

$$A_{R^{n_k} \wedge W_a} \rightarrow \{p\}$$

as $k \rightarrow \infty$ because

$$A_{R^{n_k} \wedge W_a} \subset A_{R^{n_k}} \subset \left[p - \frac{1}{m^{n_k}}, p + \frac{1}{m^{n_k}} \right].$$

This shows that $p \in \omega(c)$ because

$$f^{k*\alpha+1+\sum_{i=1}^{k-1} n_i}(c) \in A_{R^{n_k} \wedge W_a}.$$

Not only that but $p \in \omega(c)'$, the set of limit points of $\omega(c)$, because c is mapped into each $A_{R^{n_k} \wedge W_a}$ infinitely many times and since c is not periodic under f , it must be the case that it takes a different value each time it is mapped into that interval. So each interval contains a point of $\omega(c)$. To see that c is mapped infinitely often into each $A_{R^{n_k} \wedge W_a}$ notice that for every pair of positive integers k and j with $k < j$,

$$f^{n_j-n_k}(A_{R^{n_j} \wedge W_a}) \subseteq A_{R^{n_k} \wedge W_a \wedge R^{n_{j-1}}}.$$

Thus for a given positive integer k and for all $j > k$,

$$f^{n_j-n_k} \circ f^{k*\alpha+1+\sum_{i=1}^{k-1} n_i}(c) \in A_{R^{n_k} \wedge W_a \wedge R^{n_{j-1}}}.$$

Since

$$\begin{aligned} A_{R^{n_k} \wedge W_a} \supset A_{R^{n_k} \wedge W_a \wedge R^{n_{k+1}}} \supset A_{R^{n_k} \wedge W_a \wedge R^{n_{k+2}}} \supset \dots \\ \dots \supset A_{R^{n_k} \wedge W_a \wedge R^{n_j}} \supset \dots \end{aligned}$$

and since $\text{diam}(A_{R^{n_k} \wedge W_a \wedge R^{n_j}}) \rightarrow 0$ as $j \rightarrow \infty$, it must be the case that each $A_{R^{n_k} \wedge W_a}$ contains exactly one point from $\omega(c)$. There is another interval of interest, A_{W_a} which contains c and a point, \hat{p} that is sent to the fixed point and has $I_f(\hat{p}) = W_a \wedge R^\infty$. This implies that

$$\begin{aligned} A_{W_a} \supset A_{W_a \wedge R^{n_1}} \supset A_{W_a \wedge R^{n_2}} \supset \dots \supset A_{W_a \wedge R^{n_i}} \supset \dots \\ \dots \supset A_{W_a \wedge R^\infty} = \{\hat{p}\}. \end{aligned}$$

Let $\{\hat{p}_i\}_{i=1}^\infty$ be a subset of $[c, 1]$ such that $f^i(\hat{p}_i) = \hat{p}$. This gives us an explicit description of the point in $A_{R^{n_k} \wedge W_a}$ that is also in $\omega(c)$.

Since

$$\{\hat{p}_{n_k}\} = A_{R^{n_k} \wedge W_a \wedge R^\infty} \subset A_{R^{n_k} \wedge W_a \wedge R^{n_j}}$$

for all positive integers j , it must be the case that $\omega(c) \cap A_{R^{n_k} \wedge W_a} = \{\hat{p}_{n_k}\}$. Since $f(\omega(c)) \subseteq \omega(c)$, $\hat{p} \in \omega(c)$ and $f^i(\hat{p}) \in \omega(c)$ for all $i \leq \alpha$, remember that $f^\alpha(\hat{p}) = p$. Thus $\{\hat{p}_i\}_{i=1}^\infty \cup \{f^j(\hat{p})\}_{j=1}^\alpha \subseteq \omega(c)$. Notice that we could have used an argument involving intervals of points with the same itinerary in order to show that the forward images of \hat{p} are in $\omega(c)$. From such an argument it follows that $\omega(c) = \{\hat{p}_i\}_{i=1}^\infty \cup \{f^j(\hat{p})\}_{j=0}^\alpha$ and $\omega(c)' = \{p\}$. Thus $c \notin \omega(c)$.

Lemma 3.4. *Let f be the core of a tent map with critical point c . If x is a point in $\varprojlim\{[0, 1], f\}$ such that there is a natural number N with the property that for every $n \geq N$, there is a positive number δ_n with $|x_n - y| > \delta_n$ for all $y \in \omega(c)$, then there is a positive number ϵ and an arc, A , in $\varprojlim\{[0, 1], f\}$ containing x with endpoints a and b such that $d(x, a) > \epsilon$ and $d(x, b) > \epsilon$. Furthermore, x has a neighborhood, U , that is homeomorphic to the product of a Cantor set and an arc.*

Proof. Let N be large enough to satisfy the assumption of the lemma. Notice that we can make N large enough also to guarantee that $x_n \notin \text{orb}(c)$ for all $n \geq N$. To see this assume that for all N there is a value $n \geq N$ such that $x_n \in \text{orb}(c)$. This implies that c is either periodic or preperiodic which implies that $\text{orb}(c) \subseteq \omega(c)$. This would contradict the assumption that for $n \geq N$, x_n is at least δ_n -far away from points in $\omega(c)$. Since $\omega(c) \cup \text{orb}(c) = \overline{\text{orb}(c)}$ where \overline{A} denotes the closure of A , for all $n \geq N$ there exists a positive number γ_n such that $|x_n - y| > \gamma_n$ for all $y \in \overline{\text{orb}(c)}$ and all $n \geq N$.

Notice that $[x_n - \gamma_n, x_n + \gamma_n] = [a_n, b_n] = A_n$ is an arc in $[0, 1]$ containing x_n but missing $\text{orb}(c)$. Thus every preimage of A_n misses $\text{orb}(c)$, and more importantly, misses $\{0, 1\}$. Thus every preimage of A_n contains a preimage of x_n in its interior. For every $m < n$, let $A_m = f^{n-m}(A_n)$ and for every $m > n$ let A_m be the component of $f^{n-m}(A_n)$ that contains x_m . Notice that $f|_{A_i}$ is monotone for every i , hence $\varprojlim\{A_i, f|_{A_i}\}$ is an arc in the inverse limit containing x with endpoints a and b . Furthermore, $d(x, a) > \frac{|x_n - a_n|}{2^n} = \frac{\gamma_n}{2^n}$ and similarly $d(x, b) > \frac{\gamma_n}{2^n}$.

To see that x has such a neighborhood U , notice that by taking an inverse limit along any path of the components of inverse images of A_n we generate an arc in the inverse limit space. Since f is the core of a tent map, we have two choices of inverse image infinitely often along this path of inverse images of A_n . This demonstrates that $\pi_n^{-1}(A_n) \cap \varprojlim\{[0, 1], f\}$ is a product of a Cantor set and an arc. □

The proof of the next lemma can be found in [10].

Lemma 3.5. *If f is the core of a tent map with critical point c and $x \in \varprojlim\{[0, 1], f\}$ has the property that for all $n \in \mathbb{N}$, $\pi_n(x) = x_n \in \omega(c)$ then every chaining of $\varprojlim\{[0, 1], f\}$ contains an essential turnlink containing x .*

Combining these two lemmas with the previous analysis of the ω -limit set of the critical point for the family of tent maps under consideration, we have the following theorem.

Theorem 3.6. *Let W_a be a word from $\{L, R\}$, and let $\{n_i\}_{i=1}^\infty$ be an increasing sequence of positive integers. Let f_m be the core of a tent map with critical point c such that*

$$K(f_m) = W_a \wedge R^{n_1} \wedge W_a \wedge R^{n_2} \dots$$

Then $\varprojlim\{[0, 1], f_m\}$ is a continuum with a countable set that is invariant under any autohomeomorphism and a point, $\bar{p} = (p_m, p_m, \dots)$, that is fixed under any autohomeomorphism.

Proof. It is clear from the previous discussion that $\omega(c)$ is a countable set, so the collection of points

$$A = \{x \in \varprojlim\{[0, 1], f_m\} : x_n \in \omega(c) \text{ for all } n \in \mathbb{N}\}$$

is countable. By Lemma 3.5 and by the fact that the homeomorphic image of an essential turnlink is an essential turnlink, [4], it follows that this set is invariant under any autohomeomorphism.

The point $\bar{p} = (p_m, p_m, \dots) \in \varprojlim\{[0, 1], f_m\}$ has the property that for every $\epsilon > 0$, $B_\epsilon(\bar{p})$ meets A . Clearly this is the only point in the inverse limit with this property. Hence \bar{p} is fixed under any autohomeomorphism. \square

It is also of interest to know the structure of the set of endpoints for such inverse limit spaces. In “Endpoints of Inverse Limit Spaces and Dynamics,” Barge and Martin prove that the inverse limit of a core of a tent map will have a non-empty set of endpoints if, and only if, the critical point is *recurrent*, i.e. $c \in \omega(c)$, [2]. We saw quite clearly that $c \notin \omega(c)$ for the collection of tent map cores with kneading sequences of the form described in Theorem 3.6. Thus even though $\varprojlim\{[0, 1], f_m\}$ has many points invariant under autohomeomorphisms and even a point that is fixed under all autohomeomorphisms, $\varprojlim\{[0, 1], f_m\}$ has no endpoints.

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