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ACTION OF CONVERGENCE GROUPS

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Abstract. This is a preliminary report on the continuous action of convergence groups on convergence spaces. In particular, the convergence structure on the homeomorphism group and its continuous action are investigated in this paper. Also, attempts have been made to establish a one-to-one correspondence between continuous action of a convergence group and its homeomorphic representation on a convergence space.

1. Introduction

In algebra, a homomorphism of a group G into the symmetric group $S(\Omega)$ of all permutations on the phase space Ω is known as a *permutation representation* of G on Ω [5]. It is also known that there exists a one-to-one correspondence between the permutation representations of G on Ω and the actions of G on Ω. If $\Omega = X$ is a compact or a locally compact topological space and G is a topological group, then there is a one-to-one correspondence between continuous homomorphisms of a topological group G into the homeomorphism group $H(X)$ and the continuous group actions of G on X [12]. However, for general topological spaces this one-to-one correspondence is not available, since in such case $H(X)$ does not have a group topology which is admissible [11].

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So in establishing such a one-to-one correspondence between continuous representations and continuous group actions of a topological group, the question of existence of an appropriate geometric structure on X which induces an admissible and compatible structure on $H(X)$ is crucial. Recently, Di Concilio [4] has proved that a Tychonoff space X, which is also rim-compact induces an admissible and compatible topological structure on $H(X)$. This result can be further generalized by replacing the topology by a special type of convergence structure [10]. In Section 4 of this paper the author tries to investigate the nature of convergence structure on $H(X)$ corresponding to the existing convergence structure on X. In Section 5 attempts have been made to establish a one-to-one correspondence between the homeomorphic representations and continuous group actions of a special type of convergence group, called limit group.

2. Preliminaries

For basic definitions and terminologies related to filters the reader is referred to [14], though a few of the definitions and notations, which are frequently used will be mentioned here. Let $\mathbf{F}(X)$ denote the collection of all filters on $\mathbf{X}$ and $\mathbf{P}(X)$ denote all the subsets of X. If B is a *base* [14] of the filter $\mathcal{F}$, then $\mathcal{F}$ is said to be *generated by* B and we write $\mathcal{F} = [B]$. For each $x \in X$, $\dot{x} = [\{x\}]$ is the fixed ultra filter containing $\{x\}$. If $\mathcal{F}$ and $\mathcal{G} \in \mathbf{F}(X)$ and $F \cap G \neq \phi$ for all $F \in \mathcal{F}$, $G \in \mathcal{G}$, then $\mathcal{F} \vee \mathcal{G}$ denotes the filter generated by $\{F \cap G : F \in \mathcal{F}$ and $G \in \mathcal{G}\}$. If $\exists F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G = \phi$, then we say that $\mathcal{F} \vee \mathcal{G}$ *fails to exist* . If $(X, .)$ is a group, then we can define a binary operation '$\circ$' on $\mathbf{F}(X)$ as $\mathcal{F}\mathcal{G} = [\{FG :\ F \in \mathcal{F}$ and $G \in \mathcal{G}]$, and $\mathcal{F}^{-1} = [\{F^{-1} : F \in \mathcal{F}\}]$. However, in general, $(\mathbf{F}(X)\ ,\ \circ)$ is a monoid, not a group , since $\dot{e} \geq \mathcal{F}\mathcal{F}^{-1}$, where e is the identity element in $(X, .)$.

For a set X, consider $q \subseteq \mathbf{F}(X) \times X$. Conventionally, for $(\mathcal{F}, x) \in \mathbf{F}(X) \times X$, we write $\mathcal{F} \rightarrow^q x$, and say that $\mathcal{F}$ '$q-$*converges* to x ', whenever $(\mathcal{F}, x) \in q$.

Definition 2.1. Let X and q be as above. For $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$ and $x \in X$, consider the following conditions.

(c$_1$) $\dot{x} \rightarrow^q x$ for each $x \in X$.

(c_2) $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{F} \to^q x \Longrightarrow \mathcal{G} \to^q x$.

(c_3) $\mathcal{F} \to^q x \Longrightarrow \mathcal{F} \cap \dot{x} \to^q x$.

(c_4) $\mathcal{F} \to^q x, G \to^q x \Longrightarrow \mathcal{F} \cap \mathcal{G} \to^q x$.

(c_5) If for each ultrafilter $\mathcal{G} \geq \mathcal{F}$, $\mathcal{G} \to^q x$, then $\mathcal{F} \to^q x$.

(c_6) For each $x \in X, \nu_q(x) \to^q x$, where $\nu_q(x) = \cap\{\mathcal{F} : \mathcal{F} \to^q x\}$.

(c_7) For each $x \in X$, and for each $V \in \nu_q(x)$, $\exists\ W \in \nu_q(x)$ such that $W \subseteq V$ and $y \in W \Rightarrow V \in \nu_q(y)$.

The function q is called a *preconvergence* structure (respectively, $convergence, limit, pseudotopology, pretopology, topology$) if it satisfies $(c_1) - (c_2)$ $(respectively, (c_1) - (c_3), (c_1) - (c_4), (c_1) - (c_5), (c_1) - (c_6), (c_1) - (c_7))$. It should be noted that all these axioms are not independent. For example, (c_1)and (c_4) imply (c_3) and (c_2)and (c_6) imply (c_5).Convergence space, limit space, pseudotopological space, pretopological space and topological space are defined likewise. More details on convergence spaces and related topics can be found in [9]. Note that limit spaces used to be called convergence spaces by Binz [2], Park [10] and several contemporary topologists.

For any $x \in X$, let $q(x) = \{\mathcal{F} \in \mathbf{F}(X) : \mathcal{F} \to^q x\}$.

A convergence space (X, q) is said to be

$\cdot$ T_2 or $Hausdorff$ iff $x = y$, whenever any filter $\mathcal{F} \to^q x, y$.

$\cdot$ *Compact* iff each ultrafilter on X converges to at least one point in X.

A mapping $f : (X, q) \to (Y, p)$ is *continuous* iff whenever $\mathcal{F} \to^q x$, $f(\mathcal{F}) \to^p f(x)$. Furthermore, a continuous mapping f is a homeomorphism, if it is bijective and f^{-1} is also continuous. The set $H(X)$ (respectively, $C(X)$) denotes the set of all self homeomorphisms (respectively, continuous functions) on X. $H(X)$ forms a group with respect to composition of maps.

Next, another important category of spaces is introduced which exists in literature mostly as a means to generalize the completion theory for uniform spaces [13]. For a detailed information on Cauchy spaces the reader is referred to [9].

Definition 2.2. Let C be a set of filters on a set X. The pair (X, C) is called a $Cauchy$ space, if

(1) $\dot{x} \in C$, for each $x \in X$.

(2) $\mathcal{F} \in C$ and $\mathcal{F} \leq \mathcal{G}$ implies $\mathcal{G} \in C$.

(3) $\mathcal{F}$, $\mathcal{G} \in C$ and $\mathcal{F} \vee \mathcal{G}$ exists imply $\mathcal{F} \cap \mathcal{G} \in C$.

Associated with each Cauchy structure C on X, there is a convergence structure q_c, which is defined as $\mathcal{F} \to^{q_c} x$, iff $\mathcal{F} \cap \dot{x} \in C$. A Cauchy space is said to be *complete*, if every $\mathcal{F} \in C$ is q_c-convergent. Note that every convergence space is not a Cauchy space. A convergence structure q on X is said to be *Cauchy compatible*, if there exists a Cauchy structure on X such that $q = q_c$.

Lemma 2.3. [9] *A convergence space (X, q) is Cauchy compatible if and only if it is a limit space and for any $x \neq y \in X$, either $q(x) = q(y)$ or $q(x) \cap q(y) = \phi$.(*)*

3. Convergence Groups

Convergence groups, especially limit groups have been studied in detail in the last half century. For clarity of notations and terminologies the reader is referred to [6, 8, 10].

Definition 3.1. A triplet $(X, q, \cdot)$ is a *$pre-convergence$ group*, if

(cg_1) (X, q) is a pre-convergence space

(cg_2) $(X, \cdot)$ is a group

(cg_3) $\mathcal{F} \to^q x$ and $\mathcal{G} \to^q y$ implies that $\mathcal{F}\mathcal{G}^{-1} \to^q xy^{-1}$.

Note that the binary operation '.' and the inverse operation in the group $(X, .)$ are continuous with respect to the pre-convergence structure q. A pre-convergence group is a convergence group (respectively, *limit group*, *pseudotopological group*, *pretopological group*, *topological group*), iff (X, q) is a convergence space (respectively, limit space, pseudotopological space, pretopological space, topological space).

Lemma 3.2. [8] *If e denotes the identity element in this group, then $x\nu_q(e) = \nu_q(e)x = \nu_q(x), \forall x \in X$.*

Lemma 3.3. [8] *The left translation $x \to ax$, for a fixed $a \in G$ (respectively, the right translation $x \to xa$), the map $x \to x^{-1}$ and the inner automorphisms $x \to axa^{-1}$ are all homeomorphisms of X onto X.*

The proof of the lemma follows from (cg_3). Note that every pre-convergence group is homogeneous. So some of the local properties can be proved at a single point.

Lemma 3.4. *The property (*) in Lemma 2.3. always holds in a pre-convergence group $(X, q, \cdot)$.*

Proof. Suppose $q(x) \cap q(y) \neq \phi$ and let $\mathcal{F}$ be such that $\mathcal{F} \rightarrow^q x$ and $\mathcal{F} \rightarrow^q y$. Let $\mathcal{G} \in q(x)$. Then, $\mathcal{F}^{-1}\mathcal{G} \rightarrow^q e$, where e is the identity in X. This implies that $\mathcal{F}\mathcal{F}^{-1}\mathcal{G} \rightarrow^q y$, since $\mathcal{F} \rightarrow^q y$. Therefore $\mathcal{G} = \dot{e}\mathcal{G} \rightarrow^q y$. Similarly, we can show that $q(y) \subseteq q(x)$. □

Lemma 3.5. *Let $\mathcal{F}$, $\mathcal{G} \in$ **F(X)**, where $(X, \cdot)$ is a group. Then $\mathcal{F} \vee \mathcal{G}$ exists implies that $\mathcal{F} \cap \mathcal{G} \geq \mathcal{F}\mathcal{G}^{-1}\mathcal{G}$.*

Proof. Let $F \in \mathcal{F}$ and $G_1, G_2 \in \mathcal{G}$. Since $G_1 \cap G_2 \neq \phi$, $F \subset FG_1^{-1}G_2$. Also, $\mathcal{F} \vee G$ exists implies that $G_2 \subset FG_1^{-1}G_2$. This proves the lemma. □

Lemma 3.6. [14] *$\mathcal{L}$ is an ultrafilter on X if and only if $A \cup B \in \mathcal{L}$ implies either A or B is in $\mathcal{L}$.*

Lemma 3.7. [3] *If $(X, q, \cdot)$ is a convergence group such that q is pretopological, then $(X, q, \cdot)$ is a topological group.*

4. Homeomorphism Groups

If (X, q) is a pre-convergence space, then the set of all self-homeomorphisms on X is denoted by $\mathbf{H}(X)$. Under the composition of maps $\mathbf{H}(X)$ forms a group, called the *homeomorphism group of* X. We can define a function σ on the filters on $\mathbf{H}(X)$ as follows:

For any filter Φ on $\mathbf{H}(X)$ and $f \in \mathbf{H}(X)$, we define $f \in \sigma(\Phi)$ or equivalently, $\Phi \rightarrow^\sigma f$, iff

(1) $\forall x \in X$, $\mathcal{F} \rightarrow^q x$ in X implies that $\Phi(\mathcal{F}) \rightarrow^q f(x)$ in X,

(2) $\mathcal{F} \rightarrow^q x$ in X implies that $\Phi^{-1}(\mathcal{F}) \rightarrow^q f^{-1}(x)$ in X,

where $\Phi^{-1} = [\{P^{-1} : P \in \Phi\}]$, and $\Phi(\mathcal{F}) = w(\Phi \times \mathcal{F})$ with w as the evaluation map, $w : \mathbf{H}(X) \times X \rightarrow X$. A pre-convergence structure on $\mathbf{H}(X)$ is said to be *admissible*, if the evaluation map is continuous. Note that if X is a locally compact (respectively, compact topological space), then σ coincides with the $g-$ topology [1] (respectively, $\gamma-$topology [1]) .

In 1972, Park [10] has shown that if q is a limit structure on X, then $(\mathbf{H}(X), \sigma)$ is a limit group and σ is the coarsest admissible limit structure on $\mathbf{H}(X)$. We intend to study the convergence structures on X which induce a suitable admissible convergence structure on the homeomorphism group $H(X)$. The following corollary can be derived from the proof of Theorem 5 [10].

Corollary 4.1. *If q is a pre-convergence structure on X, then $\mathbf{H}(X)$ is a pre-convergence group and σ is the coarsest admissible preconvergence structure on $\mathbf{H}(X)$.*

Proposition 4.2. *If $(X, q, \cdot)$ is a convergence group, then σ is the coarsest admissible group convergence structure on $\mathbf{H}(X)$.*

Proof of Proposition 4.2 follows from Lemma 3.5.

Proposition 4.3. *If q is a pseudotopology on X, then σ is the coarsest admissible pseudotopology on $\mathbf{H}(X)$.*

Proof. Since (X, q) is also a preconvergence space, it follows from Corollary 4.1 that $\mathbf{H}(X)$ satisfies (c_1), (c_2) of Definition 2.1 and (cg_3) of Definition 3.1. So $(\mathbf{H}(X), \sigma, \circ)$ is a preconvergence group. So it is enough to show that σ is a pseudotopology on $\mathbf{H}(X)$, whenever q is a pseudotopology on X. Let Φ be a filter on $\mathbf{H}(X)$ such that Φ is not σ -convergent, but for any ultrafilter $\Psi \geq \Phi$, $\Psi \rightarrow^{\sigma} f$ for some $f \in \mathbf{H}(X)$. Since $\Phi \nrightarrow^{\sigma} f$, there exists a filter $\mathcal{F} \rightarrow^{q} x$ in X such that $\Phi(\mathcal{F}) \nrightarrow^{q} f(x)$. Since q is a pseudotopology, there exists an ultrafilter $\Re \geq \Phi(\mathcal{F})$ such that $\Re \nrightarrow^{q} f(x)$.

Claim: If $\Re$ is an ultrafilter on X such that $\Re \geq \Phi(\mathcal{F})$, then $\exists$ an ultrafilter $\mathcal{L} \geq \Phi$ such that $\Re \geq \mathcal{L}(\mathcal{F})$.

Proof of the claim: Let $\Theta = \{\Lambda : \Lambda \text{ is a filter on } \mathbf{H}(X) \textit{ with } \Lambda \geq \Phi \text{ and } \Re \geq \Lambda(\mathcal{F})\}$. A standard Zorn's lemma argument shows that Θ has a maximal element. Let $\mathcal{L}$ be the maximal element in Θ. So it is enough to show that $\mathcal{L}$ is an ultrafilter. Let $A \cup B \in \mathcal{L}$, we need to show that either A or B is in $\mathcal{L}$. If neither A nor B is in $\mathcal{L}$, then

$$\mathcal{L}_A = [\{A \cap M : M \in \mathcal{L}\}] \gneqq \mathcal{L}, \mathcal{L}_B = [\{B \cap M : M \in \mathcal{L}\}] \gneqq \mathcal{L}.$$

Since $\mathcal{L}$ is maximal in Θ, $\mathcal{L}_A$, $\mathcal{L}_B$ both fail to be in Θ. So there exist M_1, $M_2 \in \mathcal{L}$ and there exist F_1, $F_2 \in \mathcal{F}$ such that $(M_1 \cap A)(F_1)$, $(M_2 \cap B)(F_2) \notin \Re$. Let $M = M_1 \cap M_2$ and $F = F_1 \cap F_2$. Since $A \cup B \in \mathcal{L}$, $(M \cap (A \cup B))(F) \in \mathcal{L}(\mathcal{F})$, which implies that $(M \cap (A \cup B))(F) \in \Re$. However, $(M \cap (A \cup B))(F) \subseteq (M_1 \cap A)(F_1) \cup (M_2 \cap B)(F_2) \notin \Re$, since $\Re$ is an ultrafilter. So $(M \cap (A \cup B))(F) \notin \Re$, which leads to a contradiction. This proves the claim.

Since $\mathcal{L}$ is an ultrafilter $\geq \Phi$, $\mathcal{L} \rightarrow^{\sigma} f$ and therefore $\mathcal{L}(\mathcal{F}) \rightarrow^{q} f(x)$. This implies that $\Re \rightarrow^{q} f(x)$, which leads to a contradiction.

This proves Proposition 4.3. $\square$

Note that in the special case when X is a locally compact T_2 topological space, $\mathbf{H}(X)$ is a topological group and σ coincides with the $g-$topology [1].

The proof of the following proposition follows immediately from Theorem 5 [10], Lemma 2.3, Lemma 3.4 and Corollary 4.1.

Proposition 4.4. *If* $(X, q, \cdot)$ *is a limit group, then* $\mathbf{H}(X)$ *is a complete Cauchy space.*

The Cauchy structure on $\mathbf{H}(X)$ can be explicitly given as $C^\sigma =$ all $\sigma-$convergent filters on $\mathbf{H}(X)$.

5. Continuous Group Action

Consider a function on $X \times G$ into X, denoted by $(x, g) \rightarrow x^g$. For any $F \subseteq X$, $g \in G$, $F^g = \{f^g : f \in F\}$ and for any $A \subseteq G$, $F^A = \cup\{F^g : g \in A\}$. If $\mathcal{F} \in \mathbf{F}(X)$ and $g \in G$, then $\mathcal{F}^g = [\{F^g : F \in \mathcal{F}\}]$, and for any subset $A \subseteq G$, $\mathcal{F}^A = [\{F^A : F \in \mathcal{F}\}]$. If $\kappa \in \mathbf{F}(G)$,then $\mathcal{F}^\kappa = [\{F^A : F \in \mathcal{F},\ A \in \kappa\}]$.

Definition 5.1. Let $(G, \Lambda, .)$ be a convergence group and (X, q) be a convergence space. G is said to *act continuously* on X, if the function $X \times G \rightarrow X$, denoted by $(x, g) \rightarrow x^g$, $\forall x \in X$and $\forall g \in G$, satisfies the following conditions :

(a1) $\forall\ \mathcal{F} \rightarrow^q x$ and $\forall\ \kappa \rightarrow^\Lambda g$, $\mathcal{F}^\kappa \rightarrow^q x^g$, where $\mathcal{F} \in \mathbf{F}(X)$ and $\kappa \in \mathbf{F}(G)$.

(a2) $x^e = x, \ \forall x \in X$.

(a3) $(x^g)^h \ = x^{gh}, \forall x \in X$ and $\forall\ \ g, h \in G$.

Note that condition (a1) implies the continuity of the group action at each $x \in X$ and $g \in G$. The action is said to be *effective*, if $x^g = x, \forall\ x \in X$ implies that $g = e$. The action is *free*, if for some x, $x^g = x, \Leftrightarrow g = e$. An action of a group G is *transitive on* X, if for any $x, y \in X$, $\exists g \in G$ such that $x^g = y$.

Lemma 5.2. *The conditions* (a2) *and* (a3) *are equivalent to* (a2$'$) $\mathcal{F}^e = \mathcal{F}$ *and* (a3$'$) $(\mathcal{F}^g)^h = \mathcal{F}^{gh}$, $\forall \mathcal{F} \in \mathbf{F}(X)$ *and* $\forall\ \ g, h \in G$.

This can be verified by substituting $\dot{x}$ for $\mathcal{F}$ in (a2$'$) and (a3$'$). So it follows that if group G algebraically acts on a set X, then G algebraically acts on the set $\mathbf{F}(X)$ of all filters on X, the action being defined as $(\mathcal{F}, g) \longmapsto \mathcal{F}^g$ on $\mathbf{F}(X)$.

Example 5.3. **T**he homeomorphism group $H(X)$ acts continuously on (X, q), the action being defined by the map $(x, f) \mapsto x^f = f^{-1}(x)$, for all $x \in X$ and $f \in H(X)$.

Example 5.4. Any convergence group $(G, \Lambda, .)$ acts on itself continuously, the action being defined by the *right multiplication*, $(a, x) \mapsto a^x = ax$, $\forall a, x \in G$. This follows from (cg$_3$) and (a$_1$). The left multiplication $(a, x) = xa$, $\forall a, x \in G$ is not an action, but $(a, x) \mapsto a^x = x^{-1}a$, $\forall a, x \in G$ is an action of G on itself. In case of right regular representation, if a_1, $a_2 \in G$, then there exists $x = a_1^{-1}\ a_2 \in G$ such that $a_1^x = a_2$. Similarly, in the latter action $a_1^y = a_2$, by choosing $y = a_1\ a_2^{-1}$. So both these actions are transitive. G also acts on itself by conjugation, i.e., $a^x = x^{-1}ax$. But this is not transitive in general.

Next, we try to construct new group actions from a given group action of a convergence group G on a convergence space X.

Example 5.5. If G acts continuously on X then G acts continuously on $Y = X \times X$, where the action is defined by $(x_1, x_2)^g = (x_1^g, x_2^g)$, $\forall\ \ g \in G$ and $x_1, x_2 \in X$. Here it is assumed that Y has the product convergence structure [8].

Proposition 5.6. *Let* (X, q) *and* (Y, p) *be two limit spaces. If* G *acts continuously on* (X, q) *then* G *acts continuously on* $C(X, Y) = \{f : f : X \to Y,\ f$ *is continuous*$\}$*, with respect to the continuous convergence structure* Λ *[2] on* $C(X, Y)$.

Proof. Define the action as $(f, g) \longmapsto f^g$, where $f^g(x) = f(x^{g^{-1}})$, $\forall x \in X$ and $g \in G$. Since for any $\mathcal{F} \to^q x$ in X, $f(\mathcal{F}) \to^p f(x)$ and $f^g(\mathcal{F}) = f(\mathcal{F}^{g^{-1}})$, it follows from (a1) that $f^g(\mathcal{F}) \to^p f(x^{g^{-1}})$, which shows $f^g \in C(X, Y)$. Since $f^e(x) = f(x^e) = f(x), \forall x \in X$, $f^e = f$ and $(f^g)^h(x) = f^g(x^{h^{-1}}) = f((x^{h^{-1}})^{g^{-1}}) = f(x^{(gh)^{-1}}) = f^{gh}(x) \forall x \in X \Rightarrow (f^g)^h = f^{gh}$. This proves (a2) and (a3). To prove (a1) let $\Phi \to^\Lambda f$ in $C(X, Y)$ and $\Re \to^\gamma g$ in G. Then $\Re^{-1} \to^\gamma g^{-1}$ in G and by the continuous action of G on X, for any $\mathcal{F} \to^q x$ in X, $\mathcal{F}^{\Re^{-1}} \to^q x^{g^{-1}}$ in X. So, by definition of Λ, $\Phi(\mathcal{F}^{\Re^{-1}}) \to^p f(x^{g^{-1}})$, in Y, which implies that $\Phi^{\Re}(\mathcal{F}) \to^p f^g(x), \forall \mathcal{F} \to^q x$ and therefore, $\Phi^{\Re} \to^\Lambda f^g$. This proves Proposition 5.6. $\square$

In particular, G acts continuously on $C(X)$, the space of self-continuous maps on X,whenever G acts continuously on (X, q).

Proposition 5.7. *Let (X,q) and (Y,p) be two limit spaces. If G acts continuously on (Y,p) then G acts continuously on $C(X,Y)=\{f : f : X \to Y,\ f$ is continuous$\}$, with respect to the continuous convergence structure Λ [2] on $C(X,Y)$.*

Proof. Define the action as $(f,g) \longmapsto f^g$, where $f^g(x) = (f(x))^g$, $\forall x \in X$ and $g \in G$. Since $f^e(x) = (f(x))^e = f(x)$ and $(f^g)^h(x) = (f(x))^{gh}, \forall x \in X$, (a2),(a3) are satisfied. Also for any $\mathcal{F} \to^q x$, since $f(\mathcal{F}) \to^p f(x)$ and $f^g(\mathcal{F}) = (f(\mathcal{F}))^g$, it follows from (a1) that $f^g(\mathcal{F}) \to^p (f(x))^g$, which shows $f^g \in C(X,Y)$. The continuity of the action follows from the admissibility of the continuous convergence structure and (a1) in Definition 5.1 by arguments similar to those of Proposition 5.6. □

Proposition 5.8. *If $(G, \gamma, \cdot)$ acts on the limit space (X,q), then G acts on $(H(X),\sigma)$, the group of self-homeomorphisms with respect to the double convergence σ.*

Proof. For any $f \in H(X)$ and $g \in G$ define the action as $(f,g) \longmapsto f^g$, where $f^g(x) = f(x^{g^{-1}})$, $\forall x \in X$. We first show that $f^g \in H(X)$ for any $f \in H(X)$. Let $f^g(x_1) = f^g(x_2)$, i.e., $f(x_1^{g^{-1}}) = f(x_2^{g^{-1}})$. This implies that $x_1^{g^{-1}} = x_2^{g^{-1}}$, since f is bijective and therefore $x_1 = x_2$. Next, for any $y \in X$, $\exists x^g \in X$, such that $f^g(x^g) = f((x^g)^{g^{-1}}) = f(x) = y$, this implies that f^g is bijective. The continuity of f^g can be proved by following the same line of argument as in Proposition 5.6. Define $(f^g)^{-1}(x) = (f^{-1}(x))^g$, for all $x \in X$. Note that

$$(f^g)^{-1}(f^g(x)) = (f^{-1}(f^g(x))^g = (f^{-1}(f(x^{g^{-1}}))^g = x$$

and

$$f^g((f^g)^{-1}(x)) = f(((f^{-1}(x))^g)^{g^{-1}}) = f(f^{-1}(x)) = x.$$

To show that $(f^g)^{-1}$ is continuous, let $\mathcal{F} \to^q x$ in X. Then $f^{-1}(\mathcal{F}) \to^q f^{-1}(x)$, which implies that $(f^{-1}(\mathcal{F}))^g \to^q (f^{-1}(x))^g$, since f^{-1} is continuous and G acts continuously on X. So, $(f^g)^{-1}(\mathcal{F}) \to^q (f^g)^{-1}(x)$.

Now, we are going to show that the action on $H(X)$ is continuous. Let $\Phi \to^\sigma f$ in $H(X)$ and $\Re \to^\gamma g$ in G, then by arguments similar to those of Proposition 5.6, we can show that $\Phi^{\Re}(\mathcal{F}) \to^q f^g(x)$ for all $\mathcal{F} \to^q x$ in X. Also, $(\Phi^{\Re})^{-1}(\mathcal{F}) = (\Phi^{-1}(\mathcal{F}))^{\Re}$ and since $\Phi^{-1}(\mathcal{F}) \to^q f^{-1}(x)$, $(\Phi^{-1}(\mathcal{F}))^{\Re} \to^q (f^{-1}(x))^g$. So we have $(\Phi^{\Re})^{-1}(\mathcal{F}) \to^q (f^g)^{-1}(x)$, which implies that $\Phi^{\Re} \to^\sigma f^g$. □

Next, a one-to-one correspondence has been established between the homeomorphic representations of a convergence group and its continuous actions on a limit space.

Lemma 5.9. *If a convergence group G acts continuously on a convergence space X, then for each $g \in G$, the mapping $\bar{g} : X \to X$, defined by $(x)\bar{g} = x^g$ is a homeomorphism.*

Proof. It is routine to show that $\bar{g}$ is a bijection and $\bar{g}^{\ -1} = \overline{g^{-1}}$. The fact that $\bar{g}$ is continuous on X for each $g \in G$, follows from (a1), if we substitute $\dot{g}$ for κ. Since $(x)\bar{g}^{\ -1} = (x)^{g^{-1}}$, $\bar{g}^{\ -1}$ is also continuous. This proves the lemma. □

We have seen that $(\mathbf{H}(X), \sigma)$ the homeomorphism group of X acts continuously on X . But every convergence group G acting continuously on X is not necessarily a homeomorphism group, though it is very close to a homeomorphism group when X is a limit space. Define a mapping $\rho : G \to \mathbf{H}(X)$ by $\rho(g) = \bar{g}, \forall g \in G$. Let $\rho(A) = \bar{A}$ $= \{\bar{g} : g \in A\}$ and for any filter κ on G, $\bar{\kappa} = \rho(\kappa) = [\{\bar{K} : K \in \kappa\}]$.

Proposition 5.10. *Let $(X, \Im)$ be a limit space and $(G, \Lambda, .)$ be a convergence group which acts continuously on X. Then, the map ρ is a continuous group homomorphism.*

Proof. Since $\overline{gh} = \bar{g}\ \bar{h}, \rho$ is a group homomorphism. To prove the continuity of ρ, we consider the limit structure σ on $\mathbf{H}(X)$. Let g in G and $\kappa \to^{\Lambda} g$ in G. We need to show that $\rho(\kappa) \to^{\sigma} \bar{g}$. Let $\mathcal{F} \to^{\Im} x$ in X. It suffices to show that $\bar{\kappa}(\mathcal{F}) \to^{\Im} x^g$, and $\bar{\kappa}^{-1}(\mathcal{F}) \to^{\Im} x^{g^{-1}}$, because $\bar{\kappa}^{-1} = \overline{\kappa^{-1}}$. Since $\mathcal{F} \to^{\Im} x$ and $\kappa \to^{\Lambda} g$, by (a1) in Definition 5.1, $\bar{\kappa}(\mathcal{F}) = \mathcal{F}^{\kappa} \to^{\Im} \bar{x}$. Since $\kappa \to^{\Lambda} g$ in G, by (cg$_3$), $\kappa^{-1} \to^{\Lambda} g^{-1}$. Applying (a1) again we can show that $\bar{\kappa}^{-1}(\mathcal{F}) \to^{\Im} x^{g^{-1}}$. This proves Proposition 5.10. □

The mapping ρ may be called the *homeomorphic representation* of the convergence group G on the limit space X. So every continuous group action on X is associated with a homeomorphic representation. If the continuous homomorphism ρ is one to one, i.e., $\text{Kernel}_{\rho} = \{e\}$, then it can be easily checked that $G \approx \rho(G)$.

Next, we show that any continuous group homomorphism $\theta : G \to \mathbf{H}(X)$ on G induces a continuous group action of G on X.

Proposition 5.11. *Let $\theta : G \to \mathbf{H}(X)$, be a continuous homomorphism. Then, G acts continuously on X.*

Proof. Let us define a function on $X \times G \to X$ by $(x,g) \mapsto x^{\theta(g^{-1})}$, where $x^{\theta(g^{-1})} = x^{(\theta(g))^{-1}} = (\theta(g))^{-1}(x)$. Since θ is a homomorphism, $\theta(e) = I_d(X)$, and $\theta(gh) = \theta(g)\theta(h)$. So $x^{\theta(e^{-1})}$ $=\theta(e^{-1})(x) = x$ and $x^{\theta((gh)^{-1})} = x^{\theta(h^{-1}g^{-1})} = x^{\theta(h^{-1})\theta(g^{-1})} = \theta(h^{-1})\theta(g^{-1})(x) = \theta(h^{-1})(x^{\theta(g^{-1})}) = (x^{\theta(g^{-1})})^{\theta(h^{-1})}$ which imply that (a_2), (a_3) hold. So it remains to prove (a_1). Let $\mathcal{F} \to^q x$ and $\forall\ \kappa \to^\Lambda g$, $\mathcal{F}^\kappa \to^q x^g$, where $\mathcal{F} \in \mathbf{F}(X)$ and $\kappa \in \mathbf{F}(G)$. Then, $\mathcal{F}^\kappa = \mathcal{F}^{\theta(\kappa^{-1})} = (\theta(\kappa))^{-1}(\mathcal{F}) \to^q x^{\theta(g^{-1})}$, since θ is continuous and the evaluation map is continuous with respect to σ on $H(X)$. This proves Proposition 5.11. $\square$

In Proposition 5.10 and 5.11, a one-to-one correspondence between homeomorphic representations of a convergence group G and its continuous action on a limit space X has been established. The question whether such a one-to-one correspondence exists for topological groups acting on general topological spaces still remains as an open question. Also, the transitive actions of a convergence group and actions of the quotient convergence group [7] are yet to be investigated.

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