Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 27, No. 2, 2003
Pages 613–641

FREE PARATOPOLOGICAL GROUPS

S. ROMAGUERA, M. SANCHIS, AND M. TKACHENKO*

ABSTRACT. We prove the existence of a free paratopological group $FP(X,\mathcal{U})$ and a free Abelian paratopological group $AP(X,\mathcal{U})$ on every quasi-uniform space $(X,\mathcal{U})$ such that both groups contain $(X,\mathcal{U})$ as a left quasiuniform subspace. The main tool in our proof is an extension of quasi-uniformly continuous quasi-pseudometrics from $(X,\mathcal{U})$ to continuous left invariant quasi-pseudometrics on both $FP(X,\mathcal{U})$ and $AP(X,\mathcal{U})$. Then we show that the paratopological groups $FP(X,\mathcal{U})$ and $AP(X,\mathcal{U})$ are 2-Hausdorff and X is a 2-closed subset of these groups if and only if the uniform space $(X,\mathcal{U} \vee \mathcal{U}^{-1})$ is Hausdorff.

1. INTRODUCTION

The role of free topological groups in the study of general topological groups is extremally important. Free topological groups serve, on the one side, as a source of numerous examples and counterexamples of topological groups with special properties and, on the other side, they proved to be a flexible tool for establishing new

2000 *Mathematics Subject Classification.* Primary 22A30, 54H11, 54E15; Secondary 54D10, 54D15, 54D30, 54E55.

Key words and phrases. Quasi-pseudometric, quasi-uniformity, paratopological group, bitopological space, 2-Hausdorff, 2-closed.

*This author was supported by the Ministerio de la Educación y Cultura of Spain. He thanks his hosts for hospitality and generous support during his sabbatical leave to the Universidad Jaume I, Castellón, Spain, during the academic year 1999/2000.

and subtle results valid for wide classes of topological groups. It suffices to mention Markov's embedding theorem in [16] (every Tychonoff space is homeomorphic to a closed subspace of a Hausdorff topological group) and Arhangel'skii's theorem in [1] on a representation of every topological group as a quotient of a zero-dimensional topological group. Both results are essentially based on the use of free topological groups.

Recent publications in the field of topological algebra show an increasing activity in the study of semitopological and paratopological groups. We recall that paratopological groups are groups with a topology that makes the group multiplication (but not necessarily the inverse operation) continuous. These "topologically asymmetric" groups have been studied by many authors and the terminology is due to Bourbaki's [4]. The main problem in a number of articles has been to find conditions under which a paratopological group becomes a topological group. The reader will find a lot of information on the subject in the works of Numakura [18], Wallace [24], Ellis [6, 7], Zelazko [25], Mukherjea and Tserpes [17], Fletcher and Lindgren [9], Raghavan and Reilly [20], Brand [5], Pfister [19], Grant [12], Reznichenko [21], etc.

Differently, in [2] and [3], Arhangel'skii and Hušek started to investigate the topological properties of paratopological and semitopological groups. Here we continue the study of paratopological groups in their own right. As in free topological groups, it seems natural to define free paratopological groups, investigate their properties, and then apply the new tool for the study of general paratopological groups. In fact, we believe that free paratopological groups is an appropriate tool for solving the long-standing problem on the regularity of Hausdorff paratopological groups.

There are at least three different proofs of the existence of the free topological group $F(X)$ on a Tychonoff space X presented in [16, 13, 11]. Markov in [16] applied a complicated method based on the use of multinorms which required more than fifty pages. Kakutani's proof given in [13] is the shortest one, but it does not contribute much to understanding the properties of $F(X)$. We feel that the most fruitful approach was Graev's idea of extending continuous pseudometrics from X to $F(X)$ [11]. Here we modify Graev's technique and extend quasi-pseudometrics from the set of generators X to the abstract free group $F_a(X)$ and the abstract free Abelian group $A_a(X)$ over X which in turn enables us to prove the existence of a free paratopological group $FP(X)$ and a free Abelian paratopological group $AP(X)$ over an *arbitrary* space X such that both groups contain X as a subspace and are algebraically free over X (Theorem 2.4). The topological asymmetry of the object (that is, the discontinuity of the inverse) forces our argument to be 20% longer than that in the case of free topological groups (see [11, Theorem 1]). More generally, our approach also works to show the existence of a free paratopological group $FP(X,\mathcal{U})$ and a free Abelian paratopological group $AP(X,\mathcal{U})$ for every quasi-uniform space $(X,\mathcal{U})$ such that these groups contain $(X,\mathcal{U})$ as a left quasiuniform subspace and are algebraically free over X (Theorem 3.4).

Then we make the first step towards the study of the topological properties of the paratopological groups $FP(X,\mathcal{U})$ and $AP(X,\mathcal{U})$ and consider the problems as to whether X is closed in these groups or whether the natural group topologies (the upper bound of the original topology and its conjugate) on $FP(X,\mathcal{U})$ and $AP(X,\mathcal{U})$ are Hausdorff. We show in Theorem 4.4 that this is the case iff the uniform space $(X,\mathcal{U}\vee\mathcal{U}^{-1})$ is Hausdorff.

2. Preliminaries

We recall that a *quasi-metric* d on a set X is a non-negative real-valued function on $X\times X$ which satisfies the triangle inequality and is equal to zero only on the diagonal of $X \times X$. In other words, d satisfies the following conditions for all $x, y, z \in X$:

(QM1) $d(x,z) \le d(x,y) + d(y,z)$;

(QM2) $d(x,y) = 0$ iff $x = y$.

If d satisfies only the triangle inequality, then it is called a *quasi-pseudometric*.

If one omits the symmetry and Hausdorff conditions in the definition of a uniform space $(X,\mathcal{U})$, the concept of a *quasi-uniform space* comes. Therefore, a family $\mathcal{U}$ of entourages of the diagonal in $X \times X$ is a *quasiuniformity* on X if the family $\mathcal{U}$ has the following property:

(QU1) for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.

Every quasi-uniformity on a set X is generated by an appropriate family of quasi-pseudometrics on X (see [10, 14]); this corresponds to the fact that every uniformity on X is generated by a certain family of pseudometrics [8, 8.5.5].

Let us turn to paratopological groups. The following description of a complete neighborhood base at the identity of a paratopological group is well known (see [15]).

Proposition 2.1. *Let G be a paratopological group and $\mathcal{N}$ be a base at the identity e of G. Then the family $\mathcal{N}$ has the following four properties:*

(1) *for every $U, V \in \mathcal{N}$, there exists $W \in \mathcal{N}$ with $W \subseteq U \cap V$;*
(2) *for every $U \in \mathcal{N}$ there exists $V \in \mathcal{N}$ such that $V \cdot V \subseteq U$;*
(3) *for every $U \in \mathcal{N}$ and $g \in U$, there exists $V \in \mathcal{N}$ such that $gV \subseteq U$ and $Vg \subseteq U$;*
(4) *for every $U \in \mathcal{N}$ and $g \in G$, there exists $V \in \mathcal{N}$ such that $gVg^{-1} \subseteq U$.*

Conversely, if $\mathcal{N}$ is a family of subsets of an abstract group G containing the identity e of G and satisfying $(1)-(4)$, then G admits the unique topology $\mathcal{T}$ that makes it a paratopological group with $\mathcal{N}$ being a base at e. In addition, if $\{e\} = \cap\mathcal{N}$, then the topology $\mathcal{T}$ satisfies the T_1-separation axiom.

In contrast with the case of topological groups, we can define the *free paratopological group* $FP(X)$ on every space X. Let us give the corresponding definition.

Definition 2.2. Let X be a subspace of a paratopological group G with identity e such that $e \in X$. Suppose that

(1) X algebraically generates G, that is, $\langle X \rangle = G$;
(2) every continuous mapping $f \colon X \to H$ of X to an arbitrary paratopological group H satisfying $f(e) = e_H$ extends to a continuous homomorphism $\tilde{f} \colon G \to H$.

Then G is called the *free paratopological group* on (X, e) and is denoted by $FP(X, e)$.

If all paratopological groups in Definition 2.2 are Abelian, then we obtain the definition of the *free Abelian paratopological group* on (X, e) which will be denoted by $AP(X, e)$. We shall show in Theorem 2.4 that the groups $FP(X, e)$ and $AP(X, e)$ always exist. Similarly to the case of free topological groups, it turns out that the groups $FP(X, e)$ and $AP(X, e)$ do not depend on the choice of the point $e \in X$.

Proposition 2.3. *Let e_1 and e_2 be arbitrary points of a space X. Then the free paratopological groups $FP(X, e_1)$ and $FP(X, e_2)$ are topologically isomorphic. The same is true for $AP(X, e_1)$ and $AP(X, e_2)$.*

Proof. Define the mapping $\varphi\colon X \to FP(X, e_2)$ by $\varphi(x) = x \cdot e_1^{-1}$ for each $x \in X$. Then φ is continuous and $\varphi(e_1)$ is the identity of $FP(X, e_2)$, so φ extends to a continuous homomorphism $\tilde{\varphi}\colon FP(X, e_1) \to FP(X, e_2)$. Similarly, consider the mapping $\psi\colon X \to FP(X, e_1)$ defined by $\psi(x) = xe_2^{-1}$ for each $x \in X$. Then ψ is continuous and $\psi(e_2)$ is equal to the identity of the group $FP(X, e_1)$. Therefore, ψ extends to a continuous homomorphism $\tilde{\psi}\colon FP(X, e_2) \to FP(X, e_1)$.

For every point $x \in X \subseteq FP(X, e_1)$, we have

$$(\tilde{\psi} \circ \tilde{\varphi})(x) = \tilde{\psi}(x \cdot e_1^{-1}) = \tilde{\psi}(x) \cdot (\tilde{\psi}(e_1))^{-1} = (x \cdot e_2^{-1}) \cdot (e_1 \cdot e_2^{-1})^{-1} = x,$$

since e_1 is the identity of $FP(X, e_1)$. So, the composition $f = \tilde{\psi} \circ \tilde{\varphi}$ is a continuous homomorphism of $FP(X, e_1)$ to itself whose restriction to X is the identity mapping. As X algebraically generates the group $FP(X, e_1)$, f must be the identity automorphism. Similarly, the composition $\tilde{\varphi} \circ \tilde{\psi}$ is the identity mapping, hence both $\tilde{\varphi}$ and $\tilde{\psi}$ are topological isomorphisms. The same argument works for $AP(X, e_1)$ and $AP(X, e_2)$. □

The result just proved permits us to denote the paratopological groups $FP(X, e)$ and $AP(X, e)$ simply by $FP(X)$ and $AP(X)$.

Theorem 2.4. *The free paratopological group $FP(X)$ and the free Abelian paratopological group $AP(X)$ on X exist for every space X.*

Our proof of Theorem 2.4 is based on a Graev-type extension of quasi-pseudometrics from X to left invariant quasi-pseudometrics on the abstract groups $F_a(X)$ and $A_a(X)$ (see Theorem 3.2).

3. Extending quasi-pseudometrics from X to $FP(X)$

We show in Theorem 3.2 that every quasi-pseudometric ϱ on a non-empty set X extends to a left invariant quasi-pseudometric $\widehat{\varrho}$ on the abstract free group $F_a(X)$. As usual, an analogous assertion remains valid for the abstract free Abelian group $A_a(X)$. Left invariance of $\widehat{\varrho}$ on $F_a(X)$ means that

$$\widehat{\varrho}(xg, xh) = \widehat{\varrho}(g, h)$$

for all $g, h, x \in F_a(X)$. The proof of Theorem 3.2 is based on a combinatorial work with the words that form the groups $F_a(X)$ and $A_a(X)$. First, we need the notion of a *scheme* that plays a crucial role in our construction. Let A be a subset of $\mathbb{N}$ such that $|A| = 2n$ for some $n \geq 1$. Roughly speaking, a scheme on A is a partition of A to pairs $\{a_i, b_i\}$, with $a_i < b_i$, such that every two intervals $[a_i, b_i]$ and $[a_j, b_j]$ in $\mathbb{N}$ are either disjoint or one contains the other. To be precise, a scheme for A is a bijection $\varphi\colon A \to A$ satisfying the following conditions:

(1) if $i \in A$ and $j = \varphi(i)$, then $j \neq i$ and $\varphi(j) = i$;
(2) there are no $i, j \in A$ such that $i < j < \varphi(i) < \varphi(j)$.

Therefore, a scheme for A is an idempotent permutation of A without fixed points that satisfies (2). Note that (2) is equivalent to saying that no $i, j \in A$ satisfy $i < \varphi(j) < \varphi(i) < j$. The following simple fact follows from our definition of a scheme.

Proposition 3.1. *If n is a positive integer and φ is a scheme for $\{1, 2, \ldots, 2n\}$, then $\varphi(i) = i + 1$ for some $i < 2n$.*

Theorem 3.2. *Every quasi-pseudometric ϱ on a non-empty set X extends to a left invariant quasi-pseudometric $\widehat{\varrho}$ ($\widehat{\varrho}_A$) on the abstract group $F_a(X)$ (respectively, $A_a(X)$). In addition, if (X, ϱ) is a quasi-pseudometric space, then the quasi-pseudometrics $\widehat{\varrho}$ and $\widehat{\varrho}_A$ generate paratopological group topologies on $F_a(X)$ and $A_a(X)$, respectively. In both cases, these topologies induce on X the original topology generated by ϱ.*

Proof. We consider in detail the case of the group $F_a(X)$, and then indicate the necessary changes for the group $A_a(X)$.

Let us fix a point $e \in X$ which is identified with the identity of the group $F_a(X)$. The first step is to extend ϱ to a quasi-pseudometric ϱ^* on the subset $\tilde{X} = X \cup X^{-1}$ of $F_a(X)$. For $x, y \in X$, define the distances $\varrho^*(x, y)$, $\varrho^*(x^{-1}, y^{-1})$, $\varrho^*(x^{-1}, y)$ and $\varrho^*(x, y^{-1})$ by

$$\varrho^*(x, y) = \varrho(x, y), \ \varrho^*(x^{-1}, y^{-1}) = \varrho(y, x),$$

$$\varrho^*(x^{-1}, y) = \varrho(e, x) + \varrho(e, y), \ \varrho^*(x, y^{-1}) = \varrho(x, e) + \varrho(y, e).$$

It immediately follows from our definition that $\varrho^*(z, t) \geq 0$ for all $z, t \in \tilde{X}$. Let us verify that ϱ^* satisfies the triangle inequality

$$\varrho^*(u, w) \leq \varrho^*(u, v) + \varrho^*(v, w)$$

for all $u, v, w \in \tilde{X}$. Suppose that $u = x^\varepsilon$, $v = y^\delta$ and $w = z^\nu$, where $x, y, z \in X$ and $\varepsilon, \delta, \nu = \pm 1$. We have to consider the following cases:

$$\begin{array}{llll} \text{(a)} & u, v, w \in X; & \text{(a}'\text{)} & u, v, w \in X^{-1}; \\ \text{(b)} & u, v \in X,\ w \in X^{-1}; & \text{(b}'\text{)} & u, v \in X^{-1},\ w \in X; \\ \text{(c)} & u, w \in X,\ v \in X^{-1}; & \text{(c}'\text{)} & u, w \in X^{-1},\ v \in X; \\ \text{(d)} & v, w \in X,\ u \in X^{-1}; & \text{(d}'\text{)} & v, w \in X^{-1},\ u \in X. \end{array}$$

The cases (a) and (a′) are clear. By the symmetry argument, it suffices to restrict our attention to the cases (b), (c) and (d). In case (b), we have

$$\begin{aligned} \varrho^*(u,v) + \varrho^*(v,w) &= \varrho^*(x,y) + \varrho^*(y,z^{-1}) \\ &= \varrho(x,y) + \varrho(y,e) + \varrho(z,e) \\ &\geq \varrho(x,e) + \varrho(z,e) = \varrho^*(x,z^{-1}) = \varrho^*(u,w). \end{aligned}$$

Similarly, in case (c) we have

$$\begin{aligned} \varrho^*(u,v) + \varrho^*(v,w) &= \varrho^*(x,y^{-1}) + \varrho^*(y^{-1},z) \\ &= \varrho(x,e) + \varrho(y,e) + \varrho(e,y) + \varrho(e,z) \\ &\geq \varrho(x,e) + \varrho(e,z) \geq \varrho(x,z) = \varrho^*(u,w). \end{aligned}$$

Finally, in case (d) we can write

$$\begin{aligned} \varrho^*(u,v) + \varrho^*(v,w) &= \varrho^*(x^{-1},y) + \varrho^*(y,z) \\ &= \varrho(e,x) + \varrho(e,y) + \varrho(y,z) \\ &\geq \varrho(e,x) + \varrho(e,z) = \varrho^*(x^{-1},z) = \varrho^*(u,w). \end{aligned}$$

We have thus proved that ϱ^* is a quasi-pseudometric on $\tilde{X}$. The following property of ϱ^* is immediate.

Claim 1. $\varrho^*(u^{-1}, v^{-1}) = \varrho^*(v, u)$ for all $u, v \in \tilde{X}$.

Now we have to extend ϱ^* from $\tilde{X}$ to the whole group $F_a(X)$. Let g be a reduced element of $F_a(X)$, and suppose that $\mathcal{X} = x_1 x_2 \ldots x_{2n}$ is a word in the alphabet $\tilde{X}$ of even length $l(\mathcal{X}) = 2n$ such that all possible cancellations in $\mathcal{X}$ transform it to g or, in symbols, $[\mathcal{X}] = g$. Denote by $\mathcal{S}_n$ the family of all schemes for $\{1, 2, \ldots, 2n\}$. For every $\varphi \in \mathcal{S}_n$, put

$$\Gamma_\varrho(\mathcal{X}, \varphi) = \frac{1}{2} \sum_{i=1}^{2n} \varrho^*(x_i^{-1}, x_{\varphi(i)}).$$

The factor $1/2$ appears in the above expression due to the fact that we count every pair $\{x_i, x_j\}$ with $j = \varphi(i)$ twice, and $\varrho^*(x_i^{-1}, x_j) = \varrho^*(x_j^{-1}, x_i)$ by Claim 1. Then we define a number $N_\varrho(g)$ by $N_\varrho(g) = 0$ if $g = e$, and

$$N_\varrho(g) = \inf\{\Gamma_\varrho(\mathcal{X}, \varphi) : [\mathcal{X}] = g,\ l(\mathcal{X}) = 2n,\ \varphi \in \mathcal{S}_n,\ n \in \mathbb{N}^+\}$$

for every $g \in F_a(X)$ distinct from e. It is clear that $N_\varrho(g) \geq 0$ for each $g \in F_a(X)$. We divide the rest of the proof into several steps.

Given a word $\mathcal{X}$ in the alphabet $\tilde{X}$, we say that $\mathcal{X}$ is *almost irreducible* if $\mathcal{X}$ does not contain two consecutive symbols of the form u, u^{-1} or u^{-1}, u (but $\mathcal{X}$ may contain several letters equal to e). Since $e^{-1} = e$, an almost irreducible word of length $2k$ can contain at most k letters equal to e. This simple observation is used in the proof of the following claim.

Claim 2. For every $g \in F_a(X)$ distinct from e, there exist an almost irreducible word $\mathcal{X}_g$ of even length $2n \geq 2$ in the alphabet $\tilde{X}$ and a scheme $\varphi_g \in \mathcal{S}_n$ that satisfy the following conditions:

(i) each letter of $\mathcal{X}_g$ either belongs to g or is equal to e;
(ii) $[\mathcal{X}_g] = g$ and $l(\mathcal{X}_g) \leq 2l(g)$;
(iii) $N_\varrho(g) = \Gamma_\varrho(\mathcal{X}_g, \varphi_g)$.

Indeed, let $\mathcal{X}$ be a word of length $l(\mathcal{X}) = 2n$ with $[\mathcal{X}] = g$ and $\varphi \in \mathcal{S}_n$ be a scheme. Suppose that $\mathcal{X} = x_1 x_2 \dots x_{2n}$, where $x_1, x_2, \dots, x_{2n} \in \tilde{X}$. First, we show that one can find an almost irreducible word $\mathcal{X}_1$ of length $2m$ with $1 \leq m \leq n$ obtained after several cancellations in $\mathcal{X}$ and a scheme $\varphi_1 \in \mathcal{S}_m$ that satisfy (i), (ii) and

(iv) $\Gamma_\varrho(\mathcal{X}_1, \varphi_1) \leq \Gamma_\varrho(\mathcal{X}, \varphi)$.

If $\mathcal{X}$ is reduced, there is nothing to prove. Suppose, therefore, that $\mathcal{X}$ contains either two adjacent symbols of the form uu^{-1} or three adjacent symbols ueu^{-1}, for some $u \in \tilde{X}$ (in the latter case, deleting e from $\mathcal{X}$ produces a new cancellation).

Case I. $\mathcal{X}$ contains two adjacent symbols uu^{-1} for some $u \in \tilde{X}$. Then $x_i = u$ and $x_{i+1} = u^{-1}$ for some $i < 2n$. Let us consider two subcases: $\varphi(i) = i + 1$ or $\varphi(i) \neq i + 1$. If $\varphi(i) = i + 1$, we delete uu^{-1} from $\mathcal{X}$, thus obtaining a word $\mathcal{X}'$, and define φ' as the restriction of φ to $\{1, \dots, i-1, i+2, \dots, 2n\}$. It is clear that $\Gamma_\varrho(\mathcal{X}', \varphi') \leq \Gamma_\varrho(\mathcal{X}, \varphi)$. If $\varphi(i) \neq i + 1$, put $r = \varphi(i)$ and $s = \varphi(i + 1)$. Then $\{r, s\} \cap \{i, i + 1\} = \emptyset$. Again, we delete uu^{-1}

from $\mathcal{X}$, thus obtaining a new word $\mathcal{X}'$, and define a bijection φ' of $A = \{1, \dots, i-1, i+2, \dots, 2n\}$ onto itself by $\varphi'(m) = \varphi(m)$ if $m \notin \{r, s\}$ and $\varphi'(r) = s$, $\varphi'(s) = r$. One easily verifies that φ' is a scheme for A, and it follows from

$$\rho^*(x_r^{-1}, x_s) \le \rho^*(x_r^{-1}, x_i) + \rho^*(x_i, x_s) = \rho^*(x_i^{-1}, x_r) + \rho^*(x_{i+1}^{-1}, x_s)$$

that $\Gamma_\varrho(\mathcal{X}', \varphi') \le \Gamma_\varrho(\mathcal{X}, \varphi)$. Evidently, each letter of $\mathcal{X}'$ is also a letter of $\mathcal{X}$.

Case II. The word $\mathcal{X}$ contains three adjacent symbols $u^{-1}eu$ for some $u \in \tilde{X}$. Then there exists i with $1 < i < 2n$ such that $u^{-1} = x_{i-1}$, $e = x_i$ and $u = x_{i+1}$. Let $r = \varphi(i-1)$, $s = \varphi(i)$ and $t = \varphi(i+1)$. As in Case I, there are two possible subcases: $s \in \{i-1, i+1\}$ or $s \notin \{i-1, i+1\}$. In the former subcase, we can assume without loss of generality that $s = i+1$ and $r < i-1$. Then $\mathcal{X} \equiv Ax_rBu^{-1}euC$, where the words A, B and C have the lengthes $r-1$, $i-r-2$ and $2n-i-1$, respectively. Put $\mathcal{X}' \equiv Ax_rx_iBC$. Since $x_i = e$, we have $[\mathcal{X}'] = [\mathcal{X}] = g$, $l(\mathcal{X}') = l(\mathcal{X}) - 2$, and each letter of $\mathcal{X}'$ is also a letter of $\mathcal{X}$. Let φ' be a bijection of the set $K = \{1, \dots, i-1, i+2, \dots, 2n\}$ which coincides with φ on $K \setminus \{r, i-1\}$ and satisfies $\varphi'(r) = i$, $\varphi'(i) = r$. Then φ' is a scheme on K and an easy calculation shows that

$$\Gamma_\varrho(\mathcal{X}, \varphi) - \Gamma_\varrho(\mathcal{X}', \varphi') = \varrho(x_r, u) + \varrho(u, e) - \varrho(x_r, e) \ge 0.$$

Hence we conclude that $\Gamma_\varrho(\mathcal{X}', \varphi') \le \Gamma_\varrho(\mathcal{X}, \varphi)$.

Suppose now that $s \notin \{i-1, i+1\}$. Since φ is a scheme, neither of the inequalities $r < t < i-1$, $i-1 < t < r$, $i+1 < t < s$, $s < t < i+1$ is possible. Suppose, for example, that $r < i-1 < i+1 < s < t$. Then $\mathcal{X} \equiv Ax_rBu^{-1}euCx_s^{-1}Dx_tE$, where A, B, C, D and E are words of the lengthes $r-1$, $i-r-2$, $s-i-2$, $t-s-1$ and $2n-t$, respectively. Let $\mathcal{X}' \equiv Ax_rBCx_s^{-1}Dx_ix_tE$ be the word obtained from $\mathcal{X}$ by deleting uu^{-1} and translating $e = x_i$ to the letter x_t. It is clear that $[\mathcal{X}'] = g$. Let also φ' be a bijection of the set $K = \{1, \dots, i-2, i, i+2, \dots, 2n\}$ which coincides with φ on $K \setminus \{r, i, s, t\}$ and satisfies $\varphi'(r) = s$, $\varphi'(s) = r$, $\varphi(i) = t$ and $\varphi(t) = i$. One easily verifies that φ' is a scheme on K (which connects x_r with x_s and $x_i = e$ with x_t). Our definition of φ' implies that

$$\Gamma_\varrho(\mathcal{X}, \varphi) - \Gamma_\varrho(\mathcal{X}', \varphi') = \varrho(x_r, u) + \varrho(u, x_s) - \varrho(x_r, x_s) \ge 0,$$

whence $\Gamma_\varrho(\mathcal{X}', \varphi') \le \Gamma_\varrho(\mathcal{X}, \varphi)$. Notice that each letter of $\mathcal{X}'$ is a letter of $\mathcal{X}$. The two more cases when $s < r < i-1$ or $i+1 < s < r$ are similar to the one just considered and, therefore, are left to the reader.

In each of Cases I and II, the length of $\mathcal{X}'$ is strictly less than the length of $\mathcal{X}$ and, in addition, $\mathcal{X}'$ does not contain "new" letters. If the word $\mathcal{X}'$ again fails to be almost irreducible, we apply one of the operations described in Case I and Case II to $\mathcal{X}'$, thus obtaining a word $\mathcal{X}''$ and a scheme φ'' for $\mathcal{X}''$ such that $[\mathcal{X}''] = g$, $\Gamma_\varrho(\mathcal{X}'', \varphi'') \le \Gamma_\varrho(\mathcal{X}', \varphi')$, and so on. Since $g = [\mathcal{X}] = [\mathcal{X}'] = [\mathcal{X}''] = \ldots$ and $\Gamma_\varrho(\mathcal{X}, \varphi) \ge \Gamma_\varrho(\mathcal{X}', \varphi') \ge \Gamma_\varrho(\mathcal{X}'', \varphi'') \ge \ldots$, we finally obtain an almost irreducible word $\mathcal{X}_1$ of even length and a scheme φ_1 for $\mathcal{X}_1$ satisfying (i), (ii) and (iv). Notice that the inequality $l(\mathcal{X}_1) \le 2l(g)$ in (ii) is a consequence of the fact that the word $\mathcal{X}_1$ is almost irreducible.

Finally, for a given element $g \in F_a(X)$, there exist only finitely many pairs $(\mathcal{X}_1, \varphi_1)$ satisfying (i) and (ii). Therefore one of these pairs, say, $(\mathcal{X}_g, \varphi_g)$ satisfies (i)–(iii). This implies Claim 2.

Claim 3. The function N_ϱ is an invariant quasi-prenorm on the group $F_a(X)$. In other words, N_ϱ has the following properties:

(1) $N_\varrho(e) = 0$ and $N_\varrho(g) \ge 0$ for each $g \in F_a(X)$;
(2) $N_\varrho(g \cdot h) \le N_\varrho(g) + N_\varrho(h)$ for all $g, h \in F_a(X)$;
(3) $N_\varrho(h^{-1}gh) = N_\varrho(g)$ for all $g, h \in F_a(X)$.

The property (1) is evident. Now we verify (2). Let g and h be elements of $F_a(X)$ and suppose that $\mathcal{X} = x_1 x_2 \ldots x_{2n}$ and $\mathcal{Y} = y_1 y_2 \ldots y_{2m}$ are words in the alphabet $\tilde{X}$ such that

$$[\mathcal{X}] = g, \;\; [\mathcal{Y}] = h, \;\; N_\varrho(g) = \Gamma_\varrho(\mathcal{X}, \varphi) \text{ and } N_\varrho(h) = \Gamma_\varrho(\mathcal{Y}, \psi),$$

where $\varphi \in \mathcal{S}_n$ and $\psi \in \mathcal{S}_m$. Put $\mathcal{Z} = \mathcal{X}\mathcal{Y} = x_1 \ldots x_{2n} y_1 \ldots y_{2m}$ and rewrite $\mathcal{Z}$ in the form $\mathcal{Z} = z_1 \ldots z_{2n} z_{2n+1} \ldots z_{2n+2m}$, where $z_i = x_i$ if $1 \le i \le 2n$ and $z_{2n+j} = y_j$ if $1 \le j \le 2m$. Define a scheme $\sigma \in \mathcal{S}_{n+m}$ by the formula

$$\sigma(k) = \begin{cases} \varphi(k) & \text{if } 1 \le k \le 2n; \\ 2n + \psi(k - 2n) & \text{if } 2n < k \le 2n + 2m. \end{cases}$$

It is clear that $[\mathcal{Z}] = g \cdot h$ and

$$\Gamma_\varrho(\mathcal{Z}, \sigma) = \Gamma_\varrho(\mathcal{X}, \varphi) + \Gamma_\varrho(\mathcal{Y}, \psi) = N_\rho(g) + N_\rho(h).$$

Therefore,

$$N_\rho(gh) \le N_\rho(g) + N_\rho(h).$$

This proves that N_ϱ is a quasi-prenorm on the group $F_a(X)$. It remain to verify that N_ϱ is invariant, i.e., satisfies (3). In fact, it suffices to verify the above equality in the case when h has the length 1, say, $h = x \in \tilde{X}$. Let $\mathcal{X} = x_1 \dots x_{2n}$ be a word in the alphabet $\tilde{X}$ such that $[\mathcal{X}] = g$ and $\Gamma_\varrho(\mathcal{X}, \varphi) = N_\varrho(g)$ for some $\varphi \in \mathcal{S}_n$. Consider the word $\mathcal{Y} = y_1 y_2 \dots y_{2n+1} y_{2n+2}$, where $y_1 = x^{-1}$, $y_{2n+2} = x$ and $y_k = x_{k-1}$ if $2 \le k \le 2n+1$. Define the scheme $\psi \in \mathcal{S}_{n+1}$ by the formula

$$\psi(k) = \begin{cases} 2n+2 & \text{if } k = 1; \\ 1 & \text{if } k = 2n+2; \\ \varphi(k-1)+1 & \text{if } 2 \le k \le 2n+1. \end{cases}$$

Then $[\mathcal{Y}] = x^{-1}gx$, and hence

$$N_\varrho(x^{-1}gx) \le \Gamma_\varrho(\mathcal{Y}, \psi) = \Gamma_\varrho(\mathcal{X}, \varphi) = N_\varrho(g).$$

Replace x by x^{-1} and g by $x^{-1}gx$ in the above inequality to obtain $N_\varrho(g) \le N_\varrho(x^{-1}gx)$. The two inequalities imply that $N_\varrho(x^{-1}gx) = N_\varrho(g)$. This proves Claim 3.

Claim 4. $N_\varrho(x^{-1}y) = \varrho(x,y) = N_\varrho(yx^{-1})$ for all $x, y \in X$.

Fix two elements $x, y \in X$. Note that by Claim 3,

$$N_\varrho(x^{-1}y) = N_\varrho(xx^{-1}yx^{-1}) = N_\varrho(yx^{-1}),$$

so it suffices to show that $N_\varrho(x^{-1}y) = \varrho(x,y)$. If $x = y$, then clearly

$$\varrho(x,y) = 0 \text{ and } N_\varrho(x^{-1}y) = \Gamma_\varrho(\mathcal{X}, \varphi) = 0,$$

where $\mathcal{X} = x_1x_2$ is the word with $x_1 = x^{-1}$, $x_2 = x$ and $\varphi \in \mathcal{S}_1$ is a transposition of elements $1, 2$. Suppose, therefore, that $x \neq y$.

Put $g = x^{-1}y$. By Claim 2, one can find an almost irreducible word $\mathcal{Y}$ of length $2m \le 4$ in the alphabet $\tilde{X}$ such that $[\mathcal{Y}] = g$ and $\Gamma_\varrho(\mathcal{Y}, \varphi) = N_\varrho(g)$ for some $\varphi \in \mathcal{S}_m$. Since $l(g) = 2$, we have that $m = 1$ or $m = 2$. If $m = 1$, then $\mathcal{Y} = x^{-1}y$ and φ is the transposition of $1, 2$, whence it follows that

$$N_\varrho(x^{-1}y) = \Gamma_\varrho(\mathcal{Y}, \varphi) = \varrho(x,y).$$

Suppose that $m = 2$. Then $\mathcal{Y}$ coincides with one of the words

$$ex^{-1}ey, \ x^{-1}eye, \ ex^{-1}ye.$$

Suppose, for example, that $\mathcal{Y} \equiv ex^{-1}ey \equiv x_1x_2^{-1}x_3x_4$. There are only two distinct schemes for $m = 2$, so that either $\varphi(1) = 2$ and $\varphi(3) = 4$, or $\varphi(1) = 4$ and $\varphi(2) = 3$. In the first case, we have

$$\Gamma_\varrho(\mathcal{Y}, \varphi) = \varrho^*(e^{-1}, x^{-1}) + \varrho^*(e^{-1}, y) = \varrho(x, e) + \varrho(e, y) \geq \varrho(x, y).$$

In the second case,

$$\Gamma_\varrho(\mathcal{Y}, \varphi) = \varrho^*(e^{-1}, y) + \varrho^*(x, e) = \varrho(x, e) + \varrho(e, y) \geq \varrho(x, y).$$

A similar argument shows that $\Gamma_\varrho(\mathcal{Y}, \varphi) \geq \varrho(x, y)$ if $\mathcal{Y}$ is one of the words $x^{-1}eye$ or $ex^{-1}ye$. Therefore, $\Gamma_\varrho(\mathcal{Y}, \varphi) = \varrho(x, y)$, as claimed. This proves Claim 4.

Define a quasi-pseudometric $\widehat{\varrho}$ on $F_a(X)$ by $\widehat{\varrho}(g, h) = N_\varrho(g^{-1}h)$ for all $g, h \in F_a(X)$.

Claim 5. The quasi-pseudometric $\widehat{\varrho}$ is left invariant on $F_a(X)$ and its restriction to X coincides with ϱ.

Indeed, by definition of $\widehat{\varrho}$ we have

$$\widehat{\varrho}(xg, xh) = N_\varrho(g^{-1}x^{-1}xh) = N_\varrho(g^{-1}h) = \widehat{\varrho}(g, h)$$

for all $x, g, h \in F_a(X)$. We conclude, therefore, that the quasi-pseudometric $\widehat{\varrho}$ is left invariant. Claim 4 immediately implies that the restriction of $\widehat{\varrho}$ to X coincides with ϱ, so Claim 5 is proved.

The next step is to see that $\widehat{\varrho}$ generates a paratopological group topology on $F_a(X)$ whose restriction to X coincides with the topology of the quasi-pseudometric space (X, ϱ). For every $\varepsilon > 0$, put

$$U_\varrho(\varepsilon) = \{g \in F_a(X) : N_\varrho(g) < \varepsilon\}.$$

Claim 6. The family $\mathcal{N} = \{U_\varrho(\varepsilon) : \varepsilon > 0\}$ is a base at the identity e for a paratopological group topology $\mathcal{T}_\varrho$ on $F_a(X)$. The restriction of $\mathcal{T}_\varrho$ to X coincides with the topology of the space X generated by ϱ.

We have to verify that the family $\mathcal{T}_\varrho$ satisfies the four conditions of the complete neighborhood system at e (see Proposition 2.1). Let us do this step by step.

1) For every $U, V \in \mathcal{N}$, there exists $W \in \mathcal{N}$ with $W \subseteq U \cap V$. Indeed, if $U = U_\varrho(\varepsilon)$, $V = U_\varrho(\delta)$ and $\varepsilon \leq \delta$, then $U \cap V = U$.

2) For every $U \in \mathcal{N}$ there exists $V \in \mathcal{N}$ such that $V \cdot V \subseteq U$.

Suppose that $U = U_\varrho(\varepsilon)$. Put $\delta = \varepsilon/2$ and $V = U_\varrho(\delta)$. If $g, h \in V$, then we have

$$N_\varrho(gh) \leq N_\varrho(g) + N_\varrho(h) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This immediately implies that $V \cdot V \subseteq U$.

3) For every $U \in \mathcal{N}$ and $g \in U$ there exists $V \in \mathcal{N}$ such that $gV \subseteq U$ and $Vg \subseteq U$.

Again, suppose that $U = U_\varrho(\varepsilon)$, and take an arbitrary element $g \in U$. Then $\varepsilon_1 = N_\varrho(g) < \varepsilon$ and hence $\delta = \varepsilon - \varepsilon_1 > 0$. Put $V = U_\varrho(\delta)$. If $h \in V$, then

$$N_\varrho(gh) \leq N_\varrho(g) + N_\varrho(h) < \varepsilon_1 + \delta = \varepsilon,$$

whence it follows that $gV \subseteq U$. A similar argument implies that $Vg \subseteq U$.

4) For every $N \in \mathcal{N}$ and $g \in F_a(X)$, there exists $V \in \mathcal{N}$ such that $g^{-1}Vg \subseteq U$.

Indeed, one can take $V = U$. If $U = U_\varrho(\varepsilon)$ and $h \in U$, then $N_\varrho(g^{-1}hg) = N_\varrho(h) < \varepsilon$ by Claim 3, so $g^{-1}hg \in U$. This proves that $g^{-1}Ug \subseteq U$.

Therefore, the first part of Claim 6 follows directly from Proposition 2.1. In addition, the restriction of $\widehat{\varrho}$ to X coincides with ϱ, so left invariance of $\widehat{\varrho}$ on $F_a(X)$ implies the second part of Claim 6. In strict terms, it suffices to show that

$$X \cap xU_\varrho(\varepsilon) = \{y \in X : \varrho(x, y) < \varepsilon\}$$

for every $x \in X$ and $\varepsilon > 0$. If $x, y \in X$ and $\varrho(x, y) < \varepsilon$, then $N_\varrho(y^{-1}x) = \varrho(x, y)$ by Claim 4, whence $y = x \cdot (x^{-1}y) \in xU_\varrho(\varepsilon)$. This proves that $\{y \in X : \varrho(x, y) < \varepsilon\} \subseteq xU_\varrho(\varepsilon)$. The inverse inclusion is also clear: if $y \in X \cap xU_\varrho(\varepsilon)$, then $x^{-1}y \in U_\varrho(\varepsilon)$, whence $\varrho(x, y) = N_\varrho(x^{-1}y) < \varepsilon$.

Since the sets of the form $gU_\varrho(\varepsilon)$ form a base of the paratopological group $(F_a(X), \mathcal{T}_\varrho)$, the equality just proved means that the restriction of $\mathcal{T}_\varrho$ to X coincides with the topology of X generated by the quasi-pseudometric ϱ. This proves Claim 6.

The argument for the Abelian group $A_a(X)$ is similar to that just given, but it requires one important change in the definition of a scheme. Given a finite subset B of $\mathbb{N}$ with $|B| = 2n \geq 2$, we say that a bijection $\varphi\colon B \to B$ is an *Abelian scheme* for B if φ is an involution without fixed points, that is, $\varphi(i) = j$ always implies $j \neq i$ and $\varphi(j) = i$. Then one defines an invariant quasi-prenorm N_ϱ^A on the abstract group $A_a(X)$ using Abelian schemes. For example, if $x, y, z, t \in X$ and $g = x - y + z - t$, then $N_\varrho^A(g)$ is equal to the minimum of the numbers $\varrho(y, x) + \varrho(t, z)$ and $\varrho(t, x) + \varrho(y, z)$. The rest of the proof goes the same way, thus giving us a left invariant quasi-pseudometric $\widehat{\varrho}_A$ on $A_a(X)$ extending ϱ. The theorem is proved. $\square$

In what follows we call the left invariant quasi-pseudometric $\widehat{\varrho}$ on $F_a(X)$ (respectively, $\widehat{\varrho}_A$ on $A_a(X)$) defined in Theorem 3.2 the *Graev extension of* ϱ.

Proof of Theorem 2.4. Let X be an arbitrary space. The family $\mathcal{Q}$ of all continuous quasi-pseudometrics ϱ on X generates the original topology τ_X of X ([10], p. 28). For every $\varrho \in \mathcal{Q}$, denote by $\widehat{\varrho}$ the Graev extension of ϱ to a left invariant quasi-pseudometric on $F_a(Y)$. Let $\mathcal{T}_\varrho$ be the paratopological group topology on $F_a(X)$ generated by $\widehat{\varrho}$. Denote by $\mathcal{T}_X$ the supremum of all topologies $\mathcal{T}_\varrho$ on $F_a(X)$, with $\varrho \in \mathcal{Q}$. Then $\mathcal{T}_X$ is also a paratopological group topology on $F_a(X)$, and a simple verification shows that this topology induces on X the original topology τ_X of the space X. We conclude, therefore, that the abstract group $F_a(X)$ admits at least one paratopological group topology whose restriction to X coincides with τ_X.

Denote by $\mathcal{T}$ the supremum of *all* paratopological group topologies on $F_a(X)$ whose restrictions to X coincide with τ_X. This is obviously the finest paratopological group topology on $F_a(X)$ with this property, so that $FP(X) = (F_a(X), \mathcal{T})$. Indeed, let $f\colon X \to H$ be a continuous mapping of X to a paratopological group H whose topology is τ_H, and suppose that $f(e) = e_H$. Let $\tilde{f}\colon F_a(X) \to H$ be a homomorphism extending f. Then the family

$$\mathcal{T}_H = \{\tilde{f}^{-1}(V) : V \in \tau_H\}$$

is a paratopological group topology on $F_a(X)$, and the supremum t of $\mathcal{T}_H$ and $\mathcal{T}_X$ is also a paratopological group topology on $F_a(X)$. Since f is continuous, the topology t induces on X its original topology τ_X. Therefore, t is coarser than $\mathcal{T}$. This means that the homomorphism $\tilde{f}\colon (F_a(X), \mathcal{T}) \to H$ is continuous. In other words, every continuous mapping of X to a paratopological H extends to a continuous homomorphism $\tilde{f}\colon (F_a(X), \mathcal{T}) \to H$, and hence $(F_a(X), \mathcal{T}) = FP(X)$. $\square$

Let us now extend Theorem 2.4 to free paratopological groups on quasi-uniform spaces. First we recall several notions related to quasi-uniform spaces and paratopological groups. Every quasi-uniformity $\mathcal{U}$ on a non-empty set X induces a topology on X defined as follows. For $U \in \mathcal{U}$ and $x \in X$, put

$$U(x) = \{y \in X : (x, y) \in U\}.$$

Then the family

$$\{U(x) : U \in \mathcal{U},\ x \in X\}$$

is a base of neighborhoods for the topology associated with $\mathcal{U}$. It is known that every topology on X is generated by some quasi-uniformity on X [10]. Given a quasi-uniform space $(X,\mathcal{U})$, one defines the conjugate quasi-uniformity $\mathcal{U}^{-1}$ on X with the base $\{U^{-1} : U \in \mathcal{U}\}$, where $U^{-1} = \{(y, x) \in X \times X : (x, y) \in U\}$. In contrast to the case of uniformities, the topologies on X associated with $\mathcal{U}$ and $\mathcal{U}^{-1}$ can be different.

Every paratopological group H admits two natural quasi-uniformities $\mathcal{L}_H$ and $\mathcal{R}_H$ which can be described in terms of the neighborhood base $\mathcal{N}(e_H)$ of the identity e_H of H. For every $U \in \mathcal{N}(e_H)$, put

$$W_U^l = \{(x, y) \in H \times H : x^{-1}y \in U\}$$

and

$$W_U^r = \{(x, y) \in H \times H : xy^{-1} \in U\}.$$

Then the families

$$\{W_U^l : U \in \mathcal{N}(e_H)\} \text{ and } \{W_U^r : U \in \mathcal{N}(e_H)\}$$

are bases for the left and right quasi-uniformities $\mathcal{L}_H$ and $\mathcal{R}_H$ on H, respectively.

Definition 3.3. Suppose that X is a subspace of a paratopological group G with identity e such that $e \in X$, and let $\mathcal{U}$ be the quasi-uniformity on X generated by the left quasi-uniformity $\mathcal{L}_G$ of G. Suppose also that X and G satisfy the following conditions:

(1) X algebraically generates G;
(2) if a quasi-uniformly continuous mapping $f{:}(X,\mathcal{U}) \to (H, \mathcal{L}_H)$ of X to a paratopological group H satisfies $f(e) = e_H$, then f extends to a continuous homomorphism $\tilde{f}\colon G \to H$.

Then G is called the free left paratopological group (or simply free paratopological group) over the quasi-uniform space $(X, e, \mathcal{U})$ and is denoted by $FP(X, e, \mathcal{U})$.

Similarly to Proposition 2.3, one can show that if e_1 and e_2 are points of a quasi-uniform space $(X, \mathcal{U})$ and $G_i = FP(X, e_i, \mathcal{U})$ is the free paratopological group over the quasi-uniform space $(X, e_i, \mathcal{U})$, $i = 1, 2$, then G_1 and G_2 are topologically isomorphic. This enables us to denote $F(X, e, \mathcal{U})$ simply by $F(X, \mathcal{U})$. In addition, if all paratopological groups in the above definition are assumed to be Abelian, we get the definition of the *free Abelian paratopological group* $AP(X, \mathcal{U})$ over $(X, \mathcal{U})$.

Let $(X, \mathcal{U})$ be a quasi-uniform space. Then X carries the topology $\tau_X = T(\mathcal{U})$ generated by the quasi-uniformity $\mathcal{U}$ whose base of neighborhoods consists of the sets

$$U(x) = \{y \in X : (x, y) \in U\},$$

with $U \in \mathcal{U}$ and $x \in X$. The conjugated quasi-uniformity $\mathcal{U}^{-1} = \{V \subseteq X \times X : V^{-1} \in \mathcal{U}\}$ generates the topology $\tau_X^{-1} = T(\mathcal{U}^{-1})$ on X. It is clear that $\tau_X^{-1} = \{W \subseteq X : W^{-1} \in \tau_X\}$. We also say that a topology t_X and a quasi-uniformity $\mathcal{U}$ on X are *compatible* if $t_X = T(\mathcal{U})$. Since every topological space X admits a compatible quasi-uniformity [10], the following result generalizes Theorem 2.4.

Theorem 3.4. *The free paratopological group $FP(X, \mathcal{U})$ and the free Abelian paratopological group $AP(X, \mathcal{U})$ exist for every quasi-uniform space $(X, \mathcal{U})$.*

Proof. It suffices to prove the existence of $FP(X, \mathcal{U})$. Denote by $\mathcal{U}_u$ the upper quasi-uniformity on $\mathbb{R}$ the standard base of which consists of the sets

$$U_r = \{(x, y) : y < x + r\},$$

where r is an arbitrary positive real number. By [23, Chap. 3, Th. 2.5], the quasiuniformity $\mathcal{U}$ on X is generated by the family $\mathcal{D}$ of all quasi-pseudometrics d on X such that the mapping $d\colon (X \times X, \mathcal{U}^{-1} \times \mathcal{U}) \to (\mathbb{R}, \mathcal{U}_u)$ is quasi-uniformly continuous. Note that by [23, Chap. 3, Prop. 2.4], the condition $d \in \mathcal{D}$ is equivalent to saying that the set

$$V_d(r) = \{(x, y) \in X \times X : d(x, y) < r\}$$

belongs to $\mathcal{U}$ for each $r > 0$ or, in other words, d is $\mathcal{U}$-quasi-uniformly continuous.

For every $d \in \mathcal{D}$, denote by $\mathcal{T}_d$ the paratopological group topology on $F_a(X)$ generated by the extension $\widehat{d}$ of d to $F_a(X)$ (see Theorem 3.2). Let $\mathcal{T}$ be the supremum of the topologies $\mathcal{T}_d$, with $d \in \mathcal{D}$. Then $H = (F_a(X), \mathcal{T})$ is a paratopological group which contains X as a subspace. In addition, the left quasi-uniformity $\mathcal{L}_H$ of H induce on X the uniformity $\mathcal{U}$. Indeed, each topology $\mathcal{T}_d$ is coarser than $\mathcal{T}$ and the left quasi-uniformity $\mathcal{L}_d$ of the paratopological group $(F_a(X), \mathcal{T}_d)$ induces on X a quasi-uniformity $\mathcal{U}_d$ which coincides with the one generated by the quasi-pseudometric d. Since each $d \in \mathcal{D}$ is $\mathcal{U}$-quasi-uniformly continuous, $\mathcal{U}_d$ is coarser than $\mathcal{U}$. Clearly, the quasi-uniformity $\mathcal{V}$ on X induced by $\mathcal{L}_H$ is the supremum of the quasi-uniformities $\mathcal{U}_d$, $d \in \mathcal{D}$, and hence $\mathcal{V} = \mathcal{U}$.

We have thus proved that there exists at least one paratopological group topology $\mathcal{T}$ on $F_a(X)$ such that the left quasi-uniformity $\mathcal{L}(\mathcal{T})$ of the paratopological group $(F_a(X), \mathcal{T})$ induces on X its original quasi-uniformity $\mathcal{U}$. Similarly to the proof of Theorem 2.4, denote by $\mathcal{T}^*$ the supremum topology of the family $\mathcal{P}$ of all paratopological group topologies $\mathcal{T}$ on $F_a(X)$ with this property. Clearly, $\mathcal{T}^*$ is also a paratopological group topology on $F_a(X)$ which induces on X the quasi-uniformity $\mathcal{U}$, that is, $\mathcal{T}^*$ is the maximal element of $\mathcal{P}$. It remains to note that $G = (F_a(X), \mathcal{T}^*)$ is the free paratopological group over $(X, \mathcal{U})$. Indeed, let $f\colon (X, \mathcal{U}) \to (H, \mathcal{L}_H)$ be a quasi-uniformly continuous mapping of X to a paratopological group H whose topology is t_H. Extend f to a homomorphism $\tilde{f}\colon F_a(X) \to H$ and put

$$\mathcal{T}_H = \{\tilde{f}^{-1}(O) : O \in t_H\}.$$

Then $\mathcal{T}_H$ is a paratopological group topology on $F_a(X)$ and from the choice of f it follows that the left quasi-uniformity $\mathcal{L}$ of the paratopological group $(F_a(X), \mathcal{T}_H)$ induces on X a quasi-uniformity coarser that $\mathcal{U}$. Therefore, the supremum of the topologies $\mathcal{T}^*$ and $\mathcal{T}_H$ is an element of $\mathcal{P}$, and hence $\mathcal{T}_H$ is coarser than $\mathcal{T}^*$. This means that the mapping $\tilde{f}\colon (F_a(X), \mathcal{T}^*) \to H$ is continuous. The proof is complete. $\square$

4. Separation properties

It is natural, after Theorem 3.4, to study the topological properties of the paratopological groups $FP(X, \mathcal{U})$ and $AP(X, \mathcal{U})$. Every paratopological group G has three natural topologies: the original topology $\mathcal{T}_G$ of G, its conjugate $\mathcal{T}_G^{-1} = \{U \subseteq G : U^{-1} \in \mathcal{T}\}$ and the supremum of both, $\mathcal{T}_G^* = \mathcal{T}_G \vee \mathcal{T}_G^{-1}$, the latter being a group topology on G. We call $\mathcal{T}_G^*$ the *fine topology* of G. It is clear that $\mathcal{T}_G^*$ is the coarsest group topology on G that contains the original topology $\mathcal{T}$ of G. Therefore, the first question to ask is whether the fine topology $\mathcal{T}_G^*$ is Hausdorff. If this is so, we shall say that the paratopological group G is 2-*Hausdorff*. We characterize the quasi-uniform spaces $(X,\mathcal{U})$ for which the paratopological groups $FP(X,\mathcal{U})$ and $AP(X,\mathcal{U})$ are 2-Hausdorff in Theorem 4.4. Its proof requires several notions and auxiliary results.

For every $\varepsilon > 0$, put

$$U_u(\varepsilon)=\{(x,y)\in\mathbb{R}^2 : y < x+\varepsilon\} \text{ and } U_l(\varepsilon)=\{(x,y)\in\mathbb{R}^2 : x < y+\varepsilon\}.$$

The families $\{U_u(\varepsilon) : \varepsilon > 0\}$ and $\{U_l(\varepsilon) : \varepsilon > 0\}$ constitute bases of the *upper* and *lower* quasi-uniformities $\mathcal{U}_u$ and $\mathcal{U}_l$ on $\mathbb{R}$, respectively.

The category of quasi-uniform spaces has products (see [10], 1.16). If $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in A\}$ is a family of quasi-uniform spaces, then the product $(X,\mathcal{U}) = \prod_{\alpha\in A}(X_\alpha,\mathcal{U}_\alpha)$ is defined as follows: $X = \prod_{\alpha\in A} X_\alpha$, and the quasi-uniformity $\mathcal{U}$ on X has the base

$$\{W(\alpha_1,\dots,\alpha_n,U_1,\dots,U_n) : n \in \mathbb{N}^+,$$

$$\alpha_i \in A,\ U_i \in \mathcal{U}_{\alpha_i} \text{ for each } i = 1,\dots,n\},$$

where

$$W(\alpha_1,\dots,\alpha_n,U_1,\dots,U_n) =$$

$$\{(x,y) \in X \times X : (x_{\alpha_i}, y_{\alpha_i}) \in U_i \text{ for each } i = 1,\dots,n\}.$$

If all members of the family $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in A\}$ coincide, say, with $(Y,\mathcal{V})$, then we use $(Y^A,\mathcal{V}^A)$ instead of $\prod_{\alpha\in A}(X_\alpha,\mathcal{U}_\alpha)$.

Let us start with the simplest paratopological groups.

Lemma 4.1. *The paratopological groups $FP(\mathbb{R}^n,\mathcal{U}_u^n)$ and $AP(\mathbb{R}^n,\mathcal{U}_u^n)$ are 2-Hausdorff for each $n \in \mathbb{N}^+$.*

Proof. Let $n \in \mathbb{N}^+$ be arbitrary. We prove the lemma only for $FP(\mathbb{R}^n,\mathcal{U}_u^n)$ since the argument for the group $AP(\mathbb{R}^n,\mathcal{U}_u^n)$ is analogous. Let d_u be the usual quasi-pseudometric on $\mathbb{R}$ that generates the quasi-uniformity $\mathcal{U}_u$:

$$d_u(x,y) = \begin{cases} y - x & \text{if } x < y; \\ 0 & \text{if } y \le x. \end{cases}$$

Then the left quasi-uniformity $\mathcal{U}_l$ on $\mathbb{R}$ is generated by the quasi-pseudometric d_l, where $d_l(x,y) = d_u(y,x)$ for all $x,y \in \mathbb{R}$. If $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$, we put $d_{u,n}(x,y) = \sum_{i=1}^n d_u(x_i,y_i)$ and $d_{l,n}(x,y) = \sum_{i=1}^n d_l(x_i,y_i)$. One easily verifies that the quasi-pseudometrics $d_{u,n}$ and $d_{l,n}$ generate the quasi-uniformities $\mathcal{U}_u^n$ and $\mathcal{U}_l^n$ on $\mathbb{R}^n$, respectively.

Denote the Graev extensions of $d_{u,n}$ and $d_{l,n}$ to $F_a(\mathbb{R}^n)$ by $\widehat{d}_{u,n}$ and $\widehat{d}_{l,n}$, respectively. By Theorem 3.2, $\widehat{d}_{u,n}$ is continuous on $FP(\mathbb{R}^n,\mathcal{U}_u^n)$ and $\widehat{d}_{l,n}$ is continuous on $FP(\mathbb{R}^n,\mathcal{U}_l^n)$. Let $\mathcal{T}$ be the topology of $FP(\mathbb{R}^n,\mathcal{U}_u^n)$ and $\mathcal{T}^{-1}$ be its conjugate. We have to show that the topology $\mathcal{T}^* = \mathcal{T} \vee \mathcal{T}^{-1}$ on $F_a(\mathbb{R}^n)$ is Hausdorff. For every $\varepsilon > 0$, put

$$B_u(\varepsilon) = \{g \in F_a(\mathbb{R}^n) : \widehat{d}_{u,n}(e,g) < \varepsilon\}$$

and

$$B_l(\varepsilon) = \{g \in F_a(\mathbb{R}^n) : \widehat{d}_{l,n}(e,g) < \varepsilon\}.$$

It is easy to see that $B_l(\varepsilon) = B_u(\varepsilon)^{-1}$. Indeed, we have $d_l(x,y) = d_u(y,x)$ for all $x,y \in \mathbb{R}$, whence $d_{l,n}(x,y) = d_{u,n}(y,x)$ for all $x,y \in \mathbb{R}^n$. Therefore, $\widehat{d}_{l,n}(g,h) = \widehat{d}_{u,n}(h,g)$ for all $g,h \in F_a(\mathbb{R}^n)$. Consequently, for every $g \in F_a(\mathbb{R}^n)$, we have

$$\widehat{d}_u(e,g^{-1}) < \varepsilon \Longleftrightarrow \widehat{d}_l(g^{-1},e) < \varepsilon \Longleftrightarrow \widehat{d}_l(e,g) < \varepsilon,$$

whence the equality $B_l(\varepsilon) = B_u(\varepsilon)^{-1}$ follows. We conclude, therefore, that the left invariant quasi-pseudometric $\widehat{d}_{l,n}$ is continuous on $(F_a(\mathbb{R}^n),\mathcal{T}^{-1})$. Hence the sum $\varrho = \widehat{d}_{u,n} + \widehat{d}_{l,n}$ is a continuous left invariant quasi-pseudometric on $(F_a(\mathbb{R}^n),\mathcal{T}^*)$. To finish the proof, it suffices to show that ϱ is a metric (and hence generates a Hausdorff topology on $F_a(\mathbb{R}^n)$ weaker than $\mathcal{T}^*$).

For every $g,h \in F_a(\mathbb{R}^n)$, we have

$$\varrho(g,h) = \widehat{d}_{u,n}(g,h) + \widehat{d}_{l,n}(g,h) = \widehat{d}_{l,n}(h,g) + \widehat{d}_{u,n}(h,g) = \varrho(h,g),$$

whence it follows that ϱ is symmetric. It remains to verify that $\varrho(e,g) > 0$ for every element $g \in F_a(\mathbb{R}^n)$ distinct from e. First, we note that the set

$$N = \{h \in F_a(\mathbb{R}^n) : \varrho(e,h) = 0\}$$

satisfies the equality $N \cdot N = N$. Indeed, let D denote either $\widehat{d}_{u,n}$ or $\widehat{d}_{l,n}$. Using left invariance of D, we obtain

$$D(e, gh) \leq D(e, g) + D(g, gh) = D(e, g) + D(e, h)$$

for arbitrary $g, h \in F_a(\mathbb{R}^n)$. Therefore, if $g, h \in N$, then

$$\begin{aligned} \varrho(e, gh) &= \widehat{d}_{u,n}(e, gh) + \widehat{d}_{l,n}(e, gh) \\ &\leq \widehat{d}_{u,n}(e, g) + \widehat{d}_{u,n}(e, h) + \widehat{d}_{l,n}(e, g) + \widehat{d}_{l,n}(e, h) \\ &= \varrho(e, g) + \varrho(e, h) = 0. \end{aligned}$$

Suppose to the contrary that $\varrho(e, g) = 0$ for some $g \in F_a(\mathbb{R}^n) \setminus \{e\}$, and let $g = x_1 \dots x_n$ be an irreducible representation of g with $x_1, \dots, x_n \in \mathbb{R}^n \cup (\mathbb{R}^n)^{-1}$. We can assume without loss of generality that n is even, say, $n = 2k$ (otherwise replace g by $g \cdot g \in N$). Clearly, $\widehat{d}_u(e, g) = 0$ and $\widehat{d}_l(e, g) = 0$. By the definition of Graev's extension of quasi-pseudometrics and Claim 2 in the proof of Theorem 3.2, one can find schemes $\varphi, \psi \in S_{2k}$ such that

$$(1)\quad \frac{1}{2}\sum_{i=1}^{2k} d^*_{u,n}(x_i^{-1}, x_{\varphi(i)}) = N_u(g) = 0 = N_l(g) = \frac{1}{2}\sum_{i=1}^{2k} d^*_{l,n}(x_i^{-1}, x_{\psi(i)}),$$

where $d^*_{u,n}$ and $d^*_{l,n}$ are extension of $d_{u,n}$ and $d_{l,n}$ to $\mathbb{R}^n \cup (\mathbb{R}^n)^{-1}$, respectively (see Claim 1 in the proof of Theorem 3.2). By Proposition 3.1, there exists an integer r with $1 \leq r \leq 2k$ such that $\varphi(r) = r + 1$. By assumption, the word $g = x_1 \cdots x_{2k}$ is irreducible, so that $x_r^{-1} \neq x_{r+1}$. Now we define a sequence of integers $i_1, j_1, i_2, j_2, \dots$ by $i_1 = r$, $j_1 = \varphi(i_1)$, $i_2 = \psi(j_1)$, $j_2 = \varphi(i_2)$, etc. Let us show that $i_1 = \psi(j_q) = i_{q+1}$ for some q with $1 \leq q \leq 2k$. First, $j_1 = \varphi(i_1) \neq i_1$ by definition of a scheme. Suppose that the numbers $i_1, j_1, \dots, i_p, j_p$ are pairwise distinct for some $p \geq 1$. Then $\psi(j_p) \neq j_m$ for each $m < p$. Indeed, $\psi(j_m) = i_{m+1} \neq j_p$ by our assumption, so $\psi(i_{m+1}) = j_m$. Since ψ is a bijection of $\{1, \dots, 2k\}$ and $i_{m+1} \neq j_p$, we conclude that $\psi(j_p) \neq j_m$. In addition, we claim that $\psi(j_p) \neq i_m$ for each $m = 2, \dots, p$. Indeed, $\psi(j_{m-1}) = i_m$ and by assumption, $j_p \neq j_{m-1}$. Therefore, $\psi(j_p) \neq i_m$. We have thus proved that either $\psi(j_p) \notin \{i_1, j_1, \dots, i_p, j_p\}$ or $\psi(j_p) = i_1$. Since the set $\{1, \dots, 2k\}$ has $2k$ elements, the first possibility can happen at most k times. In other words, we must have $\psi(j_q) = i_1$ for some $q \leq 2k$.

We claim that

$$d^*_{u,n}(x_r^{-1}, x_{r+1}) = 0 \text{ and } d^*_{l,n}(x_r^{-1}, x_{r+1}) = 0. \tag{2}$$

Indeed, since $\varphi(r) = r+1$, the left equality follows directly from (1). Again by (1), we have $d^*_{u,n}(x^{-1}_{i_m}, x_{j_m}) = 0 = d^*_{l,n}(x^{-1}_{j_m}, x_{i_{m+1}})$ for each $m = 1, \ldots, 2k$. In addition, $d^*_{u,n}(x^{-1}, y) = d^*_{l,n}(y, x^{-1}) = d^*_{l,n}(x, y^{-1})$ for all $x, y \in \mathbb{R}^n \cup (\mathbb{R}^n)^{-1}$. Therefore, we have

$$\begin{aligned} d^*_{l,n}(x^{-1}_{j_1}, x_{i_1}) &\leq d^*_{l,n}(x^{-1}_{j_1}, x_{i_2}) + d^*_{l,n}(x_{i_2}, x^{-1}_{j_2}) + \cdots \\ &\cdots + d^*_{l,n}(x_{i_q}, x^{-1}_{j_q}) + d^*_{l,n}(x^{-1}_{j_q}, x_{i_1}) \\ &= d^*_{l,n}(x^{-1}_{j_1}, x_{i_2}) + d^*_{u,n}(x^{-1}_{i_2}, x_{j_2}) + \cdots \\ &\cdots + d^*_{u,n}(x^{-1}_{i_q}, x_{j_q}) + d^*_{l,n}(x^{-1}_{j_q}, x_{i_1}) \\ &= 0. \end{aligned} \tag{3}$$

Therefore, from (3) and the equalities $i_1 = r$ and $j_1 = r+1$ it follows that

$$0 = d^*_{l,n}(x^{-1}_{j_1}, x_{i_1}) = d^*_{l,n}(x^{-1}_{r+1}, x_r) = d^*_{l,n}(x^{-1}_r, x_{r+1}).$$

This proves (2). Let us show that (2) implies $x^{-1}_r = x_{r+1}$, thus contradicting the fact that the word $g = x_1 \ldots x_{2k}$ is irreducible. Denote by d the usual metric d on $\mathbb{R}$, $d(x,y) = |x-y|$. Let d_n be the metric on $\mathbb{R}^n$ defined by $d_n(x,y) = \sum_{i=1}^n d(x_i, y_i)$ for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$. Note that $d = d_u + d_l$. Therefore, $d_n = d_{u,n} + d_{l,n}$, and the sum $d^*_{u,n} + d^*_{l,n}$ coincides with the extension d^*_n of d_n to $\mathbb{R}^n \cup (\mathbb{R}^n)^{-1}$ (see the construction before Claim 1 in the proof of Theorem 3.2). Since d_n is a metric, so is d^*_n. Thus, (2) implies that $d^*_n(x^{-1}_r, x_{r+1}) = 0$, whence $x^{-1}_r = x_{r+1}$. This contradiction completes the proof. $\square$

Let us extend Lemma 4.1 to arbitrary powers of the quasi-uniform space $(\mathbb{R}, \mathcal{U}_u)$. We denote the upper and lower quasi-uniformities on $\mathbb{I}$ inherited from $\mathbb{R}$ by the same letters $\mathcal{U}_u$ and $\mathcal{U}_l$.

Lemma 4.2. *The free paratopological groups $FP(\mathbb{R}^A, \mathcal{U}_u^A)$, $FP(\mathbb{I}^A, \mathcal{U}_u^A)$, $AP(\mathbb{R}^A, \mathcal{U}_u^A)$ and $AP(\mathbb{I}^A, \mathcal{U}_u^A)$ are 2-Hausdorff for every non-empty set A.*

Proof. Again, we prove the lemma only for the groups $FP(\mathbb{R}^A, \mathcal{U}_u^A)$ and $FP(\mathbb{I}^A, \mathcal{U}_u^A)$. Let $\mathcal{T}^*$ be the fine topology of $FP(\mathbb{R}^A, \mathcal{U}_u^A)$. Since $(F_a(\mathbb{R}^A), \mathcal{T}^*)$ is a topological group, it suffices to show that $\mathcal{T}^*$ satisfies the T_1-separation axiom.

For every finite subset B of A, let $p_B\colon \mathbb{R}^A \to \mathbb{R}^B$ be the projection. Note that the map $p_B\colon (\mathbb{R}^A,\mathcal{U}_u^A) \to (\mathbb{R}^B,\mathcal{U}_u^B)$ is quasi-uniformly continuous and $(\mathbb{R}^B,\mathcal{U}_u^B)$ is a subspace of $(FP(\mathbb{R}^B,\mathcal{U}_u^B),\mathcal{L})$, where $\mathcal{L}$ is the left quasi-uniformity of $FP(\mathbb{R}^B,\mathcal{U}_u^B)$. Therefore, p_B extends to a continuous homomorphism $\widetilde{p}_B\colon FP(\mathbb{R}^A,\mathcal{U}_u^A) \to FP(\mathbb{R}^B,\mathcal{U}_u^B)$.

Consider an arbitrary element $g \in FP(\mathbb{R}^A,\mathcal{U}_u^A)$ distinct from the identity e. Let $g = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ be the irreducible form of g, where $x_1,\dots,x_n \in \mathbb{R}^A$ and $\varepsilon_1,\dots,\varepsilon_n = \pm 1$. For every $i = 1,\dots,n-1$, there exists $\alpha_i \in A$ such that $x_i(\alpha_i)^{-\varepsilon_i} \neq x_{i+1}(\alpha_i)^{\varepsilon_{i+1}}$. Also, choose $\alpha_n \in A$ such that $x_n(\alpha_n) \neq 0$ (we identify $0 \in \mathbb{R}$ with the identity of $FP(\mathbb{R},\mathcal{U}_u)$). Put $B = \{\alpha_1,\dots,\alpha_n\}$. One easily verifies that $g_B = \widetilde{p}_B(g) \neq e_B$, where e_B is the identity of $FP(\mathbb{R}^B,\mathcal{U}_u^B)$. By Lemma 4.1, the paratopological group $FP(\mathbb{R}^B,\mathcal{U}_u^B)$ is 2-Hausdorff, so we can find a $\mathcal{T}_B^*$-open neighborhood V of e_B in $F_a(\mathbb{R}^B)$ such that $g_B \notin V$, where $\mathcal{T}_B^*$ is the fine topology of $FP(\mathbb{R}^B,\mathcal{U}_u^B)$. Since the homomorphism $\widetilde{p}_B\colon (F_a(\mathbb{R}^A),\mathcal{T}^*) \to (F_a(\mathbb{R}^B),\mathcal{T}_B^*)$ is continuous, $\widetilde{p}_B^{-1}(V)$ is a $\mathcal{T}^*$-open neighborhood of e in $F_a(\mathbb{R}^A)$ which does not contain g. This proves that the group topology $\mathcal{T}^*$ is T_1, and hence Hausdorff.

The natural embedding φ of $(\mathbb{I}^A,\mathcal{U}_u^A)$ into $(\mathbb{R}^A,\mathcal{U}_u^A)$ is quasi-uniformly continuous, so it extends to a continuous injective homomorphism $\widetilde{\varphi}\colon FP(\mathbb{I}^A,\mathcal{U}_u^A) \to FP(\mathbb{R}^A,\mathcal{U}_u^A)$. This monomorphism remains continuous when both groups are considered with their fine topologies, so the fine topology of $FP(\mathbb{I}^A,\mathcal{U}_u^A)$ is Hausdorff by the similar fact established for $FP(\mathbb{R}^A,\mathcal{U}_u^A)$ in the first part of the proof. □

Let $\mathcal{T}^*$ be the fine topology of a paratopological group G. We say that a subset B of G is 2-*closed* in G if B is $\mathcal{T}^*$-closed in G. Similarly, B is 2-*compact* in G if B is a compact subset of $(G,\mathcal{T}^*)$. The second important question is whether X is 2-closed in $FP(X,\mathcal{U})$. In Theorem 4.4 we characterize the quasi-uniform spaces $(X,\mathcal{U})$ with the property that X is 2-closed in $FP(X,\mathcal{U})$. First, we deduce the following corollary of Lemma 4.2.

Corollary 4.3. *The set $\mathbb{I}^A$ is 2-compact and hence 2-closed in the paratopological groups $FP(\mathbb{I}^A,\mathcal{U}_u^A)$ and $AP(\mathbb{I}^A,\mathcal{U}_u^A)$ for each nonempty set A.*

Proof. Let G be one of the groups $FP(\mathbb{I}^A, \mathcal{U}_u^A)$ or $AP(\mathbb{I}^A, \mathcal{U}_u^A)$. Clearly, $(\mathbb{I}^A, \mathcal{U}_u^A)$ is a quasi-uniform subspace of $(G, \mathcal{L})$, where $\mathcal{L}$ is the left quasi-uniformity of G. So the topology t_u on $\mathbb{I}^A$ induced by the quasi-uniformity $\mathcal{U}_u^A$ coincides with the one induced on X by the quasi-uniformity $\mathcal{L}$. Let $\mathcal{T}$ be the topology of G and $\mathcal{T}^{-1}$ be its conjugate (so that $\mathcal{T}^* = \mathcal{T} \vee \mathcal{T}^{-1}$). Since the topology $\mathcal{T}$ of G is induced by the quasi-uniformity $\mathcal{L}$, we conclude that $t_u = \mathcal{T} \restriction \mathbb{I}^A$. Similarly, the topology t_l on $\mathbb{I}^A$ induced by the quasi-uniformity $\mathcal{U}_l^A$ coincides with the restriction of $\mathcal{T}^{-1}$ to $\mathbb{I}^A$. Therefore, the restriction of $\mathcal{T}^*$ to $\mathbb{I}^A$ is exactly the topology $t_u \vee t_l$. An easy verification shows that $t_u \vee t_l$ is the usual product topology of $\mathbb{I}^A$, which is compact by the Tychonoff compactness theorem. Since the fine topology $\mathcal{T}^*$ of G is Hausdorff by Lemma 4.2, we conclude that the set $\mathbb{I}^A$ is closed in $(G, \mathcal{T}^*)$ or, equivalently, $\mathbb{I}^A$ is 2-compact and 2-closed in G. □

To formulate and prove Theorem 4.4, we need some notions introduced in [23] and [15]. Let $(X, \mathcal{U})$ be a quasi-uniform space. Denote by $\tau_X = T(\mathcal{U})$ and $\tau_X^{-1} = T(\mathcal{U}^{-1})$ the compatible topology on X and its conjugate, respectively. This gives rise to the *bitopological space* (X, τ_X, τ_X^{-1}). Similarly to the case of paratopological groups, we call $\tau_X^* = \tau_X \vee \tau_X^{-1}$ the *fine topology* of the quasi-uniform space $(X, \mathcal{U})$. If the fine topology τ_X^* is Hausdorff, we say that $(X, \mathcal{U})$ is 2-*Hausdorff*.

Denote by u and l the upper and lower topologies on the real line $\mathbb{R}$ which consist of the sets $(-\infty, r)$ and $(r, +\infty)$ (with $r \in \mathbb{R}$), respectively. Then $(\mathbb{R}, u, l)$ is a bitopological space and $u \vee l$ is the usual interval topology of $\mathbb{R}$. A map $f\colon (X, \tau_1, \tau_2) \to (\mathbb{R}, u, l)$ is called *bicontinuous* if both maps $f\colon (X, \tau_1) \to (\mathbb{R}, u)$ and $(X, \tau_2) \to (\mathbb{R}, l)$ are continuous. A bitopological space (X, τ_1, τ_2) is called 2-*completely regular* if the topologies τ_1, τ_2 are initial with respect to the family of all bicontinuous maps $f\colon (X, \tau_1, \tau_2) \to (\mathbb{R}, u, l)$. This is equivalent to saying that for every $x_0 \in X$ and a τ_1-closed set P in X with $x_0 \notin P$, there exists a bicontinuous function $f\colon (X, \tau_1, \tau_2) \to (\mathbb{R}, u, l)$ such that $f(x_0) = 0$ and $f(x) = 1$ on P; and for every τ_2-closed set Q in X not containing x_0, there is a bicontinuous function $g\colon (X, \tau_1, \tau_2) \to (\mathbb{R}, u, l)$ such that $g(x_0) = 1$ and $g(x) = 0$ on Q (see [23, Chap. 1, Prop. 2.2.2]).

The bitopological space (X, τ_1, τ_2) is said to be *quasi-uniformizable* if there exists a quasi-uniformity $\mathcal{U}$ on X such that $\tau_1 = T(\mathcal{U})$ and $\tau_2 = T(\mathcal{U}^{-1})$. It is well known that a bitopological space is quasi-uniformizable iff it is 2-completely regular (see [14]). It is also known that if G is a paratopological group with their natural topologies $\mathcal{T}$ and $\mathcal{T}^{-1}$, then the bitopological space $(G, \mathcal{T}, \mathcal{T}^{-1})$ is quasi-uniformizable [15, Theorem 1]. It seems interesting to remark that every bitopological space (X, τ_1, τ_2) has a finest compatible quasi-uniformity denoted by $\mathcal{BFN}$ such that every bicontinuous function $f\colon (X, \mathcal{B}FN, \mathcal{B}FN^{-1}) \longrightarrow (\mathbb{R}, \mathcal{U}_u, \mathcal{U}_l)$ satisfies that both $f\colon (X, \mathcal{B}FN) \longrightarrow (\mathbb{R}, \mathcal{U}_u)$ and $f\colon (X, \mathcal{B}FN^{-1}) \longrightarrow (\mathbb{R}, \mathcal{U}_l)$ are quasi-uniformly continuous (see [22]). So, it is easy to see that the free paratopological group $FP(X, \mathcal{B}FN)$ (respectively, the free Abelian paratopological group $AP(X, \mathcal{B}FN)$) is defined by the property that, for every paratopological group H, each bicontinuous function $f\colon (X, \mathcal{B}FN, \mathcal{B}FN^{-1}) \longrightarrow (H, \mathcal{L}_H, \mathcal{L}_H^{-1})$ which satisfies $f(e) = e_H$ admits a bicontinuous extension to a homomorphism from $FP(X, \mathcal{B}FN)$ (respectively, from $AP(X, \mathcal{B}FN)$) into H.

Finally, a bitopological space (X, τ_1, τ_2) is said to be 2-T_0 if the topology $\tau_1 \vee \tau_2$ on X satisfies the T_0-separation axiom. Similarly, a quasi-uniform space $(X, \mathcal{U})$ is called 2-T_0 if the corresponding bitopological space $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is 2-T_0. Clearly, if a quasi-uniform space is 2-T_0, then it is 2-Hausdorff, since every T_0 uniform space is Hausdorff. This fact will be used without special mention. The theorem below is the main result of the article.

Theorem 4.4. *The following conditions are equivalent for a quasi-uniform space $(X, \mathcal{U})$:*

(1) *the free paratopological group $FP(X, \mathcal{U})$ is 2-Hausdorff;*
(2) *the free Abelian paratopological group $AP(X, \mathcal{U})$ is 2-Hausdorff;*
(3) *X is 2-closed in $FP(X, \mathcal{U})$;*
(4) *X is 2-closed in $AP(X, \mathcal{U})$;*
(5) *the quasi-uniform space $(X, \mathcal{U})$ is 2-Hausdorff.*

Proof. We shall show that (5) implies (1)–(4) and vice versa, each of (1), (2), (3) and (4) implies (5).

Let us show that (5) implies (1). Suppose that $(X, \mathcal{U})$ is 2-Hausdorff and consider the family $\mathcal{M}$ of all quasi-uniformly continuous maps $f\colon (X, \mathcal{U}) \to (\mathbb{I}, \mathcal{U}_u)$. Let x_0, y_0 be two distinct points of X. Since $T(\mathcal{U} \vee \mathcal{U}^{-1})$ is Hausdorff, there exists $U \in \mathcal{U}$ such that either $y_0 \notin U(x_0)$ or $y_0 \notin U(x_0)$. We can assume without loss of generality that $y_0 \notin U(x_0)$. By [23, Chap. 3, Prop. 2.4], there exists a quasi-pseudometric d on X which is quasi-uniformly continuous as a map $d\colon (X \times X, \mathcal{U}^{-1} \times \mathcal{U}) \to (\mathbb{R}, \mathcal{U}_u)$ and satisfies

$$\{(x, y) \in X \times X : d(x, y) < 1\} \subseteq U.$$

Put $\varrho = \min\{d, 1\}$. Then the quasi-pseudometric

$$\varrho\colon (X \times X, \mathcal{U}^{-1} \times \mathcal{U}) \to (\mathbb{I}, \mathcal{U}_u)$$

is also quasi-uniformly continuous and satisfies the similar condition

$$\{(x, y) \in X \times X : \varrho(x, y) < 1\} \subseteq U.$$

Let f be a function on X defined by $f(y) = \varrho(x_0, y)$. It is clear that $f\colon (X, \mathcal{U}) \to (\mathbb{I}, \mathcal{U}_u)$ is a quasi-uniformly continuous map, that is, $f \in \mathcal{M}$. In addition, from $y_0 \notin U(x_0)$ and our choice of d it follows that $d(x_0, y_0) \geq 1$, whence $f(y_0) = \varrho(x_0, y_0) = 1$. Since $f(x_0) = \varrho(x_0, x_0) = 0$, we conclude that $f(x_0) \neq f(y_0)$. In other words, the maps from $\mathcal{M}$ separate the points of X.

Enumerate $\mathcal{M}$, say, $\mathcal{M} = \{f_\alpha : \alpha \in A\}$, and let φ be the diagonal product of the maps from $\mathcal{M}$, $\varphi\colon X \to \mathbb{I}^A$. Then φ is injective. Since each map $f_\alpha\colon (X, \mathcal{U}) \to (\mathbb{I}, \mathcal{U}_u)$ is quasi-uniformly continuous, the map $\varphi\colon (X, \mathcal{U}) \to (\mathbb{I}^A, \mathcal{U}_u^A)$ is also quasi-uniformly continuous, where $\mathcal{U}_u^A$ is the product of "A copies" of the quasi-uniformity $\mathcal{U}_u$. Let $FP(\mathbb{I}^A, \mathcal{U}_u^A)$ be the free paratopological group over $(\mathbb{I}^A, \mathcal{U}_u^A)$. Then φ extends to a continuous homomorphism $\widetilde{\varphi}\colon FP(X, \mathcal{U}) \to FP(\mathbb{I}^A, \mathcal{U}_u^A)$. Clearly, $\mathbb{I}^A$ is a free basis of the group $F_a(\mathbb{I}^A)$, and hence the homomorphism $\widetilde{\varphi}$ is injective. Since $FP(\mathbb{I}^A, \mathcal{U}_u^A)$ is 2-Hausdorff by Lemma 4.3, we conclude that $FP(X, \mathcal{U})$ is also 2-Hausdorff. The proof of the implication (5) $\Rightarrow$ (2) goes the same way.

It is also clear that (5) implies (3). Indeed, since the homomorphism $\widetilde{\varphi}$ is injective, we have

$$\varphi(X) = \widetilde{\varphi}(FP(X, \mathcal{U})) \cap \mathbb{I}^A.$$

The set $\mathbb{I}^A$ is 2-closed in $FP(\mathbb{I}^A, \mathcal{U}_u^A)$ by Corollary 4.3, so the above equality implies that $\varphi(X)$ is 2-closed in $\widetilde{\varphi}(FP(X, \mathcal{U}))$. Using injectivity of $\widetilde{\varphi}$ once again, we conclude that X is 2-closed in $FP(X, \mathcal{U})$. The same argument shows that (5) implies (4).

Now we show that (3) $\Rightarrow$ (5). Suppose to the contrary that the quasi-uniform space $(X,\mathcal{U})$ is not 2-Hausdorff. Put $\tau_X^* = T(\mathcal{U}) \vee T(\mathcal{U}^{-1})$. We can find two distinct points $x_0, y_0 \in X$ such that every τ_X^*-open set in X containing one of the points x_0, y_0 contains the other. We claim that the element $g = x_0 y_0^{-1} x_0$ of $F_a(X)$ is in the $\mathcal{T}^*$-closure of X, where $\mathcal{T}^*$ is the fine topology of $FP(X,\mathcal{U})$. Suppose not, then there exists a $\mathcal{T}^*$-open neighborhood V of the identity e in $F_a(X)$ such that $x_0 \notin gV$ or, equivalently, $x_0^{-1}y_0 \notin V$. By definition of the fine topology $\mathcal{T}^*$, there exists an open neighborhood U of the identity in $FP(X,\mathcal{U})$ such that $U \cap U^{-1} \subseteq V$. Then x_0U is a τ_X-open neighborhood of x_0, and hence $y_0 \in x_0U$. Similarly, $y_0 \in y_0U$, whence $x_0 \in y_0U$. Therefore, $x_0^{-1}y_0 \in U$ and $y_0^{-1}x_0 \in U$, so that $x_0^{-1}y_0 \in U \cap U^{-1} \subseteq V$ which in its turn implies $y_0 \in x_0V$, a contradiction. This proves that X is not 2-closed in $FP(X,\mathcal{U})$. A similar argument shows that (4) $\Rightarrow$ (5).

Finally, it remains to show that each of (1) and (2) implies (5). It suffices to verify the implication (1) $\Rightarrow$ (5). Suppose, therefore, that the paratopological group $FP(X,\mathcal{U})$ is 2-Hausdorff. Again, let $\mathcal{T}$ be the topology of $FP(X,\mathcal{U})$, $\mathcal{T}^{-1}$ be its conjugate and $\mathcal{T}^* = \mathcal{T} \vee \mathcal{T}^{-1}$ be the fine topology of $FP(X,\mathcal{U})$. Then $\tau_X = \mathcal{T}\restriction X$ and $\tau_X^{-1} = \mathcal{T}^{-1}\restriction X$, where $\tau_X = T(\mathcal{U})$ and $\tau_X^{-1} = T(\mathcal{U}^{-1})$. Put $\tau_X^* = \tau_X \vee \tau_X^{-1}$. It is clear that $\tau_X^* = \mathcal{T}^*\restriction X$, and since the topology $\mathcal{T}^*$ is Hausdorff, so is τ_X^*. This proves that the quasi-uniform space $(X,\mathcal{U})$ is 2-Hausdorff. □

5. Open problems

As is easily seen after Section 4, almost every question about free paratopological groups presents certain difficulties to answer it. We collect here a very limited number of problems the solution to which will definitely clarify the topological properties of these topologically "skew" groups.

Problem 5.1. Characterize the quasi-uniform spaces $(X,\mathcal{U})$ such that the free paratopological group $FP(X,\mathcal{U})$ (or $AP(X,\mathcal{U})$) is Hausdorff, regular or completely regular.

Problem 5.2. Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\mathcal{V} = \mathcal{U} \vee \mathcal{U}^{-1}$ be the corresponding uniformity on X. When does the fine topology of $FP(X,\mathcal{U})$ coincide with the topology of the uniform free topological group $F(X,\mathcal{V})$?

Problem 5.3. Are the free Abelian paratopological groups $AP(\mathbb{I}, \mathcal{U}_u)$ and $AP(\mathbb{R}, \mathcal{U}_u)$ complete when they carry their fine group topologies?

References

[1] Arhangel'skiĭ, A.V., *Every topological group is a quotient group of a zero-dimensional topological group*, Dokl. AN SSSR **258** (1981), 1037–1040 (*in Russian*).

[2] Arhangel'skiĭ, A.V., *On topological and algebraic structure of extremally disconnected semitopological groups*, Comment. Math. Univ. Carolin. **41** no. 4 (2000), 803–810.

[3] Arhangel'skiĭ, A.V. and M. Hušek, *Extensions of topological and semitopological groups and the product operation*, Comment. Math. Univ. Carolin. **42** no. 1 (2000), 173–186.

[4] Bourbaki, N., *Elements of Mathematics. General Topology. Part I,* Paris 1966.

[5] Brand, N., *Another note on the continuity of the inverse*, Archiv. Math. (Basel) **39** (1982), 241–245.

[6] Ellis, R., *A note on the continuity of the inverse*, Proc. Amer. Math. Soc. **8** (1957), 372–373.

[7] Ellis, R., *Locally compact transformation groups*, Duke Math. J. **24** (1957), 119–125.

[8] Engelking, R., *General Topology*, PWN, Warszawa, 1977.

[9] Fletcher, P. and W.F. Lindgren, *Some unsolved problems concerning countably compact spaces*, Rocky Mountain J. Math. **5** (1975), 95–106.

[10] Fletcher, P. and W. F. Lindgren, *Quasi Uniform spaces*, Lecture Notes in Pure Appl. Math., Marcel Dekker, New York, **77** 1982.

[11] Graev, M.I., *Free topological groups*, In: Topology and Topological Algebra, Translations Series 1, vol. 8 (1962), pp. 305–364. American Mathematical Society. Russian original in: Izvestiya Akad. Nauk SSSR Ser. Mat. **12** (1948), 279–323.

[12] Grant, D.L., *The Wallace problem and continuity of the inverse in pseudocompact groups*, General Topology and Applications, Lecture Notes in Pure and Appl. Math., Marcel Dekker, New York, **134** (1991), 93–114.

[13] Kakutani, S., *Free topological groups and direct product of topological groups*, Proc. Imp. Acad. Tokyo **40** (1944), 595–598.

[14] Lane, E.P., *Bitopological spaces and quasi-uniform spaces*, Proc. London Math. Soc. **17** (1967), 241–256.

[15] Marin, J. and S. Romaguera, *A bitopological view of quasi-topological groups*, Indian J. Pure Appl. Math. **27** (1996), 393–405.

[16] Markov, A.A., *On free topological groups*, In: Topology and Topological Algebra, Translation Series 1, vol. 8 (1962), 195–272. American Math. Society. Russian original in: Izvestiya Akad. Nauk SSSR **9** (1945), 3–64.

[17] Mukherjea, A. and N.A. Tserpes, *A note on countably compact semigroups*, J. Austral. Math. Soc. **13** (1972), 180–184.

[18] Numakura, K., *On bicompact semigroups*, Math. J. Okayama Univ. **1** (1952), 99–108.
[19] Pfister, H., *Continuity of the inverse*, Proc. Amer. Math. Soc. **95** (1985), 312–314.
[20] Raghavan, T.G. and I.L. Reilly, *On nonsymmetric topological structures*, Colloquia Math. Soc. Janos Bolyai **23** (1978), 1005–1014.
[21] Reznichenko, E.A., *Extensions of functions defined on products of pseudocompact spaces and continuity of the inverse in pseudocompact groups*, Topology Appl. **59** (1994), 233–244.
[22] Salbany, S., *Quasi-uniformities and quasi-pseudometrics*, Math. Colloq. Univ. Cape Town **6** (1970/1971), 88–102.
[23] Salbany, S., *Bitopological spaces, compactifications and completions*, Math. Monographs Univ. Cape Town, no. 1, 1974.
[24] Wallace, A.D., *The structure of topological semigroups*, Bull. Amer. Math. Soc. **61** (1955), 95–112.
[25] Zelazko, W., *Metric generalizations of Banach algebras*, Rozprawy Math. **47** (1965), Warszawa.

Universidad Politécnica de Valencia, Camino de Vera 14, Apartado 22012, 46071 Valencia, Spain
E-mail address: `sromague@mat.upv.es`

Universidad Jaume I, Campus Riu Sec s/n, 12071 Castellón, Spain
E-mail address: `sanchis@mat.uji.es`

Departamento de Matemáticas, Universidad Autónoma Metropolitana, México
E-mail address: `mich@xanum.uam.mx`