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Department of Mathematics & Statistics
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OPEN COVERS AND SYMMETRIC OPERATORS

VERA USHEVA AND STEPHEN WATSON

ABSTRACT. We show that despite the infinitary nature of discreteness, the theorem of Stone-Michael can be proved entirely outside the topological context as an essentially finitary and combinatorial theorem.

1. INTRODUCTION

We associate a natural asymmetric operator with each open cover of a topological space. If each open cover has a star refinement, then in a natural way the induced symmetric operator also admits star refinements. Surprisingly the existence of a σ -discrete refinement of an open cover then corresponds to the existence of a σ -disjoint refinement of the induced symmetric operator. Thus the classical theorem proved by Stone and Michael that *if every open cover has a star refinement which is an open cover, then every open cover has a σ -discrete open refinement which covers*, has a finitary combinatorial counterpart in this language of symmetric operators. That is we show that despite the infinitary nature of discreteness the theorem of Stone-Michael can be proved entirely outside the topological context, as an essentially finitary and combinatorial theorem: *If a class of symmetric operators admits star refinements, then every operator in that class has a σ -disjoint refinement which covers in that class.* The original topological theorem can then still be derived as a special case.

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2. PRELIMINARIES

Definition 2.1. Suppose \mathcal{A} and \mathcal{B} are subfamilies of $\mathcal{P}(X)$. Then \mathcal{B} is said to be a refinement of \mathcal{A} if for every $B \in \mathcal{B}$ there exists a $A \in \mathcal{A}$ such that $B \subset A$. The family \mathcal{B} is said to be a precise refinement of \mathcal{A} if \mathcal{B} and \mathcal{A} are indexed by the same index set S and for every $s \in S$ $B_s \subset A_s$.

Definition 2.2. Let $\mathcal{A} = \{A_s : s \in S\}$ be a subfamily of $\mathcal{P}(X)$. The star of a set $Y \subset X$ with respect to \mathcal{A} is the set $\text{St}(Y, \mathcal{A}) = \cup\{A_s : Y \cap A_s \neq \emptyset\}$.

Definition 2.3. The family $\mathcal{B} = \{B_t : t \in T\}$ is said to be a star refinement of the family $\mathcal{A} = \{A_s : s \in S\}$ if for every $t \in T$ there exists $s \in S$ such that $\text{St}(B_t, \mathcal{B}) \subset A_s$.

Definition 2.4. The family \mathcal{B} is said to be barycentric refinement of the family $\mathcal{A} = \{A_s : s \in S\}$ if $\{\text{St}(x, \mathcal{B}) : x \in X\}$ refines \mathcal{A} .

We introduce the following definitions and notation.

Definition 2.5. Any mapping $G: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ will be called an operator on X . The set of all operators on X , will be denoted $\mathcal{O}(X)$.

Definition 2.6. An operator $G \in \mathcal{O}(X)$ will be called *monotone* if $G(A) \subset G(B)$ for any $A, B \in \mathcal{P}(X)$ such that $A \subset B$. The set of all monotone operators on a given set X will be denoted $\mathcal{M}(X)$.

Definition 2.7. An operator $G \in \mathcal{O}(X)$ will be called *pointwise* if it preserves unions, i.e. $G(A) = \cup\{G(a) : a \in A\}$ for any set $A \in \mathcal{P}(X)$. The set of all pointwise operators on a given set X will be denoted $\mathcal{Pt}(X)$.

In the following we fix certain notation, which will be of constant use.

- For any operator $G \in \mathcal{O}(X)$ and any family $\mathcal{A} \in \mathcal{P}^2(X)$ we denote by $G(\mathcal{A})$ or simply $G\mathcal{A}$ the family $\{G(A) : A \in \mathcal{A}\}$. In particular for any $G \in \mathcal{O}(X)$ we denote by G also the family $\{G(x) : x \in X\}$.
- For any two operators $G, H \in \mathcal{O}(X)$ and any two families $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$ denote by $G(\mathcal{A}) < H(\mathcal{B})$ the fact that $G(\mathcal{A})$ is a precise refinement of $H(\mathcal{B})$. In further applications we often refer to the above notation as *inequality* between the considered families.

- For any two operators $G, H \in \mathcal{O}(X)$ and any two families $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$ denote by $G(\mathcal{A}) \prec H(\mathcal{B})$ the fact that $G(\mathcal{A})$ is a refinement of $H(\mathcal{B})$.
- For any two families $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$ indexed by the same index set S , we denote by \mathcal{A}_{-B} the family $\{(A_{-B})_s : s \in S\}$ where $(A_{-B})_s = A_s \setminus \cup_{t < s} B_t$ and $<$ is a well order on the index set S .

3. POINTWISE OPERATORS AND POINTWISE INVERSE

Definition 3.1. The operator G is called *inverse* to the operator H if $x \in G(y)$ if and only if $y \in H(x)$ for any two points x and y in the underlying set. In case that G is inverse to H and G is pointwise, we refer to it as the *pointwise inverse* of H .

Remark 3.2. Note that the pointwise inverse always exists and is unique among the pointwise operators.

Definition 3.3. For any operator G define by $G^-(x) = \{y \in X : x \in G(y)\}$ a pointwise operator G^- , and by $G^+(A) = \{x \in X : G(x) \subset A\}$ where $A \in \mathcal{P}(X)$ an operator G^+ .

Remark 3.4. Note that G^- is the (unique) pointwise inverse of G and that G^+ is not necessarily pointwise, but is monotone. Any pointwise operator is also monotone.

Proposition 3.5. For any two pointwise operators G and H we have $(GH)^- = H^-G^-$.

Proof. Directly from the definition of $(GH)^-$. □

Definition 3.6. An operator G will be called *symmetric* if $G = G^-$.

Note that if G is symmetric, then $x \in G(y)$ if and only if $y \in G(x)$ and that a symmetric operator is necessarily pointwise. If \mathcal{G} is an open cover of X , then defining $G(x) = st(x, \mathcal{G})$ gives a symmetric operator G . The set of all symmetric operators on a given set will be denoted $\mathcal{Sm}(X)$.

Proposition 3.7. $\mathcal{Sm}(X) \subset \mathcal{Pt}(X) \subset \mathcal{M}(X) \subset \mathcal{O}(X)$.

Proposition 3.8. For any pointwise operator G the operators GG^- and G^-G are symmetric.

Proof. For pointwise operators $(G^-)^- = G$. □

Lemma 3.9. *For any pointwise operator G and any two subsets A and B of the underlying set the intersection $A \cap GG^-(B)$ is nonempty if and only if the intersection $GG^-(A) \cap B$ is nonempty if and only if $G^-(A) \cap G^-(B)$ is nonempty.*

Proof. The intersection $G^-(A) \cap G^-(B)$ is nonempty if and only if $\exists a \in A \exists b \in B \exists x$ such that $x \in G^-(a) \cap G^-(b)$. But this is equivalent to $\exists a \in A \exists b \in B \exists x$ such that $a \in G(x)$ and $x \in G^-(b)$, i.e. $\exists a \in A \exists b \in B$ such that $a \in GG^-(b)$, i.e. the intersection $A \cap GG^-(B)$ is nonempty. \square

Remark 3.10. Suppose G is a monotone operator such that $G^-(A) \cap G^-(B) \neq \emptyset$ for some A and B . Then $GG^-(A) \cap B \neq \emptyset$ and $A \cap GG^-(B) \neq \emptyset$.

Proposition 3.11. *For any operator G the inequality $1 < G$ implies that $1 < G^-$ and $G^+ < 1$.*

Proof. For any x we have $x \in G(x)$. Then by the definition of inverse operator we obtain $x \in G^-(x)$, that is $1 < G^-$. To obtain the other inequality suppose that $x \in G^+(A)$ for some subset A of X . By definition that is $G(x) \subset A$. But $x \in G(x)$ and so $x \in A$. \square

Proposition 3.12. *For any pointwise operator G and any subset A of the underlying set the identities $G^-(A^c) = G^+(A)^c$ and $(G^-)^+(A^c) = G(A)^c$ hold.*

Proof. The point x belongs to $G^-(A^c)$ if and only if the intersection $G(x) \cap A^c$ is nonempty, which is equivalent to $x \in G^+(A)^c$. The point x belongs to $(G^-)^+(A^c)$ if and only if the intersection $G^-(x) \cap A$ is empty, which is equivalent to $x \notin G(A)$. \square

4. CROSS MULTIPLICATION

Proposition 4.1. *For any pointwise operator G and any two subsets A and B of the underlying set $G(A) \subset B$ if and only if $A \subset G^+(B)$.*

Proof. Note that by definition $G(a) \subset B$ if and only if $a \in G^+(B)$ and use the fact that G is pointwise. \square

We refer to the above Proposition as *cross multiplication* of G and G^+ .

Corollary 4.2. *For any pointwise operator G we have $GG^+ < 1$ and $1 < G^+G$.*

Proposition 4.3. *Suppose G and H are pointwise operators. Then $(GH)^+ = H^+G^+$.*

Proof. For any subset A of the underlying set we have $x \in (GH)^+(A)$ if and only if $GH(x) \subset A$, which by cross multiplication holds if and only if $H(x) \subset G^+(A)$ and again by cross multiplication this holds, if and only if $x \in H^+G^+(A)$. \square

Corollary 4.4. *Suppose A and B are pointwise operators such that $A^n < B$ for some $n \in \mathbb{N}$. Then $A^m B^+ < A^{+p}$ for any $m, p \in \mathbb{N}$ such that $m + p = n$.*

Proof. By Corollary 4.2 we have $BB^+ < 1$ and so $A^n B^+ < 1$. Apply p -times cross multiplication of A and A^+ to obtain the desired inequality. \square

Proposition 4.5. *For any pointwise operator G and any monotone operator H such that $G \prec H$ we have $GG^- < HH^-$.*

Proof. Consider arbitrary point z and $\omega \in GG^-(z)$. There exists a point $x \in G^-(z)$ such that $\omega \in G(x)$. Since $G \prec H$, there exists a point y such that $G(x) \subset H(y)$. But $z \in G(x)$ and so $z \in H(y)$. Then in particular $y \in H^-(z)$. By monotonicity we obtain $H(y) \subset HH^-(z)$. But $\omega \in H(y)$ and so $\omega \in HH^-(z)$. \square

Proposition 4.6. *Suppose G is a pointwise operator, \mathcal{A} and \mathcal{B} families such that $G^-(\mathcal{A}) < G^+(\mathcal{B})$. Then $G^-(\mathcal{A}_{-\mathcal{B}})$ is disjoint.*

Proof. Consider arbitrary elements $(A_{-B})_s$ and $(A_{-B})_t$ of $\mathcal{A}_{-\mathcal{B}}$. We can assume that $s < t$, where $<$ denotes the total ordering relation on the index set. Since $GG^-(A_s) \subset B_s$, we have $GG^-(A_{-B})_s \cap (A_{-B})_t$ is empty, which implies by Lemma 3.9 that $G^-(\mathcal{A}_{-\mathcal{B}})$ is disjoint. \square

Lemma 4.7. *Suppose $\{\mathcal{A}_i\}_{i=1}^\infty$ is a sequence in $\mathcal{P}^2(X)$ such that $\cup \cup \mathcal{A}_i = X$. Then $\cup(\mathcal{A}_{i-\mathcal{A}_{i+1}}) = X$.*

Proof. For any $x \in X$ choose the smallest index $s(x)$ such that $x \in \mathcal{A}_{i,s(x)}$ for some i . Then $x \in (\mathcal{A}_{i-\mathcal{A}_{i+1}})_{s(x)}$. \square

5. STAR AND σ -DISJOINT REFINEMENTS

Definition 5.1. A class of operators $\Theta(X)$ is said to admit star refinements, if for every $S \in \Theta(X)$ there exists $S' \in \Theta(X)$ such that $(S')^2 < S$.

Proposition 5.2. *Suppose $\Theta(X)$ is a class of monotone operators > 1 which admits star refinements. Then for every natural number $n \geq 2$ and every $S \in \Theta(X)$ there exists $S' \in \Theta(X)$ such that $(S')^n < S$.*

Proof. Consider any $S_0 = S \in \Theta(X)$ and for every $i = 0, \dots, n-1$ choose $S_{i+1} \in \Theta(X)$ such that $S_{i+1}^2 < S_i$. Then by monotonicity $S_n^{2^n} < S$ and since $S > 1$, $S_n^n < S$. \square

Remark 5.3. Note that the above proposition holds also for monotone operators which are < 1 .

Definition 5.4. A class of operators $\Theta(X)$ is said to admit σ -disjoint refinements, if for every $S \in \Theta(X)$ there exists a cover P of X , which decomposes as $P = \cup P_i$ and a family $\{S_i : i \in \omega\}$ in $\Theta(X)$ such that for each $i \in \omega$ $S_{i+1}(P_i)$ is a disjoint refinement of S .

Theorem 5.5. *Suppose $\Theta(X)$ is a class of symmetric operators > 1 which admits star refinements, then for every $S \in \Theta(X)$ and every natural number $n \geq 1$ there is a cover P , which decomposes as $P = \cup P_i$ and a family $\{S_i : i \in \omega\}$ in $\Theta(X)$ such that for each i, n $S_{i+1}^n(P_i)$ is a disjoint refinement of S . Thus $\cup S_{i+1}^n(P_i)$ is a σ -disjoint refinement of S , which is a cover.*

Proof. Consider any $S = S_0 \in \Theta(X)$ and for every $i \geq 0$ choose $S_{i+1} \in \Theta(X)$ such that $S_{i+1}^{(2n+1)} < S_i$. By Theorem 4.4 $S_{i+1}^n(S_i^+ S) < (S_{i+1}^n)^+(S_{i+1}^+ S)$. Let $P_i = S_i^+ S_{-S_{i+1}^+} S$. Then by Lemma 4.7 (applied for $\mathcal{A}_i = S_i^+ S$) $P = \cup P_i$ is a cover, since $\cup \mathcal{A}_0 = \cup S_0^+ S = X$. Furthermore by Proposition 4.6 the family $S_{i+1}^n(P_i)$ is disjoint. Since $S_{i+1}^+ < 1$, $S_{i+1}^n(P_i)$ is a refinement of S . \square

6. BARYCENTRIC REFINEMENTS

Definition 6.1. For any class $\Theta(X)$ of pointwise operators let $S_\Theta(X)$ be the class of operators of the form GG^- for G in $\Theta(X)$.

Remark 6.2. Note that $S_\Theta(X)$ consists of symmetric operators.

Definition 6.3. A class of pointwise operators $\Theta(X)$ is said to admit barycentric refinements if for every $S \in \Theta(X)$, there exists $T \in S_\Theta(X)$ such that $T \prec S$.

Theorem 6.4. *Suppose $\Theta(X)$ is a class of pointwise operators > 1 which admits barycentric refinements. Then:*

- (i) $S_\Theta(X)$ admits star refinements.
- (ii) For every $S \in \Theta(X)$ and every natural number $n \geq 2$ there exists $S' \in S_\Theta(X)$ such that $(S')^n \prec S$.
- (iii) For every $S \in \Theta(X)$ and every natural number $n \geq 2$ there exists a cover P which decomposes as $\cup P_i$ and a family of operators $\{S_i : i \in \omega\}$ in $S_\Theta(X)$ such that $S_{i+1}^n(P_i)$ is disjoint refinement of S .

Proof. Let $S \in S_\Theta(X)$. Then $S = GG^-$ for some $G \in \Theta(X)$. Since $\Theta(X)$ admits barycentric refinements there exists an $S' \in S_\Theta(X)$ such that $S' \prec G$. Then by Proposition 4.5 $(S')^2 \prec S$ and so $S_\Theta(X)$ admits star refinements. For part ii consider any $S \in \Theta(X)$. Since $\Theta(X)$ admits barycentric refinements there exists a $T \in S_\Theta(X)$ such that $T \prec S$. But by part i and Proposition 5.2 there exists $T' \in S_\Theta(X)$ such that $(T')^n \prec T$, and so $(T')^n \prec S$. Part iii follows by Theorem 5.5 applied to $S_\Theta(X)$. □

Remark 6.5. Note that 6.4.ii (for $n = 2$) is weaker than star refinements - 5.1.

7. OPEN OPERATORS

Definition 7.1. For any topological space X a pointwise operator G on X will be called *open* if $1 < G$ and $G(x)$ is open for any $x \in X$. The set of all open operators on X will be denoted by $\mathcal{T}(X)$.

Theorem 7.2. *A topological space X has the property that every open cover of X has an open barycentric refinement which covers if and only if $\mathcal{T}(X)$ admits barycentric refinements.*

Proof. Suppose A is an open operator of X . Let $\mathcal{A} = \{A(x) : x \in X\}$. Let \mathcal{B} be a given open barycentric refinement of \mathcal{A} which covers. Let B be an open operator such that $B(x) \in \mathcal{B}$ for every $x \in X$. Then $BB^- \prec A$. □

Corollary 7.3. (Classical) *If X is a topological space such that every open cover of X has an open star refinement, then every open cover has an open symmetric star refinement.*

Proof. Apply Theorem 6.4.ii to $\mathcal{T}(X)$ for $n = 3$, and note that for symmetric open operators S , $S^3(x) = \text{St}(S(x), S)$. \square

Proposition 7.4. *An open family \mathcal{A} is discrete (resp. locally finite) if and only if there exists an open operator G such that $G^-\mathcal{A}$ is disjoint (resp. point finite).*

Proof. The family \mathcal{A} is not discrete (resp. not locally finite) if and only if for every open operator G there exists a point x and a set of indexes I of cardinality 2 (resp. $\geq \omega$) such that $G(x) \cap A_i \neq \emptyset$ for every $i \in I$, which by cross multiplication holds if and only if $x \in G^-(A_i)$ for every $i \in I$, i.e. for every open operator G the family $G^-\mathcal{A}$ is not disjoint (resp. not point finite). \square

Theorem 7.5. *Suppose X is a topological space such that every open cover has an open star refinement. Then every open cover has an open σ -discrete refinement, which is a cover.*

Proof. By Theorem 7.2 $\mathcal{T}(X)$ admits barycentric refinements, and so by Theorem 6.4.iii for every $S \in \mathcal{T}(X)$ there exists a cover $P = \cup P_i$ and a family $\{S_i : i \in \omega\}$ in $S_{\mathcal{T}}(X)$ such that $S_{i+1}^2(P_i)$ is a disjoint refinement of S . But by Proposition 7.4 $S_{i+1}(P_i)$ is a discrete refinement of S , and so $\cup S_{i+1}(P_i)$ is σ -discrete and covers. \square

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FACULTY OF MATHEMATICS AND INFORMATICS, 5 JAMES BOUCHIER BLVD.,
SOFIA 1146, BULGARIA

E-mail address: vusheva@fmi.uni-sofia.bg

DEPARTMENT OF MATHEMATICS, 4700 KEELE ST., TORONTO, ONTARIO
M3J 1P3 CANADA

E-mail address: watson@msfac6.math.yorku.ca