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STRONGLY RIGID EVEN COXETER GROUPS

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Abstract. A Coxeter group W is said to be strongly rigid if any two fundamental generating sets for W are conjugate to one another. We characterize all strongly rigid even Coxeter groups W with Coxeter diagram $\mathcal{V}$ of one of the following forms:

1. $\mathcal{V}$ has no edges labeled 2,
2. $\mathcal{V}$ has no simple circuits of length less than 5.

Further, we indicate how the method of proof can be used to compute the automorphism group $Aut(W)$ for certain Coxeter groups W and to show rigidity and strong rigidity of other classes of Coxeter groups.

1. Introduction

A *Coxeter system* (W, S) is a pair consisting of a group W with a distinguished generating set $S = \{s_i\}_{i \in I}$ for which there is a presentation of the form $\langle S|R\rangle$, where

$$R = \{(s_i s_j)^{m_{ij}} \mid m_{ij} \in \{1, 2, ..., \infty\}, m_{ij} = 1 \Leftrightarrow i = j, m_{ij} = m_{ji}\}.$$

If W possesses a generating set S for which (W, S) is a Coxeter system, then W is called a *Coxeter group*. If one can choose a set S as above so that all of the exponents m_{ij} (for $i \neq j$) are either even or infinite, W is said to be *even*, and (W, S) is called an *even system*. In this paper we will be concerned almost entirely with even systems.

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We will frequently omit the word "Coxeter" when it is clear that we are speaking of a Coxeter group or Coxeter system.

Coxeter groups have a geometric genesis, arising in the 1930s in H. S. M. Coxeter's work on groups of reflections and crystallographic groups. More recently, these groups have proven useful in a number of geometric topological settings, including the theory of buildings [7] and the classification of 3-manifolds ([9], [11], for instance).

It is possible that a single group W corresponds to more than one system. For instance, one may easily show that for k odd, the dihedral group D_{2k} of order $4k$ has two Coxeter presentations:

$$\langle a, b | a^2, b^2, (ab)^{2k} \rangle$$

and

$$\langle x, y, z | x^2, y^2, z^2, (xy)^2, (xz)^2, (yz)^k \rangle.$$

Therefore, we may ask when a given Coxeter group possesses a "unique" representation in some sense. Before making this notion of "uniqueness" precise, we introduce a convenient way of representing graphically all of the information contained in a Coxeter system.

We define the *Coxeter diagram* (or simply *diagram*) $\mathcal{V}$ corresponding to the Coxeter system (W, S) to be an edge-labeled graph whose vertices are in one-to-one correspondence with the set S and for which there is an edge in $\mathcal{V}$ between the vertices s_i and s_j with label m if and only if $m = m_{ij} < \infty$. For instance, it is clear that the first system for D_{2k} given above corresponds to a single edge with label $2k$, and the second corresponds to a triangle, with edges labeled 2,2, and k. This definition differs slightly from the definition of a *Coxeter graph*, which is used heavily in the literature. In a Coxeter *graph*, an edge (labeled ∞) is included between vertices s_i and s_j if $s_i s_j$ has infinite order, and no edge is included between vertices corresponding to generators which commute with one another. We will not again refer to Coxeter graphs.

We call a Coxeter group *rigid* if, given any two systems (W, S) and (W, S'), the corresponding diagrams $\mathcal{V}$ and $\mathcal{V}'$ are isomorphic as edge-labeled graphs. This is equivalent to the existence of an automorphism α of W which satisfies $\alpha(S) = S'$. We call a Coxeter group *strongly rigid* if, given any two systems as above, the fundamental generating sets S and S' are conjugate to one another (that is, α can be taken to lie in $Inn(W)$). Clearly, if W is strongly rigid then it is rigid.

We will also be concerned with a kind of "uniqueness" of a slightly different nature. We call an element w of the system (W, S) a *reflection* if it is a conjugate of some $s \in S$. (This terminology stems from the action of a given Coxeter group on a vector space, in which all elements of the set S correspond to reflections in a given hyperplane.) A Coxeter group W is said to be *reflection independent* if any two systems (W, S) and (W, S') yield the same set of reflections.

Clearly, if W is strongly rigid, then it is reflection independent.

There are a number of results concerning the notions of uniqueness defined above. In [8], R. Charney and M. Davis show that the Coxeter groups of types HM_n and PM_n are strongly rigid by considering the group action upon a particular CAT(0) simplicial complex. In [10], A. Kaul considers a related sort of Coxeter group and realizes another class, K_n, of rigid Coxeter groups. In [16], D. Radcliffe demonstrates the rigidity of all right-angled Coxeter groups. (A Coxeter system is called *right-angled* if for every $s_i, s_j \in S$, either m_{ij} is equal to 2 or $s_i s_j$ has infinite order.) He further shows that any Coxeter group for which every exponent is either 2 or divisible by 4 is rigid. P. Bahls obtains similar results in [2]. Recently, (in [15]) B. Mühlherr and R. Weidmann have obtained a number of results concerning rigidity and strong rigidity of *large-type* Coxeter groups, those groups whose diagrams have no edges labeled 2.

In [4], Bahls and M. Mihalik provide a complete classification of all even reflection independent Coxeter systems and summarize Mihalik's result from [13] which characterizes all even rigid Coxeter groups. In [6], N. Brady, J. P. McCammond, Mühlherr, and W. D. Neumann obtain a classification of all right-angled strongly rigid groups. These three classifications will be introduced in the next section as they are of prime importance in this paper.

The main result of this paper (Theorem 2.2) will provide a classification of a large number of strongly rigid even Coxeter groups. Similar methods, applied with a bit more ingenuity, should provide a solution to the strong rigidity problem for all even Coxeter groups. Furthermore, the method introduced in order to prove Theorem 2.2 is a powerful one which lends itself to further applications. Specifically, one may use the regular circuits defined in Section 5 of this paper, coupled with the understanding of centralizers of parabolic subgroups of Coxeter groups developed in [3], in order to address questions concerning more general types of Coxeter groups. In a forthcoming paper these methods will be used to understand the structure of $Aut(W)$ for certain even Coxeter groups. Furthermore, some of the rigidity results recently obtained by Mühlherr and Weidmann can be recovered using this method. We will return to these points briefly in the conclusion of this paper.

2. The main theorem and similar classifications

In the case of even Coxeter groups, the notions of rigidity and reflecton independence are closely related. The following theorem is proven in [1] and in [4].

Theorem 2.1. *Suppose that W is even. If W is reflection independent, then W is rigid.*

Suppose that (W, S) is a Coxeter system, and let $T \subseteq S$. Denote by W_T the subgroup of W generated by T. Such a subgroup of W is called a *standard parabolic subgroup*, and any conjugate of such a subgroup is called simply a *parabolic subgroup*. It is well-known (see [5]) that (W_T, T) is a Coxeter system, for any $T \subseteq S$.

Let $G \leq W$ be a subgroup of W. We denote by $C(G)$ the centralizer of G in W, and by $Z(G)$ the center of G. If $T \subseteq S$ as above, we let $C(T) = C(W_T)$ and $Z(T) = Z(W_T)$. If T consists of a single element $t \in S$, we write $C(t)$ for $C(\{t\})$ and $Z(t)$ for $Z(\{t\})$. It is clear that $C(T) = \cap_{t \in T} C(t)$ and $Z(T) = C(T) \cap W_T$.

The following theorem is this paper's main result.

Theorem 2.2. *Suppose that (W, S) is an even Coxeter system with connected Coxeter diagram $\mathcal{V}$. If W is strongly rigid, then it is reflection independent (and therefore rigid) and $\mathcal{V}$ contains no set of vertices J so that the following are true:*

1. The full subgraph Γ on the vertices $S \setminus J$ has at least 2 connected components, and

2. there are vertices s_1 and s_2 in different connected components of Γ and an element w in $Z(J)$ such that $ws_1 \neq s_1 w$ and $ws_2 \neq s_2 w$.

Moreover, if W is an even reflection independent Coxeter group whose diagram $\mathcal{V}$ has no such set of vertices, then W is strongly *rigid provided $\mathcal{V}$ has more than two vertices and satisfies one of the following two conditions:*

3. All edges in the even diagram $\mathcal{V}$ have labels greater than 2.

4. The even diagram $\mathcal{V}$ contains no simple circuits of length 3 or 4.

If either one of these conditions holds and $\mathcal{V}$ has more than 2 vertices, then conditions (1) and (2) imply that W is strongly rigid if and only if $\mathcal{V}$ contains no edge or vertex whose removal separates the diagram into more than one connected component, if and only if W is 1-ended.

The final equivalence in the above theorem is an immediate consequence of the decomposition of Coxeter groups provided by Mihalik and S. T. Tschantz in [14].

We note that the dihedral groups D_n, where n is divisible by 4, are even and rigid, but they are not strongly rigid. Thus, we must explicitly exclude these dihedral groups from consideration.

Recently, Mühlherr and Weidmann [15] have proven this characterization for those groups whose diagrams contain no edges labeled 2. Their paper contains a number of other results concerning rigidity and strong rigidity.

A subset $J \subseteq S$ satisfying the first two conditions will be called a *junction*. Here, a *simple circuit* is a path $e_1, e_2, ..., e_k$ $(k \geq 3)$ in a graph Γ with k distinct edges $e_i = [v_i v_{i+1}]$ such that $v_1 = v_{k+1}$ and all other v_i are distinct from one another. Thus, our condition essentially says that there are no "triangles" and "squares" in the diagram $\mathcal{V}$.

In order to understand how to apply this theorem, we must be able to calculate the centralizer of a given parabolic subgroup. To this end, we have the following theorem, proven in [3].

Theorem 2.3. *Suppose (W, S) is an even Coxeter system with Coxeter presentation $\langle S \mid \mathcal{R}\rangle$, and $T \subseteq S$. Define*

$$A = S \cap C(T),$$

$$B_1 = \big\{(bt)^k \mid b, t \in T,\ (bt)^{2k} \in R,\ k > 1, \text{ and } \{b,t\} \subset C(T \setminus \{b,t\})\big\},$$

and

$$B_2 = \big\{(bt)^{k-1}b \mid b \in S \setminus T,\ t \in T \cap C(T), (bt)^{2k} \in R,\ k > 1, \text{ and } b \in C(T \setminus \{t\})\big\}.$$

Then $C(T)$ is generated by $A \cup B_1 \cup B_2$, and $(C(T), A \cup B_1 \cup B_2)$ is an even Coxeter system.

Moreover, the inclusion map $C(T) \to W$ preserves geodesics. I.e. if $u_1 \cdots u_l$ is a geodesic of length l in $(C(T), A \cup B_1 \cup B_2)$, then $u_1 \cdots u_l$ is a geodesic of length $|u_1| + \cdots + |u_l|$ in (W, S).

Notice Theorem 2.1 guarantees that, up to label-preserving isomorphisms, $\mathcal{V}$ is the unique diagram corresponding to a given even, reflection independent group W, and therefore we need only concern ourselves with this single diagram.

We need a bit more terminology to introduce some related results.

The *star* $st(x)$ of a vertex x in a diagram $\mathcal{V}$ is the collection of the vertices of $\mathcal{V}$ consisting of x, along with all vertices which are connected to x by an edge. The *2-star* $st_2(x)$ is the collection of vertices of $\mathcal{V}$ which are connected to x by an edge labeled 2, plus x. A *simplex* σ in a diagram $\mathcal{V}$ is a collection of vertices which span a complete subgraph of $\mathcal{V}$. A simplex σ is called *spherical* if the group $W_\sigma \leq W$ is finite, and σ is called a *maximal spherical simplex* if it is spherical and is not properly contained in another spherical simplex.

We now compare the main theorem with known facts regarding rigidity. First we note that the main theorem provides a partial generalization of the corresponding result for right-angled groups given in [6]:

Theorem 2.4. *Suppose that W is a right-angled Coxeter group with connected diagram $\mathcal{V}$. Then W is strongly rigid if and only if for every vertex s in $\mathcal{V}$, the following are true:*

1. The full subgraph on the set of vertices in $V \setminus st(s)$ is connected, and

2. the vertex s is the intersection of all maximal spherical simplices of $\mathcal{V}$ which contain s.

The following is a corollary of the primary result in [1]; it is frequently referred to as *Even Rigidity*:

Theorem 2.5. *Given any Coxeter group W, there is, up to label-preserving graph isomorphism, at most one even diagram corresponding to W.*

In [13], Mihalik proves the following theorem, which, with Even Rigidity, completely characterizes all even rigid Coxeter groups.

Theorem 2.6. *Suppose* (W, S) *is an even Coxeter system with corresponding diagram* $\mathcal{V}$*. Then* W *corresponds to a non-even system if and only if there is an edge* $[ab]$ *of* $\mathcal{V}$ *with label* $2(2k+1)$ *for* $k > 0$*, for each* $c \neq a$*, any edge* $[bc]$ *of* $\mathcal{V}$ *is labeled 2, and for each such edge* $[bc]$*, there is also an edge* $[ac]$ *with label 2.*

We use this last theorem to identify the even groups which have no non-even systems, and Even Rigidity then implies that these are precisely the even groups which are rigid.

We may readily determine which even systems correspond to reflection independent groups by examining the corresponding diagrams, using the following result from [4].

Theorem 2.7. *Suppose* (W, S) *is an even Coxeter system. Then* W *is reflection independent if and only if it is rigid and neither of the following conditions applies.*

1. There are distinct vertices x *and* y *in* $\mathcal{V}$ *such that* $st(x) \subseteq st_2(y)$*.*

2. There are distinct vertices $x, y, z \in \mathcal{V}$ *and an edge* $[yz]$ *in* $\mathcal{V}$ *with label* $n > 2$ *such that* y *and* z *are both contained in the intersection of all maximal spherical simplices containing* x*.*

We note that as a result of various structural lemmas given in [4], maximal spherical simplices can easily be determined from the diagram $\mathcal{V}$. Thus, both rigidity and reflection independence can be determined from a given diagram. Notice that Theorem 2.2 characterizes strong rigidity in terms of properties of the diagram and in terms of reflection independence. Therefore, strong rigidity of an even Coxeter group whose diagram is of the sort described in the statement of the theorem can be determined solely from the presentation (or, equivalently, the diagram) corresponding to the given Coxeter system.

It is not difficult to see that in case condition (3) holds in Theorem 2.2, then the forbidden separating subgraphs J are precisely those edges and vertices which separate the diagram $\mathcal{V}$, if such subgraphs exist. Therefore, we easily obtain a characterization of "large-type" even strongly rigid Coxeter groups.

3. The conditions are necessary

In this section we prove the necessity of the given conditions. Let us begin by remarking that in an even Coxeter system (W, S), no two distinct generators can be conjugate to one another. This fact will be used frequently without mention in this section and throughout the paper, and it can be proven by noting that the map which forms the quotient of the even group W by the normal closure of any subset $T \subseteq S$ is a retraction onto $W_{S \setminus T}$ (see [4]).

Proposition 3.1. *The conditions given in the statement of Theorem 2.2 are necessary for the given group W to be strongly rigid.*

In order to prove this fact, we will need a few lemmas.

If $u = a_1 \cdots a_n$ is a word in the letters of S, the *length* $|u|$ of u is defined to be the number n. If u is such a word, we denote by $\bar{u}$ the group element that this word represents. Given a group element $w \in W$, the *length* $|w|$ of w is defined to be $min\{|u| \mid \bar{u} = w\}$. A *geodesic word* u in a Coxeter system (W, S) is a word in the letters of S so that $|u| = |\bar{u}|$.

The first two lemmas are proven in [3].

Lemma 3.2. *If u_1 and u_2 are geodesic words representing the same element in a Coxeter system (W, S), then the generators which appear in u_1 are exactly the same as those that appear in u_2. Moreover, if (W, S) is even, each letter $a \in S$ appears in u_1 and u_2 the same number of times.*

Lemma 3.3. *Suppose that u is a word representing a Coxeter group element in the system (W, S) and $x \in S$ is such that $x\bar{u} = \bar{u}x$ and x does not appear in u. Then x commutes with every generator which appears in u.*

The last lemma we require is often referred to as the *Deletion Condition.* A simple geometric proof due originally to Ol'Shanskii may be found in [3], and a more elementary one in [5].

Lemma 3.4. (Deletion Condition) *Suppose that $u = a_1 a_2 ... a_n$ is a word representing the element $\bar{u}$ in the Coxeter system (W, S). If u is not geodesic, then there are indices $i < j$ so that if $u' = a_1 a_2 ... a_{i-1} a_{i+1} ... a_{j-1} a_{j+1} ... a_n$, $\bar{u} = \bar{u}'$.*

This allows us to shorten any non-geodesic representation for a given group element simply by removing two of the letters from the representative word.

Proof of Proposition 3.1: We use the same notation as in the statement of Theorem 2.2. Suppose that W corresponds to the diagram $\mathcal{V}$. Clearly W must be reflection independent if it is to be strongly rigid. By Theorem 2.1, W must also be rigid. Suppose by way of contradiction that $\mathcal{V}$ *does* contain a subset J and a group element $w \in W$ satisfying the conditions put forth in Theorem 2.2. Let K be the connected component of the graph Γ which contains the vertex s_1. Consider the following collection of group elements:

$$K' = \{s' = wsw^{-1} | s \in K\}.$$

Let $S' = (S \setminus K) \cup K'$. By the remark at the very beginning of this section, $S \setminus K$ and K' do not contain any group elements in common, and $|S'| = |S|$. We claim that S' is also a fundamental generating set of W which is not conjugate to S.

It is easy to see that S' is a fundamental generating set for a group isomorphic to W. Indeed, one may show (by considering the presentation corresponding to the system (W, S) and using the fact that J separates the diagram $\mathcal{V}$) that the map α which takes $s \in K$ to wsw^{-1} and which fixes $s \notin K$ is a homomorphism with an inverse taking $s \in K$ to $w^{-1}sw$. Thus, α is an automorphism of W, and α takes S to S'. Because the automorphic image of a fundamental generating set is another such set, S' is also a fundamental generating set.

Now suppose in order to derive a contradiction that $w_1 S w_1^{-1} = S'$ for some $w_1 \in W$.

Consider any vertex $a \in S \setminus K$. Then $w_1 a w_1^{-1} = a$, so that $w_1 \in C(a)$. Therefore, $w_1 \in C(S \setminus K)$. Denote this subgroup by C. Clearly, $w_1 \neq 1$. Thus, $C \neq \{1\}$.

Let u be a geodesic word such that $1 \neq \bar{u} \in C$. In order to understand what u may be, we consider the following cases.

Case 1. First suppose that a letter a appears in u so that $[ab]$ is an edge in $S \setminus K$ with label $n > 2$. Because $\bar{u} \in C(a) \cap C(b)$, Theorem 2.3 shows that $\bar{u}$ may be represented by a product of $(ab)^{n/2}$ and of elements $x \in S$ so that x commutes with both a and b. Moreover, from Theorem 2.3 we see that if $\bar{u}$ is represented by such a product v with a minimal number of terms, then v is geodesic in W. Therefore, both a and b must appear in u, by Lemma 3.2.

Suppose that x is any vertex in $S \setminus (K \cup \{a,b\})$. If x appears in u, then $[ax]$ and $[bx]$ are edges labeled 2. If x does not appear in u, then by Lemma 3.3 it must commute with every letter which appears in u, and thus with a and b. Therefore, $[ax]$ and $[bx]$ are edges labeled 2 in any case.

Suppose that both a and b lie in J. Since $S \setminus (J \cup K)$ is not empty (by the choice of J as a junction), there is some $x \in S \setminus (J \cup K)$. Because $[ay]$ and $[by]$ are edges labeled 2 for every vertex $y \in S \setminus (K \cup \{a,b\})$, $[ab]$ must lie in every maximal spherical simplex containing x. This contradicts the reflection independence of W, by Theorem 2.7. Therefore, either one or both of a, b lie in $S \setminus (J \cup K)$. The last paragraph shows that there are no other vertices in $S \setminus (J \cup K)$.

Returning to u, suppose x is any letter of $S \setminus (K \cup \{a,b\})$ which appears in u. (Note that $x \in J$.) If there were an edge $[xy]$ in $S \setminus K$ with label $m > 2$, applying precisely the same argument as given above (with x and y replacing a and b, respectively), we would arrive at a contradiction. (At least one of x or y would lie in $S \setminus (J \cup K)$, so without loss of generality $x = a$ or $x = b$, a contradiction.)

Therefore, if x is any other generator appearing in u, x must not lie on any edge in $S \setminus K$ with label $m > 2$. Suppose x does appear in u. Then $\bar{u} \in C(x)$. If y is any vertex of $S \setminus (K \cup \{x,a,b\})$, then either y does not appear in u, in which case $xy = yx$ by Lemma 3.3, or y does appear in u and, therefore, $xy = yx$ because of the observation which begins this paragraph. In any case, x is connected to every other vertex in $S \setminus K$ by an edge labeled 2. But as $x \in J$, this contradicts the fact that W is reflection independent, by Theorem 2.7. Thus, u contains no letters besides a and b, and $u = (ab)^{n/2}$.

Case 2. Now suppose no letter a appears in u so that there is an edge $[ab] \subseteq S \setminus K$ with label $n > 2$. Since $\bar{u} \neq 1$, there is some letter a appearing in u. Choose such an a. Arguing almost exactly as before, one may show that $S \setminus (J \cup K) = \{a\}$, and that $\bar{u} = a$.

Therefore, either $w_1 = (ab)^{n/2}$ or $w_1 = a$ for the appropriate choice of a and b, as explained above.

Because $wxw^{-1} = w_1 x w_1^{-1}$ for all $x \in J \cup K$, $w^{-1}w_1 = w_2 \in \cap_{x \in J \cup K} C(x)$. Arguments similar to those given above show that either $w_2 = (cd)^{m/2}$ or $w_2 = c$ for some choice of $c \in K$ and $d \in J \cup K$ as explained above.

Now $w = w_1 w_2^{-1}$, and it is clear that no matter the form of w_1 and w_2 (as described above), w will not lie in the center of J, as was assumed. This contradiction completes the proof. □

4. An easy case

In this section we will prove a special case of Theorem 2.2, in which the diagram $\mathcal{V}$ corresponding to a given group W is a simple circuit C_k of length $k \geq 5$, all of whose edges bear even labels. We claim that all such diagrams represent strongly rigid groups. Indeed, this can be shown as a consequence of [8]; however, we will derive a new proof which will serve as the basis for the proof in more general settings, as will be seen in later sections. Moreover, as was mentioned in the introduction, the general method developed here will prove useful in answering questions concerning different sorts of Coxeter groups.

One may show rather easily that the diagrams C_k satisfy the conditions given in the statement of Theorem 2.2. Indeed, there are no vertices $x \neq y$ so that $st(x) \subseteq st_2(y)$, and for no vertices $x \neq y$ does y lie in every maximal spherical simplex containing x; therefore, by Theorem 2.7, W is reflection independent. It is just as easy to show that there is no separating subgraph J as demanded in the statement of Theorem 2.2.

Suppose that $\mathcal{V}$ corresponds to the system (W, S), and consider any other fundamental generating set S' for W. Denote the diagram corresponding to this system by $\mathcal{V}'$. Of course, $\mathcal{V}$ and $\mathcal{V}'$ are isomorphic as labeled graphs. We know by reflection independence that every generator $a \in S$ is conjugate to a unique generator $a' \in S'$. By the main theorem in [1], we may assume that this conjugacy relation respects the isomorphism of the diagrams $\mathcal{V}$ and $\mathcal{V}'$. That is, we may choose an edge-labeled graph isomorphism $\alpha : \mathcal{V} \to \mathcal{V}'$ so that for every $a \in S$, a and $\alpha(a)$ are conjugate.

Now we consider the conjugating elements very carefully. All arithmetic done below will be modulo k.

Let $S = \{a_1, a_2, ..., a_k\}$ and $S' = \{a'_1, a'_2, ..., a'_k\}$, where a_i and a'_i are conjugate to one another for all $i = 1, ..., k$, and $[a_i a_{i+1}]$ is an edge with label n_i.

We require the following fact, which is proven in [4].

Theorem 4.1. *Let (W, S) and (W, S') be even systems corresponding to the same Coxeter group, with corresponding diagrams $\mathcal{V}$ and $\mathcal{V}'$. If σ is a maximal spherical simplex in $\mathcal{V}$, then there is some maximal spherical simplex $\sigma' \subseteq \mathcal{V}'$ so that σ and σ' are isomorphic as edge-labeled graphs, and so that the subgroups W generated by these simplices are conjugate by some element $w \in W$. That is, $wW_\sigma w^{-1} = W_{\sigma'}$.*

For any $i = 1, ..., k$, $[a_i a_{i+1}]$ is an edge with even label n_i. In this case each edge $[a_i a_{i+1}]$ is itself a maximal spherical simplex. Therefore Theorem 4.1 tells us that for every $i = 1, ..., k$ there is an element $w_i \in W$ so that w_i conjugates the generators a_i and a_{i+1} to generators of the group $W_{\{a'_i, a'_{i+1}\}}$, one of which is a conjugate of a'_i, the other a conjugate of a'_{i+1}. We may assume, modifying w_i if necessary, that $w_i a_i w_i^{-1} = a'_i$. Since $w_i a_i w_i^{-1}$ and $w_i a_{i+1} w_i^{-1}$ generate $W_{\{a'_i, a'_{i+1}\}}$, $w_i \alpha_{i+1} a_{i+1} \alpha_{i+1}^{-1} w_i^{-1} = a'_{i+1}$ for some word α_{i+1} in the letters a_i and a_{i+1}. In fact, α_{i+1} is either trivial or can be written as $(a_{i+1} a_i)^{k_i}$ or $a_i(a_{i+1} a_i)^{k_i}$, where $0 \leq k_i \leq \frac{n_i}{4} - 1$. (In particular, $\alpha_{i+1} = 1$ when $n_i = 2$.)

Therefore,

$$w_i a_i w_i^{-1} = w_{i-1} \alpha_i a_i \alpha_i^{-1} w_{i-1}^{-1} \Rightarrow w_{i-1}^{-1} w_i = \alpha_i \hat{a}_i,$$

where α_i is described above and $\hat{a}_i \in C(a_i)$ for all $i = 1, ..., k$.

Now note that

$$\alpha_1 \hat{a}_1 \alpha_2 \hat{a}_2 \cdots \alpha_k \hat{a}_k = w_k^{-1} w_1 w_1^{-1} w_2 \cdots w_{k-1}^{-1} w_k = 1.$$

We first show that we can "eliminate" the words α_i. We then consider the most general form for the words $\hat{a}_i$ and describe the free cancellation that can occur in the product of such words. This will enable us to construct van Kampen diagrams which further reduce the number of possibilities for the form of the words $\hat{a}_i$. Once we have obtained simple enough forms for these words, we will be able to describe a single group element w which conjugates a_i to a'_i for any $i = 1, ..., k$.

To understand $\hat{a}_i$, consider $C(a_i)$. By Theorem 2.3, $C(a_i)$ is the subgroup of W generated by the set $\{a_i, (a_{i-1}a_i)^{n_{i-1}/2}, (a_i a_{i+1})^{n_i/2}\}$.

Since a_i is central in this subgroup, any element of $C(a_i)$ may be written geodesically as a product of either 1 or a_i and an alternating product of the words of the form $v_{i-1,i} = (a_{i-1}a_i)^{\frac{n_{i-1}}{2}-1}a_{i-1}$ and $v_{i+1,i} = (a_{i+1}a_i)^{\frac{n_i}{2}-1}a_{i+1}$. For now let us agree to write a_i at the end of the word $\hat{a}_i$ if this letter appears outside of one of the words $v_{i-1,i}$ or $v_{i+1,i}$. Such an occurrence will be called a *loose occurrence* of a_i.

Consider the homomorphism ν which identifies all generators but a_i and a_{i+1} to 1 (we identify both a_i and a_{i+1} with their images under ν). Then by considering the form of $\hat{a}_i$ given above,

$$\begin{aligned}
&\nu(\hat{a}_{i-1}) \in \{1, a_i\},\\
&\nu(\alpha_i) \in \{1, a_i\},\\
&\nu(\hat{a}_i) \in \{1, a_i, v_{i+1,i}, a_i v_{i+1,i}\},\\
&\nu(\hat{a}_{i+1}) \in \{1, a_{i+1}, v_{i,i+1}, a_{i+1}v_{i,i+1}\},\\
&\nu(\alpha_{i+2}) \in \{1, a_{i+1}\}.
\end{aligned}$$

Therefore,

$$\nu(\hat{a}_{i-1}\alpha_i\hat{a}_i) \in \{1, a_i, v_{i+1,i}, a_i v_{i+1,i}\}$$

and

$$\nu(\hat{a}_{i+1}\alpha_{i+2}) \in \{1, a_{i+1}, v_{i,i+1}, a_{i+1}v_{i,i+1}\}.$$

Note also that $\nu(\alpha_{i+1}) = \alpha_{i+1}$, and $\nu(\alpha_j\hat{a}_j) = 1$ for all $j \notin \{i-1, i, i+1, i+2\}$. Now $\nu(\alpha_1\hat{a}_1 \cdots \alpha_k\hat{a}_k) = 1$, because $\alpha_1\hat{a}_1 \cdots \alpha_k\hat{a}_k = 1$. However, it is straightforward to show using the above computations that if α_{i+1} has length greater than 1 (so $\alpha_{i+1} \notin \{1, a_i\}$), this cannot be true. Thus, $\alpha_{i+1} \in \{1, a_i\}$. If $\alpha_{i+1} = a_i$, we can "push it back" into $\hat{a}_i$, cancelling, if necessary, with the loose occurrence of a_i which may occur there already.

Therefore, we may assume from now on that $\alpha_i = 1$ for all i.

Now we ask what free cancellation can occur when we multiply the elements $\hat{a}_i$ with one another. Note that a given letter a_i can occur only in the three words $\hat{a}_{i-1}$, $\hat{a}_i$, or $\hat{a}_{i+1}$; therefore, we need only consider cancellation between these words.

First consider the product $\hat{a}_{i-1} \cdot \hat{a}_i$. The following possibilities involve non-trivial free cancellation:

1. $\hat{a}_{i-1} = u_{i-1}v_{i-2,i-1}a_{i-1}$, $\hat{a}_i = v_{i-1,i}u_i$ (2 letters cancel);
2. $\hat{a}_{i-1} = u_{i-1}v_{i-2,i-1}$, $\hat{a}_i = a_i$ (2 letters cancel);
3. $\hat{a}_{i-1} = u_{i-1}v_{i,i-1}a_{i-1}$, $\hat{a}_i = v_{i-1,i}u_i$ ($2n_{i-1}-2$ letters cancel);

4. $\hat{a}_{i-1} = u_{i-1}v_{i,i-1}a_{i-1}$, $\hat{a}_i = v_{i-1,i}a_i$ ($2n_{i-1}$ letters cancel).

Here it is assumed that u_i either contains $v_{i+1,i}$ as a subword or does not contain a loose occurrence of a_i. Each of these formulas always obtains, whether $n_{i-1} > 2$ or $n_{i-1} = 2$.

Similar formulas obtain for the cancellation which occurs between $\hat{a}_i$ and $\hat{a}_{i+1}$. Additionally, we note that the only time there is cancellation between $\hat{a}_{i-1}$ and $\hat{a}_{i+1}$ is when $\hat{a}_i = 1$, in which case we may have

$$\hat{a}_{i-1} = u_{i-1}v_{i,i-1}, \hat{a}_{i+1} = v_{i,i+1}u_{i+1}$$

which allows cancellation of the single letter a_i.

Now let us construct a van Kampen diagram Δ whose boundary $\partial\Delta$ is labeled by the product of the words $\hat{a}_i$, and assume that the free reduction described above has been performed. (We assume knowledge of such diagrams; see [12] for example.)

We now show that both $v_{i-i,i}$ and $v_{i+1,i}$ can occur at most once in $\hat{a}_i$. Were this not true, either $v_{i-1,i}v_{i+1,i}v_{i-1,i}$ or $v_{i+1,i}v_{i-1,i}v_{i+1,i}$ would appear as a subword of $\hat{a}_i$.

In the first case, after performing free cancellation in the product $\hat{a}_{i-1}\hat{a}_i$ as described above, at least $v_{i+1,i}v_{i-1,i}$ remains, and this subword cannot be affected by further free cancellation in the product $\hat{a}_{i-1}\hat{a}_i\hat{a}_{i+1}$. Therefore, we may draw a diagram as shown in Figure 1.

We now construct "bands" within the diagram Δ. Consider any edge labeled a_{i-1} inside of the occurrence of the subword $v_{i-1,i}$ in $\hat{a}_i$ described above. This edge lies on the boundary of a unique relator cell Π in Δ, and there is a unique edge (also labeled a_{i-1}) in Π directly opposite from our initial edge. This opposite edge lies either on the boundary of Δ or inside of a unique cell $\Pi' \neq \Pi$. This cell Π' will be the next cell in the band. We continue this process until we have reached the boundary of Δ once again.

We construct a similar band, beginning at any occurrence of the letter a_{i+1} in the subword $v_{i+1,i}$ in $\hat{a}_i$ described above. Because we have assumed that the words $\hat{a}_j$ are all geodesic, neither of these bands terminates at any letter lying in $\hat{a}_i$. (Otherwise the Deletion Condition is exhibited in $\hat{a}_i$, contradicting geodesity.)

Therefore, the a_{i-1}-band must terminate in either $\hat{a}_{i-1}$ or $\hat{a}_{i-2}$, and the a_{i+1}-band must terminate in either $\hat{a}_{i+1}$ or $\hat{a}_{i+2}$. This

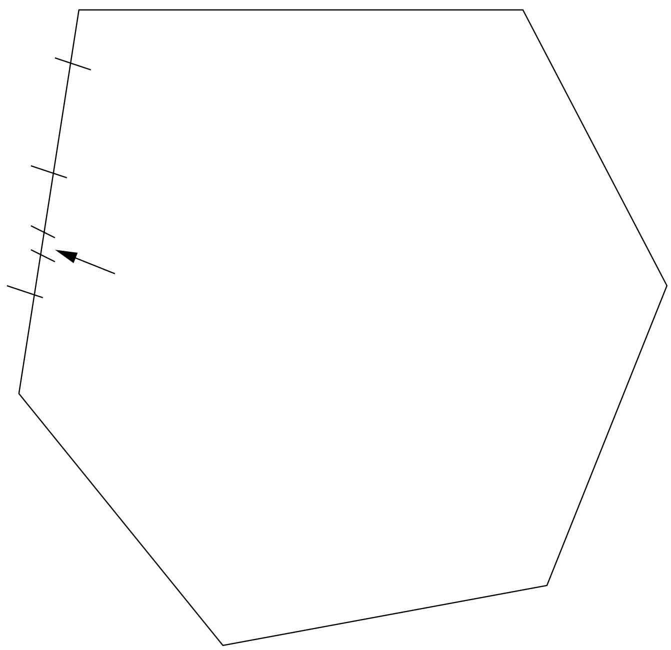

FIGURE 1. The first case, after cancellation

implies, as shown in Figure 2, that some pair of bands as described above cross each other.

However, since a_{i-1} and a_{i+1} are unrelated elements of the group, this cannot occur. A completely symmetric argument shows that $v_{i+1,i}v_{i-1,i}v_{i+1,i}$ cannot appear as a subword of $\hat{a}_i$.

Therefore, each $v_{i-1,i}$ and $v_{i+1,i}$ occurs at most once as a subword of $\hat{a}_i$, and (in light of the preceding argument) if both occur, $v_{i-1,i}v_{i+1,i}$ is a subword of $\hat{a}_i$. (That is, $v_{i-1,i}$ occurs "first.")

Let us for the moment fix $i = 2$ and consider the possibilities for the words $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$. Any occurrence of the letter a_2 in the product of the words $\hat{a}_i$ must lie in one of these three words. Since the product of all of the words $\hat{a}_i$ is trivial, the occurrences of a_2 in $\hat{a}_1\hat{a}_2\hat{a}_3$ must somehow "cancel" each other.

Consider, for instance, if we were to have $\hat{a}_1 = v_{k,1}v_{2,1}$, $\hat{a}_2 = v_{1,2}v_{3,2}a_2$, and $\hat{a}_3 = v_{2,3}v_{4,3}$. After reduction in the group W,

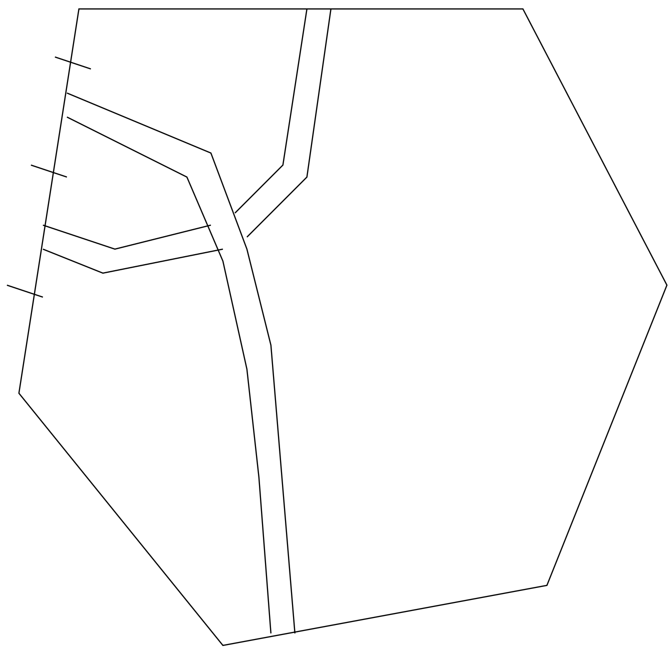

FIGURE 2. Crossing of bands

$$\hat{a}_1\hat{a}_2\hat{a}_3 = v_{k,1}a_1a_2a_3v_{4,3} \qquad (1).$$

Under the quotient homomorphism ν which sends every element of $S\setminus\{a_2\}$ to 1, the product of all of the words $\hat{a}_i$ must be taken to the identity. However, it is clear from (1) that the image of this product is represented by the word $\nu(a_2)$, implying that the generator a_2 is sent to 1 as well, which is clearly not the case because W is even. Therefore, the words $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ could not have had the forms described above, and this product is not admissible.

We may apply a similar argument to the finitely many other choices for these three words. Namely, we know that

$$\hat{a}_1 \in \{u, uv_{2,1}, uv_{2,1}a_1\}$$
$$\hat{a}_2 \in \{1, v_{1,2}, v_{3,2}, a_2, v_{1,2}v_{3,2}, v_{1,2}a_2, v_{3,2}a_2, v_{1,2}v_{3,2}a_2\}$$
$$\hat{a}_3 \in \{v, v_{2,3}v\}$$

where u and v do not contain a_2. By considering each of these choices in turn, we prove the following lemma by direct computation.

Lemma 4.2. *Let u and v retain the meanings given above. The following statements are true of the reduced word α representing any of the admissible products $\hat{a}_1\hat{a}_2\hat{a}_3$ described above.*

1. α does not contain the letter a_2.

2. α is of one of the forms uv, ua_1v, ua_3v, or ua_1a_3v. (Thus, there exist words $\alpha_1 \in W_{\{a_1,a_k\}} \cap C(a_1)$ and $\alpha_2 \in W_{\{a_3,a_4\}} \cap C(a_3)$ such that $\alpha = \alpha_1\alpha_2$.)

We can now compute all possibilities for the word v, as well as the possibilities for the portion of the word $\hat{a}_4$ which contains a_3.

We first write $\hat{a}_4$ as $\beta_1\beta_2$, where β_2 does not contain a_3. Assuming that $\hat{a}_1\hat{a}_2\hat{a}_3$ reduces to one of the forms uv or ua_1v (where u and v are as above), quotient arguments similar to that given above show that

$$n_3 > 2 \Rightarrow (v, \beta_1) \in \{(1,1), (v_{4,3}a_3, v_{3,4})\}$$

and

$$n_3 = 2 \Rightarrow (v, \beta_1) \in \{(1,1), (a_4, 1), (a_3, a_3), (a_4a_3, a_3)\}.$$

Assuming that the reduced form of the product $\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4$ is either ua_3v or ua_1a_3v, we may show

$$n_3 > 2 \Rightarrow (v, \beta_1) \in \{(a_3, 1), (v_{4,3}, v_{3,4})\}$$

and

$$n_3 = 2 \Rightarrow (v, \beta_1) \in \{(1, a_3), (a_3, 1), (a_4, a_3), (a_4a_3, 1)\}.$$

With these choices for v and β_1, we now observe that the word $\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4$ can be written as a reduced word which contains neither a_2 nor a_3. In fact, $\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4$ is equal to a word of one of the forms $u\beta_2$, $ua_1\beta_2$, $ua_4\beta_2$, $ua_1a_4\beta_2$, where u is the same word as before and β_2 is a word in the letters a_4 and a_5 which commutes with a_4.

In fact, we now know more is true: if $\hat{a}_1\hat{a}_2\hat{a}_3 \in \{uv, ua_1v\}$ with u and v as above, then our computations have shown that v commutes with both a_3 and with a_4. Similarly, if $\hat{a}_1\hat{a}_2\hat{a}_3 \in \{ua_3v, ua_1a_3v\}$, a_3v must commute with both a_3 and a_4.

We may clearly continue this procedure by considering the product $\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5$, and so on. The number of possible forms which must be considered at each step does not increase: the form of

$\hat{a}_1 \cdots \hat{a}_k$ depends only on the number of occurrences (either 0 or 1) of $\hat{a}_{k-1}$ in the "known" portion of $\hat{a}_1 \cdots \hat{a}_{k-1}$.

Inducting, we obtain the following result.

Proposition 4.3. *Suppose that the words $\hat{a}_i$ are defined as above and let $3 \leq i \leq k-1$. Then the group element represented by the product $\hat{a}_1\hat{a}_2 \cdots \hat{a}_i$ may also be represented by a word of the form $\hat{u}v_i$, where*

1. $\hat{u}$ is a word in the letters a_1 and a_k, $\hat{u}a_1 = a_1\hat{u}$ and $\hat{u}a_k = a_k\hat{u}$,

2. v_i is a word in the letters a_i and a_{i+1}, $v_ia_i = a_iv_i$ and $v_ia_{i+1} = a_{i+1}v_i$.

We emphasize that $\hat{u}$ does not depend on i; it is either ua_1 or u in the above arguments, depending on whether a_1 did or did not appear outside of u, in the reduced form for $\hat{a}_1\hat{a}_2\hat{a}_3$.

Proof: The process described above can clearly be generalized to determine $\hat{a}_1 \cdots \hat{a}_i$ given the form of $\hat{a}_1 \cdots \hat{a}_{i-1}$. It is also clear that u remains unaffected in this procedure.

The only claim not yet proven is that $\hat{u}a_k = a_k\hat{u}$. However, we can apply the argument given above to determine the form of the word $\hat{a}_k\hat{a}_1\hat{a}_2\hat{a}_3$ given the form of $\hat{a}_1\hat{a}_2\hat{a}_3$, and just as the previous argument showed that either a_3v or v (depending on whether or not a_3 appeared outside of v) commutes with a_4, the slightly modified argument shows that $\hat{u}$ must commute with a_k. □

Set $w = w_k\hat{u}$, where $\hat{u}$ is the word described in Proposition 4.3. We claim that w conjugates S to S'.

First,

$$wa_kw^{-1} = w_k\hat{u}a_k\hat{u}^{-1}w_k^{-1} = w_ka_k\hat{u}\hat{u}^{-1}w_k^{-1} = w_ka_kw_k^{-1} = a'_k$$

because $a_k\hat{u} = \hat{u}a_k$. Also, since $w_k = w_1\hat{a}_1^{-1}$,

$$wa_1w^{-1} = w_1\hat{a}_1^{-1}\hat{u}a_1\hat{u}^{-1}\hat{a}_1w_1 = w_1a_1w_1^{-1} = a'_1$$

because both $\hat{a}_1$ and $\hat{u}$ commute with a_1.

More generally,

$$w_k = w_1\hat{a}_1^{-1} = w_2\hat{a}_2^{-1}\hat{a}_1^{-1} = \cdots = w_{k-1}\hat{a}_{k-1}^{-1}\hat{a}_{k-2}^{-1} \cdots \hat{a}_1^{-1}.$$

Combining this with Proposition 4.3,

$$w = w_k\hat{u} = w_iv_i^{-1}\hat{u}^{-1}\hat{u} = w_iv_i^{-1}$$

for every $i = 3, ..., k-1$, where v_i commutes with a_i. Therefore,

$$wa_iw^{-1} = w_iv_i^{-1}a_iv_iw_i^{-1} = w_ia_iv_i^{-1}v_iw_i^{-1} = w_ia_iw_i^{-1} = a'_i,$$

as desired.

5. Regular circuits and the structure of $\mathcal{V}$

In this section we prove a couple of "structural" lemmas concerning the graph which underlies $\mathcal{V}$ and introduce a definition which will be of great importance in proving the main theorem in general. Recall that the *degree* of a vertex x in an undirected graph without loops is the number of edges for which x is an endpoint.

Lemma 5.1. *Suppose that (W, S) is a reflection independent even Coxeter system whose diagram $\mathcal{V}$ does not contain a junction J (see Theorem 2.2). Then either W is a dihedral group or $\mathcal{V}$ contains no vertices of degree 1.*

Proof: Suppose on the contrary that x is a vertex in $\mathcal{V}$ of degree 1; it is therefore incident a single edge $[xy]$. If the label on $[xy]$ is 2, clearly $st(x) \subseteq st_2(y)$, so that W is not reflection independent, by Theorem 2.7. If the label on $[xy]$ is $2(2k + 1)$ for some $k > 0$, then W is not rigid, by Theorem 2.6, and therefore not reflection independent, by Theorem 2.1.

So $[xy]$ must be labeled by an integer n, where 4 divides n. If $[xy]$ comprises the entire diagram $\mathcal{V}$, we are done, as W is then dihedral. Otherwise $J = \{y\}$ is a set which separates the diagram $\mathcal{V}$ into at least two components. Consider the group element y. If there were a generator $z \in S \setminus \{x, y\}$ such that $yz \neq zy$, J would be a junction. Since $\mathcal{V}$ does not contain a junction, $yz = zy$ for all generators $z \notin \{x, y\}$, so that $[yz]$ is an edge labeled 2 for all such vertices. Clearly this implies that for any vertex $z \notin \{x, y\}$, $st(z) \subseteq st_2(y)$, again contradicting reflection independence. □

For the remainder of this section, consider a diagram $\mathcal{V}$ which has no simple circuits of length less than 5.

If p is a path in $\mathcal{V}$ and x,y are vertices on p, we denote by $dist_p(x, y)$ the length (i.e., number of edges) of the subpath of p between these vertices. Let C be a circuit in $\mathcal{V}$ consisting of edges $\{[a_1a_2], ..., [a_na_1]\}$. We define $dist_C(a_i, a_j)$ to be the minimum of the lengths of the two paths between a_i and a_j defined by the circuit.

A collection of edges $\{[a_1a_2], [a_2a_3], ..., [a_na_1]\}$ is said to define a *regular circuit* C in $\mathcal{V}$ if the vertices a_i are all distinct, if whenever

$dist_C(a_i, a_j) > 1$, there is no edge $[a_i a_j]$ and there is no path of length 2 lying outside the given collection which connects a_i to a_j. (The fact that there are no simple circuits of length less than 5 forces $|i-j| \geq 3$ if there is to be such a path of length 2 connecting a_i and a_j, for *any* circuit.) Examples of regular and non-regular circuits are shown in Figure 3; note that the circuit

$$\{[a_1a_2], [a_2a_3], [a_3a_4], [a_4a_5], [a_5a_6], [a_6a_1]\}$$

is regular, while the circuit

$$\{[a_1a_2], [a_2a_3], [a_3a_4], [a_4b_1], [b_1b_2], [b_2a_6], [a_6a_1]\}$$

is not.

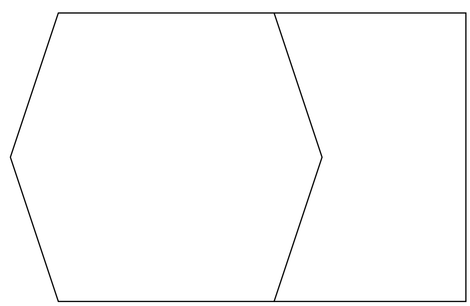

FIGURE 3. Regular and non-regular circuits

Lemma 5.2. *Suppose that (W, S) is an even reflection independent Coxeter system with corresponding diagram $\mathcal{V}$, and that $\mathcal{V}$ contains neither simple circuits of length less than 5 nor junctions. Then either W is dihedral or every vertex in $\mathcal{V}$ is contained in some regular circuit.*

Proof: We may assume that $|S| > 2$. Let us first show that every vertex is contained in a circuit, regular or non-regular.

Consider a vertex $x \in \mathcal{V}$. First suppose that removal of x and all edges incident x separates the graph underlying $\mathcal{V}$. In this case, since $\mathcal{V}$ does not contain a junction, x must commute with every generator in some component of the graph that results from removing x. However, this contradicts the assumption that (W, S) is reflection independent, as in the proof of the previous lemma.

Therefore, removal of x and all edges incident x does not separate the graph underlying $\mathcal{V}$. Consider any two vertices y, z so that there are edges $[xy], [xz]$ (this is possible, by Lemma 5.1). Because removal of x and all incident edges does not disconnect the graph underlying $\mathcal{V}$, there is a reduced path from y to z which does not contain the vertex x, and in fact does not contain the edges $[xy]$ and $[xz]$. Concatenating this path with these edges gives the desired circuit.

Now suppose we have found a circuit $C = \{[a_1a_2], ..., [a_na_1]\}$ which contains the vertex x, where $a_1 = x$. If C is regular, we are done. Otherwise, there are indices $i < j$ such that $dist_C(a_i, a_j) > 1$ and either there is an edge $[a_ia_j]$ in the graph underlying $\mathcal{V}$ or there is a path of length 2 from a_i to a_j lying outside of C. In either case, we can find a strictly shorter circuit which contains x. In the first case, where $1 \leq i < j$, the circuit

$$C' = \{[a_1a_2], ..., [a_{i-1}a_i], [a_ia_j], [a_ja_{j+1}], ..., [a_na_1]\}$$

is such a circuit. In the second case, where again $1 \leq i < j$, given that $[a_ib], [ba_j]$ is a path lying outside of C,

$$C' = \{[a_1a_2], ..., [a_{i-1}a_i], [a_ib], [ba_j], [a_ja_{j+1}], ..., [a_na_1]\}$$

is the desired circuit. An obvious induction completes the proof. □

6. No triangles, no squares

Let us now prove the main theorem under the assumption that the diagram $\mathcal{V}$ and the isomorphic diagram $\mathcal{V}'$ contain neither "triangles" nor "squares," i.e., neither simple circuits of length 3 nor of length 4.

Consider any regular circuit $C = \{[a_1a_2], ..., [a_ka_1]\}$. Since $\mathcal{V}$ contains neither triangles nor squares, the length of this circuit is at least 5. This circuit corresponds to a circuit $C' = \{[a'_1a'_2], ..., [a'_ka'_1]\}$ in $\mathcal{V}'$ so that for every $i = 1, ..., k$, the generators a_i and a'_i are conjugate to one another.

Our argument proceeds much as in Section 4. For each edge $[a_ia_{i+1}]$ (addition modulo k) there is an element $w_i \in W$ such that both $w_ia_iw_i^{-1} = a'_i$ and $w_i\alpha_{i+1}a_{i+1}w_i^{-1} = a'_{i+1}$ hold, where α_{i+1} is some word in the letters a_i and a_{i+1}. As in Section 4 we define geodesic words $\hat{a}_i \in w_{i-1}^{-1}w_i \in C(a_i)$ and note that $\prod_{i=1}^k \alpha_i\hat{a}_i = 1$.

Now it is conceivable that $\hat{a}_i$ could contain letters which do not appear in the circuit C. (For instance, if $[a_i b]$ is an edge labeled m, where b does not lie on the circuit C, then $(a_i b)^{m/2} \in C(a_i)$, and thus this word could occur as a subword of $\hat{a}_i$.)

Suppose that the letter b occurs in $\hat{a}_i$, where b does not lie on C. Because C is assumed to be regular, b cannot be adjacent to any vertex a_j where $|j - i| > 2$. But since $\mathcal{V}$ contains neither triangles nor squares, b cannot be adjacent to any of $a_{i-2}, a_{i-1}, a_{i+1}$, or a_{i+2}. Thus, b may appear only in $\hat{a}_i$. Under free cancellation in the product of the words $\alpha_i \hat{a}_i$, any occurrence of this generator must remain. Therefore, we are able to construct a van Kampen diagram Δ for the relation $\alpha_1 \hat{a}_1 \cdots \alpha_k \hat{a}_k = 1$ which has at least one edge labeled b on its boundary, and every such edge lies within the portion of the boundary corresponding to $\hat{a}_i$. If we now construct a b-band which begins at such an occurrence of b, it must end at another occurrence of b inside $\hat{a}_i$, allowing us to construct a new (and shorter) word representing the same group element as $\hat{a}_i$, contradicting the geodesity of $\hat{a}_i$.

Therefore, no $\hat{a}_i$ contains letters which do not lie on the circuit C, and the structure of the words $\hat{a}_i$ may be determined in exactly the same manner as was done in Section 4. Therefore, we may construct a group element w_C so that $w_C a_i w_C^{-1} = a_i'$ for all $i = 1, ..., k$.

Since every vertex x in $\mathcal{V}$ lies on some regular circuit, the union of all of the regular circuits in $\mathcal{V}$ contains every vertex of $\mathcal{V}$. We now show that whenever any two regular circuits C_1 and C_2 intersect, $w_{C_1} = w_{C_2}$. This will imply the desired result.

Consider two regular circuits C_1 and C_2 in $\mathcal{V}$ which intersect. Let C_1' and C_2' be the corresponding circuits in $\mathcal{V}'$. Let w_1 and w_2 be elements of W which conjugate the vertices of C_1 and C_2 to the vertices of C_1' and C_2', respectively. Define Γ to be the set of vertices in the intersection $C_1 \cap C_2$. For any vertex $x \in \Gamma$, $w_1^{-1} w_2 \in C(x)$. Therefore, if $K = \bigcap_{x \in \Gamma} C(x)$ is trivial, we are done.

When is K not trivial? It is easy to show (using Theorem 2.3 and assuming that C_1 and C_2 are regular and that the diagram $\mathcal{V}$ has no triangles and squares) that if the word u represents an element of K, then every letter which appears in u lies in Γ. For a generator in Γ to appear in a geodesic word representing an element of K requires that Γ consist of consecutive vertices of C_1, no more than 3 in number: $1 \le |\Gamma| \le 3$. We consider the three cases for $|\Gamma|$ below.

The argument will be similar in all three cases. We will construct a "chain" of regular circuits which connects C_1 to C_2, each of which shares at least a single edge with the previous one. We then consider the product of the corresponding conjugating words for these circuits, indicating how this product forces the triviality of $w_1^{-1}w_2$.

Case 1. $|\Gamma = 2|$. Suppose $\Gamma = \{a, b\}$. There are two possibilities: $[ab]$ has label $n > 2$, and $[ab]$ has label 2. The proof is almost identical in both of these cases. Although the second of these cases is used in proving the first, the first is slightly easier to formulate and visualize than the second, so that case will be considered initially.

Subcase 1.a. The edge $[ab]$ has label $n > 2$. Here $K = C(\Gamma) = \{1, (ab)^{n/2}\}$. We need to show that $w_1^{-1}w_2 \neq (ab)^{n/2}$.

Because we assume that $\mathcal{V}$ has no junctions, removal of the vertices a and b and any edges incident either of these vertices (including $[ab]$) does not disconnect the graph underlying $\mathcal{V}$. Therefore, there is a path p from $C_1 \setminus \{a, b\}$ to $C_2 \setminus \{a, b\}$ which contains neither a nor b.

Suppose this path begins at $c_1 \in C_1 \setminus \{a, b\}$ and ends at $c_2 \in C_2 \setminus \{a, b\}$, and let q be a geodesic path in $C_1 \cup C_2$ connecting c_1 and c_2. Then concatenating p and q yields a circuit, $D = pq$. We may choose this circuit to be "almost regular" in the following manner.

For any path $p = \{[p_0p_1], [p_1p_2], ..., [p_{k-1}p_k]\}$ directed from $C_1 \setminus \{a, b\}$ to $C_2 \setminus \{a, b\}$, there is a first point at which p "leaves" C_1 (i.e., $p_i \in C_1 \setminus \{a, b\}$, $p_{i+1} \notin C_1 \setminus \{a, b\}$). Similarly there is a final point p_j at which p "enters" C_2 (i.e., $p_{j-1} \notin C_2 \setminus \{a, b\}$, $p_j \in C_2 \setminus \{a, b\}$). The path $\{[p_ip_{i+1}], ..., [p_{j-1}p_j]\}$ is also a path connecting $C_1 \setminus \{a, b\}$ and $C_2 \setminus \{a, b\}$, and we may therefore assume that $i = 0$ and $j = k$.

First consider pairs (c_1, c_2) where $c_1 \in C_1 \setminus \{a, b\}$ and $c_2 \in C_2 \setminus \{a, b\}$ for which there is a path p as above satisfying $p_0 = c_1$ and $p_k = c_2$ so that $dist_{C_1 \cup C_2}(c_1, c_2)$ is minimal among all such c_1, c_2. Among all paths p as above between such pairs of vertices, select a pair (c_1, c_2) connected by a path p as above of minimal length.

Having chosen c_1, c_2, and p, let q be a geodesic path in $C_1 \cup C_2$ from c_2 to c_1. Then $D = pq$ is a circuit. It is easily seen to be a simple circuit, as otherwise we would contradict the choice of either (c_1, c_2) or p. (As in Figure 4, p may intersect $C_1 \cup C_2$ in many places.)

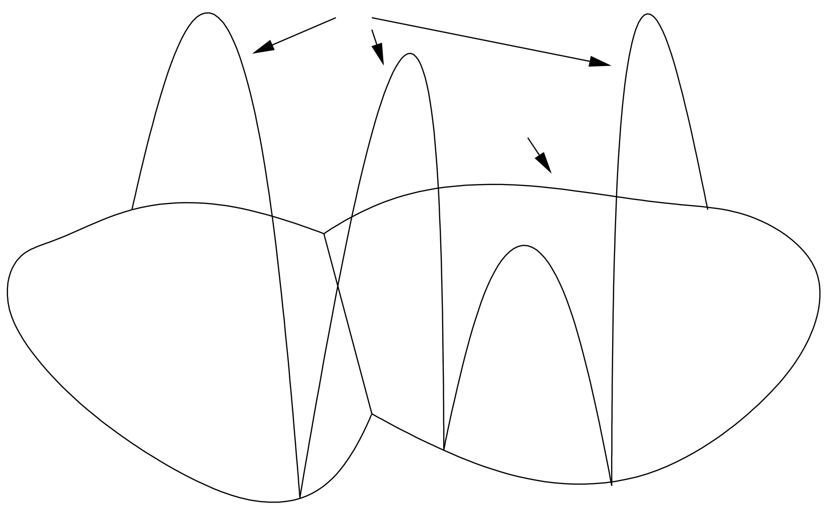

FIGURE 4

Moreover, if p_i and p_j are vertices in p and $|i - j| > 1$, the minimality of p shows that there is no edge $[p_ip_j]$ in $\mathcal{V}$, and no path $\{[p_ix], [xp_j]\}$ of length 2 for $x \notin p$. If p_i is a vertex of p and d is a vertex in q, $d \notin \{a, b, c_1, c_2\}$, the choice of (c_1, c_2) shows that there is no edge $[dp_i]$ in $\mathcal{V}$, and no path $\{[p_ix], [xd]\}$ of length 2 for $x \notin D$. Finally, if d_1 and d_2 are vertices in q (and $d_1, d_2 \notin \{a, b, c_1, c_2\}$), the regularity of C_1 and C_2 and the choice of (c_1, c_2) shows that there is no edge $[d_1d_2]$, and no path $\{[d_1x], [xd_2]\}$ of length 2, $x \notin D$.

Therefore, $D = pq$ is "almost regular." Because C_1 and C_2 are both regular, the only way in which D may fail to be regular is if there is a vertex $p_i \in p$ so that either $[p_ia]$ or $[p_ib]$ is an edge, or there is some path $\{[p_ix], [xa]\}$ or $\{[p_ix], [xb]\}$ of length 2, where $x \notin D$. (This is the point at which the present case is easier to visualize than the case in which $[ab]$ has label 2.)

Unfortunately, as Figure 5 demonstrates, this failure of regularity may be enough to allow a D to be very "non-regular."

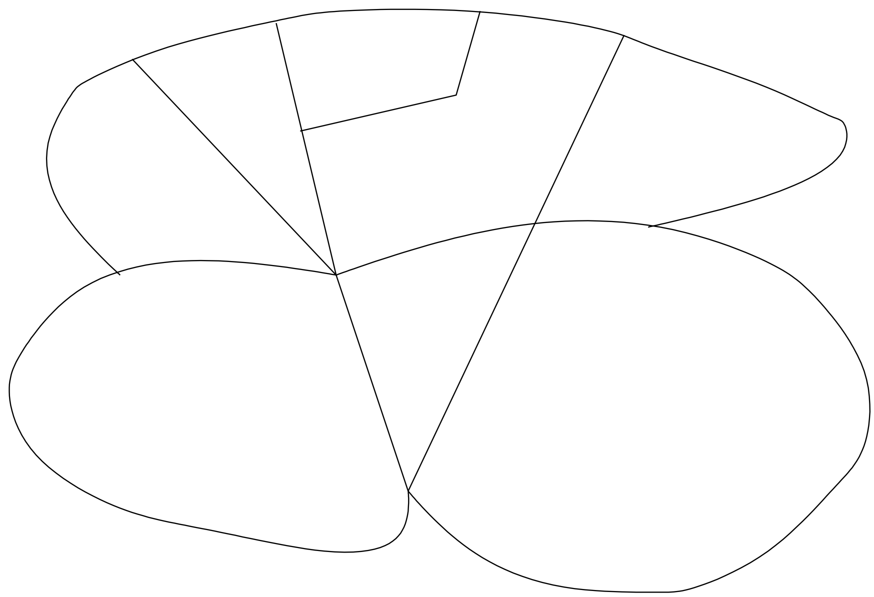

FIGURE 5. Trouble with D

However, we still have some control over D; it can be subdivided into regular circuits, each of which shares at least an edge with some other circuit in the subdivision.

For now let us assume that q contains a and not b, and that $q = q_2q_1$, where q_2 connects c_2 to a and q_1 connects a to c_1. (The arguments below are almost identical in case both a and b lie on q, and symmetric in case b lies on q and a does not.)

Let us begin with $D = pq$ as above. If D is regular, we are finished. If D is not regular, there are vertices d_1, d_2 in D so that $dist_D(d_1, d_2) \geq 1$ and either $[d_1d_2]$ is an edge or $\{[d_1x], [xd_2]\}$ is a path of length 2, $x \notin D$. We may suppose that $d_1 = a$ and $d_2 = p_i$. Let q' be a path of length ≤ 2 from p_i to a. From D we create 2 circuits, $D_1 = \{q_2, [p_0p_1], ..., [p_{i-1}p_i], q'\}$ and $D_2 = \{q_1^{-1}, [p_kp_{k-1}], ..., [p_{i+1}p_i], q'\}$. (For a path α, α^{-1} represents the path obtained by traversing α in the opposite direction.) Both of these circuits are strictly shorter than D.

If both of these circuits are regular, we are done. Otherwise, we subdivide again. For instance, if D_1 is not regular, we may find x_1, x_2 on D_1 so that $dist_{D_1}(x_1, x_2) > 1$ and there is some path q' of length ≤ 2 from x_1 to x_2 which does not lie inside D_1. Concatenating the appropriate subpaths of D_1 with this path yields two *new* circuits, D_{11} and D_{12}, say, of length strictly shorter than D_1. If these circuits are both regular, we are done. Otherwise, we subdivide again.

We continue this process until every circuit so obtained is regular. Clearly this is possible: every circuit of length 5 is regular, and it is easy to show that the average length of the circuits present after r subdivisions have been performed is a strictly decreasing function of r.

Suppose that this process of subdivision yields regular circuits $\{D_1, ..., D_m\}$, and let u_i be the conjugating word associated with the circuit D_i.

For any subset $\{D_{i_1}, ..., D_{i_r}\}$ of the collection $\{D_i\}$,

$$w_1^{-1}u_{i_1} \cdot u_{i_1}^{-1}u_{i_2} \cdots u_{i_{r-1}}^{-1}u_{i_r} \cdot u_{i_r}^{-1}w_2 \cdot w_2^{-1}w_1 = 1 \qquad (2).$$

Moreover, $u_{i_{j-1}}^{-1}u_{i_j} \in C(\Gamma_j)$, where Γ_j is the set of vertices in $D_{i_{j-1}} \cap D_{i_j}$. In order to make use of this information, we must find a "chain" of circuits D_i from C_1 to C_2.

Consider a vertex p_i in p. Before any subdivision has been performed, p_i is incident two edges, e_1 and e_2, in D. If $i \notin \{0, k\}$, $e_1 = [p_{i-1}p_i]$ and $e_2 = [p_ip_{i+1}]$. If $i = 0$, then $p_0 = c_1$, $e_1 = [dc_1]$, and $e_2 = [c_1p_1]$, where d is the second-to-last vertex in q. For $i = k$, then $p_k = c_2$, $e_1 = [p_{k-1}c_2]$, and $e_2 = [c_kd]$, where d is the second vertex in q. Clearly, both e_1 and e_2 lie in the circuit D.

Suppose that the first step in the subdivision described above involves the vertex p_i: we create two new circuits by considering a path q' of length ≤ 2 from p_i to either a or b. There are now three edges in $D_1 \cup D_2$ incident p_i. If e is the first edge in the path q', then we may assume that e_1 and e both lie on the circuit D_1 and that e and e_2 both lie on the circuit D_2, so we may proceed from e_1 to e_2 by "stepping through" the circuits D_1 and D_2.

In fact, at each subdivision described above, the number of edges incident p_i may increase. Ultimately, p_i may be incident a large number of edges. However, we claim that we may still proceed from the first edge to the last by "stepping through" the intervening circuits. Let $\{e_1, ..., e_r\}$ denote the collection of edges in $D_1 \cup \cdots \cup$

D_m which are incident the vertex p_i, where e_1 and e_r are the two edges which lie in the original circuit D.

Lemma 6.1. *There exists a subset* $\{e_{i_1} = e_1, e_{i_2}, ..., e_{i_{s-1}}, e_{i_s} = e_r\}$ *of the edges incident* p_i *and a subset* $\{D_{i_1}, ..., D_{i_{s-1}}\}$ *of the circuits in the subdivision of* D *so that for every* j, D_{i_j} *contains both* e_{i_j} *and* $e_{i_{j+1}}$.

Proof: We induct on the number of steps performed in the subdivision of D. The result is clearly true if no subdivision is necessary. Suppose that the result is true in case n subdivisions have been performed, and suppose that $n + 1$ subdivisions are necessary.

Clearly, the argument does not depend upon the regularity of the circuits involved. Therefore, we may assume that after n subdivisions have been performed on D, the result is true for p_i, relative to the $n + 1$ circuits into which D has now been divided. Therefore, there are edges $\{e_{i_1}, ..., e_{i_s}\}$ and circuits $\{D_{i_1}, ..., D_{i_{s-1}}\}$ as in the statement of the lemma. Let us consider the effect of the final subdivision, which consists of dividing a fixed circuit, $\bar{D}$, say, into two shorter circuits.

If $\bar{D} \neq D_{i_j}$ for all $j = 1, ..., s - 1$, we may retain the collection of edges and the collection of circuits listed above. (Note that even if $\bar{D} \neq D_{i_j}$ for all j, the vertex p_i, or even an edge e_i, may lie in $\bar{D}$. This does not affect our argument.)

Suppose $\bar{D} = D_{i_j}$ for some j. The circuit $\bar{D}$ is subdivided into circuits $\bar{D}_1$ and $\bar{D}_2$ by adding a path q' between vertices x_1, x_2 in $\bar{D}$. If neither x_1 nor x_2 is p_i, then one or the other of the two circuits $\bar{D}_i$ contains both edges e_{i_j} and $e_{i_{j+1}}$. Suppose $\bar{D}_1$ does. In this case we retain the collection of edges given above and merely replace $\bar{D}$ with $\bar{D}_1$.

Finally, suppose $x_1 = p_i$. Then $\bar{D}$ is subdivided into two circuits $\bar{D}_1$ and $\bar{D}_2$, and we may assume that e_{i_j} lies in $\bar{D}_1$ and $e_{i_{j+1}}$ lies in $\bar{D}_2$. Let e be the edge in q' which contains p_i. Then consider the new collections

$$\{e_{i_1}, ..., e_{i_j}, e, e_{i_{j+1}}, ..., e_{i_s}\}$$

and

$$\{D_{i_1}, ..., D_{i_{j-1}}, \bar{D}_1, \bar{D}_2, D_{i_{j+1}}, ..., D_{i_{s-1}}\}.$$

Reindex these collections, ordering them as written. These collections satisfy the statement of the lemma, since it is clear that e_{i_1} and e_{i_s} remain fixed throughout. □

We are ready to construct a "chain" of regular circuits from C_1 to C_2. To each vertex p_j in p associate the collections $\{e_{j,i_1}, ..., e_{j,i_s}\}$ and $\{D_{j,i_1}, ..., D_{j,i_{s-1}}\}$ as described above, where s depends on j.

Consider the sequence of circuits

$$(D_{1,i_1}, D_{1,i_2}, ..., D_{1,i_{s-1}}, D_{2,i_1}, ..., D_{k,i_1}, ..., D_{k,i_{s-1}}).$$

It is possible that $D_{j,i_{s-1}} = D_{j+1,i_1}$ for some j. In this case we merely omit repeated occurrences of this circuit in the sequence.

Reindex this sequence, denoting it merely by $(D_{i_1}, D_{i_2}, ..., D_{i_l})$. For every $j = 1, ..., l-1$, D_{i_j} and $D_{i_{j+1}}$ share at least an edge, and therefore, $\Gamma_j = D_{i_j} \cap D_{i_{j+1}}$ has a small centralizer. If u_j is the group element which conjugates each vertex of D_{i_j} to the corresponding vertex in $\mathcal{V}'$, then $u_j^{-1}u_{j+1} \in C(\Gamma_j)$. Let $\Gamma_0 = D_{i_1} \cap C_1$ and $\Gamma_l = D_{i_l} \cap C_2$; thus, $w_1^{-1}u_1 \in C(\Gamma_0)$ and $u_l^{-1}w_2 \in C(\Gamma_l)$.

We now apply equation (2) above to obtain

$$w_1^{-1}u_1u_1^{-1}u_2 \cdots u_{l-1}^{-1}u_lu_l^{-1}w_2w_2^{-1}w_1 = 1 \qquad (3).$$

If neither a nor b appears in any word in $C(\Gamma_j)$ for any j, then no quotient $u_j^{-1}u_{j+1}$ contains a or b; likewise, neither $w_1^{-1}u_1$ nor $u_l^{-1}w_2$ contains a or b. Therefore, from (3), the quotient $w_2^{-1}w_1$ cannot contain either a or b, and $w_1 = w_2$.

Suppose Γ_j consists of the edge $[p_ia]$ with label $n_j > 2$, so that $C(\Gamma_j) = \{1, (p_ia)^{n_j/2}\}$. Suppose $u_j^{-1}u_{j+1} = (p_ia)^{n_j/2}$. Applying the quotient map ν which identifies every vertex but p_i and a to 1 to the lefthand side of (3) yields the trivial element of the group $\nu(W)$.

However, because of the way the circuits D_{i_j} are defined, it is clear that the lefthand side of (3) may be written $v_1v_2(p_ia)^{n_j/2}v_3v_4$, where neither v_1 nor v_4 contains p_i and neither v_2 nor v_3 contains a. Thus, $\nu(v_1), \nu(v_4) \in \{1, a\}$ and $\nu(v_2), \nu(v_3) \in \{1, p_i\}$. Since

$$\nu(v_1v_2(p_ia)^{n_j/2}v_3v_4) = \nu(v_1)\nu(v_2)(p_ia)^{n_j/2}\nu(v_3)\nu(v_4),$$

we have obtained a contradiction. Thus, $u_j^{-1}u_{j+1} = 1$, and this quotient cannot contribute an occurrence of a to the product in

(3). An identical argument holds for those j for which $\Gamma_j = [p_i b]$ is an edge labeled 2.

Therefore, we need only consider the case in which Γ_j consists of an edge $[p_i a]$ (or $[p_i b]$) labeled 2. Suppose this is so. We now make use of Subcase 1.b to conclude that $u_j = u_{j+1}$. Yet again, neither a nor b can appear in $u_j^{-1} u_{j+1}$.

Subcase 1.b. $\Gamma = \{a, b\}$, and the edge $[ab]$ has label 2. Thus, $w_2^{-1} w_1 \in \{1, a, b, ab\}$.

We argue almost exactly as in the previous subcase; indeed, the only difference is the way in which we define the initial circuit D which is to be subdivided.

Because $\mathcal{V}$ contains no junctions, the removal of $st_2(a)$ and any edges incident the vertices in this set must not disconnect the graph underlying $\mathcal{V}$. Thus, there is a path p from $C_1 \setminus st_2(a)$ to $C_2 \setminus st_2(a)$ which does not contain any vertices from $st_2(a)$. As in the previous subcase, we may choose p to be a path of minimal length from c_1 in C_1 to c_2 in C_2. Concatenating p with a geodesic q between c_1 and c_2 in $C_1 \cup C_2$ yields a simple circuit D which can be subdivided exactly as before.

Lemma 6.1 still allows us to construct a sequence $(D_{i_1}, ..., D_{i_l})$ of regular circuits from C_1 to C_2 with the same properties as before. Moreover, since no vertex p_i in p is contained in the 2-star of a, there is no edge $[p_i a]$ labeled 2, for any p_i, and the final possibility in the previous subcase cannot occur. The other possibilities are handled as in Subcase 1.a.

Thus, a does not appear in any of the quotients $w_1^{-1} u_1$, $u_l^{-1} w_2$, or $u_j^{-1} u_{j+1}$, $j = 1, ..., l-1$. From equation (3), $w_2^{-1} w_1$ cannot contain an occurrence of the generator a, and $w_2^{-1} w_1 \in \{1, b\}$.

A completely symmetric argument, removing $st_2(b)$ instead of $st_2(a)$, shows that $w_2^{-1} w_1 \in \{1, a\}$, so $w_2^{-1} w_1 = 1$, and $w_1 = w_2$.

Note that we do not use Subcase 1.a at any step in this proof!

Case 2. $|\Gamma| = 3$. Suppose $\Gamma = \{a, b, c\}$, where $[ab]$ and $[bc]$ are edges labeled 2. Thus, $C(\Gamma) = \{1, b\}$, and we need only show $w_2^{-1} w_1 \neq b$.

Because $\mathcal{V}$ contains no junctions, the removal of $st_2(b)$ and any edges incident any vertex in this set does not disconnect the graph underlying $\mathcal{V}$. As in Case 1, we may construct a simple circuit D

which can be subdivided into regular circuits D_i, some sequence $(D_{i_1}, ..., D_{i_l})$ of which gives a "chain" from C_1 to C_2. Exactly as in Case 1.b, we may show that $w_2^{-1}w_1$ cannot contain the generator b, and therefore, $w_1 = w_2$.

Case 3. $|\Gamma| = 1$. Suppose $\Gamma = \{a\}$. Since $\mathcal{V}$ has no junctions, removing $st_2(a)$ and any edges incident any vertex in this set does not disconnect the graph underlying $\mathcal{V}$. As before, we construct a simple circuit D which may be subdivided into regular circuits D_i, some sequence $(D_{i_1}, ..., D_{i_l})$ of which creates a "chain" from C_1 to C_2. Here, the most important property of this sequence is that for every j, D_{i_j} and $D_{i_{j+1}}$ share an edge. Therefore, from cases 1 and 2, $u_j = u_{j+1}$ for every $j = 1, ..., l-1$. Also, since C_1 and D_{i_1} share an edge and C_2 and D_{i_l} share an edge, $w_1 = u_1$ and $w_2 = u_l$. Therefore,

$$w_1 = u_1 = u_2 = \cdots = u_{l-1} = u_l = w_2$$

and we are done.

7. No edges labeled 2

The proof of the theorem in this case will be very similar to the proof given in Section 6. We again begin with regular circuits in $\mathcal{V}$, slightly modifying what we mean for a circuit to be regular. We then prove, as before, that to each such regular circuit C there corresponds a group element $w_C \in W$ conjugating every vertex of C to the corresponding vertex in $\mathcal{V}'$. Then, as before, we piece together regular circuits in order to find a single group element w which "works" for every generator.

Suppose that $\mathcal{V}$ contains no edges with label 2. We now define a circuit $C = \{[a_1a_2, ..., [a_ka_1]\}$ to be *regular* if whenever $|i-j| \geq 2$, then there is no edge connecting a_i and a_j. (Thus, the circuit is "chord-free.")

Now suppose that $\mathcal{V}$ is an even diagram which contains no junctions so that every edge has label greater than 2. It is easy to show, almost exactly as was done in Section 5, that every vertex in $\mathcal{V}$ lies on some regular circuit. Let $C = \{[a_1a_2], ..., [a_ka_1]\}$ be a regular circuit in $\mathcal{V}$. As was done in Section 4, we may find words $\hat{a}_i \in C(a_i)$ and $\alpha_i \in W_{a_i,a_{i+1}}$ (for $i = 1, ..., k$) so that $w_{i-1}^{-1}w_i = \alpha_i\hat{a}_i$, where w_i

is a word which conjugates the group $W_{a_i,a_{i+1}}$ to the corresponding parabolic subgroup $W_{a'_i,a'_{i+1}}$ of (W,S'). Thus,

$$\alpha_1\hat{a}_1\alpha_2\hat{a}_2\cdots\alpha_k\hat{a}_k = 1 \tag{4}$$

once more. As in Section 4, we can show that α_i can be assumed to be trivial. (The argument given in that section requires only minor modification, accounting for the presence of external vertices, as defined below.)

Note that because $\mathcal{V}$ may now contain triangles and squares, it is possible that for some i any or all of the vertices a_i, a_{i+1}, and a_{i+2} may be adjacent to some vertex x which does not lie on C. That is, there could be geodesic words containing the letter x which represent elements of any one of the groups $C(a_i)$, $C(a_{i+1})$, or $C(a_{i+2})$.

Consider $\hat{a}_i$ as defined above. We call a vertex $x \in \mathcal{V}$ *external* if it does not lie on the circuit C but appears in at least one of the words $\hat{a}_i$ $(i = 1, ..., k)$. It is clear that regularity of C implies that x can occur in at most three such words. Our first goal is to show that equation (4) implies that no external vertex can appear in any $\hat{a}_i$.

Lemma 7.1. *Let C be as above. Let u_i be a geodesic word representing an element of $C(a_i)$, for $i = 1, ..., k$. Suppose $u_1u_2\cdots u_k = 1$. For $i = 1, ..., k$, no u_i contains an external vertex.*

Proof: Suppose by way of contradiction that there is some choice of words u_i for which an external vertex x appears in some u_i and so that $u_1u_2\cdots u_k = 1$. Choose such a collection of words for which the sum $\sum_{i=1}^{k}|u_i|$ is minimal.

We will show that the presence of external vertices prevents the product from being trivial. We revive the notation from Section 4: n_i will denote the label on the edge $[a_ia_{i+1}]$, $v_{i,i+1} = (a_ia_{i+1})^{n_i/2-1}a_i$, and $v_{i+1,i} = (a_{i+1}a_i)^{n_i/2-1}a_{i+1}$.

First, assume

$$u_i \in \{uv_{i+1,i}, uv_{i+1,i}a_i\},\ u_{i+1} \in \{v_{i,i+1}v, v_{i,i+1}a_{i+1}v\}$$

for some i, where u and v are any words representing elements of $C(a_i)$ and $C(a_{i+1})$, respectively. (Note that any loose occurrence of the letter a_i in u_i may be pulled to the end of that word, and any loose occurrence of a_{i+1} in u_{i+1} may be pulled to the beginning of that word, without changing the length of the words u_i and u_{i+1}. Therefore, we may assume that u and v contain no loose occurrences of a_i and a_{i+1}, respectively.)

In this case, we may reduce the product of u_i and u_{i+1}. Indeed, $u_iu_{i+1} \in \{uv, ua_iv, ua_{i+1}v, ua_ia_{i+1}v\}$. Therefore, replacing u_i by either u or ua_i and u_{i+1} by either v or $a_{i+1}v$ will yield a collection of words the sum of whose lengths is less than before, and for which some word still contains an external vertex. This contradicts the minimality of the choice of the words u_i; thus, no such reduction is possible, and the words u_i cannot have the forms described above.

Now suppose for a given i that $u_i \in \{uv_{i+1,i}, uv_{i+1,i}a_i\}$ where $\bar{u} \in C(a_i)$, and u_{i+1} does not begin with $v_{i,i+1}$. If u_{i+1} contains a loose occurrence of a_{i+1}, we may pull this occurrence to the beginning of u_{i+1}. In this case the product u_iu_{i+1} allows free cancellation of the single letter a_{i+1}, and no further cancellation or reduction. (There could be cancellation between u_i and u_{i+2}: suppose $u_{i+1} = a_{i+1}$ and the length of the circuit C is 3. If u_{i+2} begins with $v_{i,i+2}$, we may freely cancel one more letter, a_i, from each of the words u_i and u_{i+2}. There is no further reduction.) Suppose u_{i+1} does not contain a loose occurrence of a_{i+1}. If $u_{i+1} = (xa_{i+1})^m x$ for some external vertex x and some integer $m > 2$, and u_{i+2} begins with $v_{i+1,i+2}$, then we may reduce the product $u_iu_{i+1}u_{i+2}$ by removing the last letter of u_i and the first letter of u_{i+2}. Otherwise, no reduction is possible. (If $u_{i+1} = 1$, then cancellation between u_i and u_{i+2} is possible: if u_{i+2} begins with $v_{i+1,i+2}$, a single occurrence of the letter a_{i+1} may be freely cancelled from each of the words u_i and u_{i+2}. There is no further reduction.)

If u_i does not end in $v_{i+1,i}$ and $u_{i+1} \in \{v_{i,i+1}v, v_{i,i+1}a_{i+1}v\}$ where $\bar{v} \in C(a_{i+1})$, a symmetric argument shows that, at most, one letter from each word may be cancelled.

Finally, suppose that u_i does not end with $v_{i+1,i}$ and u_{i+1} does not begin with $v_{i,i+1}$. If u_i ends with a subword $(xa_i)^{m_1}x$ and u_{i+1} begins with a subword $(xa_{i+1})^{m_2}x$ for the same external vertex x, then we may freely cancel a single occurrence of the letter x. No further cancellation is possible. (As above, if u_{i+1} is trivial, there may instead be free cancellation between u_i and u_{i+2}.)

After performing all cancellations that are possible between adjacent words, we are left with a new word which represents the trivial element of W. However, this new word will not contain as a subword more than half of one of the relator words $(xy)^m$ which appears in the presentation corresponding to the system (W, S). It is a well-known fact from small cancellation theory that this is not possible (see [9], for instance), since the symmetrized presentation for (W, S) satisfies the $C'(\frac{1}{6})$ small cancellation condition.

This contradiction proves the lemma. □

By Lemma 7.1, no $\hat{a}_i$ contains external vertices. If C has length at least 5, the arguments of Section 4 provide us with a single group element w which conjugates the vertices of C appropriately. We now prove that these arguments extend to circuits of lengths 3 and 4 in case all edges in $\mathcal{V}$ have labels exceeding 2.

First, assume that C has length 4. We first show that $v_{i+1,i}v_{i-1,i}$ cannot appear as a subword of $\hat{a}_i$. Assume by way of contradiction that $v_{i+1,i}v_{i-1,i}$ does appear as a subword of $\hat{a}_i$. We now consider the quotient map ν which identifies a_{i+2} to 1 and leaves the other generators fixed. Then $\nu(\hat{a}_i) = \hat{a}_i$. Also,

$$\nu(\hat{a}_{i+1}) \in \{1, a_{i+1}, v_{i,i+1}, v_{i,i+1}a_{i+1}\},$$

$$\nu(\hat{a}_{i-1}) \in \{1, a_{i-1}, v_{i,i-1}, v_{i,i-1}a_{i-1}\},$$

$$\nu(\hat{a}_{i+2}) \in \{(a_{i-1}a_{i+1})^m a_{i-1}^{\epsilon_1}, (a_{i+1}a_{i-1})^n a_{i+1}^{\epsilon_2} | m, n \geq 0, \epsilon_j \in \{0, 1\}\}.$$

Now

$$\nu(\hat{a}_{i-1})\nu(\hat{a}_i)\nu(\hat{a}_{i+1})\nu(\hat{a}_{i+2}) = \nu(\hat{a}_{i-1}\hat{a}_i\hat{a}_{i+1}\hat{a}_{i+2}) = 1,$$

but one may show that in multiplying the words $\nu(\hat{a}_j)$ whose forms are described above, a freely reduced word is obtained which contains no subword which is more than one half of a relator word $(xy)^m$ from the presentation corresponding to (W, S). This again contradicts the properties that $(\nu(W), \{a_i, a_{i+1}, a_{i-1}\})$ enjoys due to small cancellation theory.

Therefore, $v_{i+1,i}v_{i-1,i}$ cannot appear as a subword of $\hat{a}_i$. One may now determine the form of the word w which conjugates every a_i to a_i', just as in Section 4. The argument proceeds almost exactly as before. The details are left to the reader.

One may argue in a similar fashion in order to prove the corresponding result for circuits of length 3.

We assume now that for every regular circuit C appearing in $\mathcal{V}$ there is an element $w_C \in W$ so that if x is a vertex on C, then $w_C x w_C^{-1} = x'$ for $x' \in \mathcal{V}'$.

Now an argument exactly like that given in Section 6 (indeed, easier!) shows that any time two regular circuits C and C' intersect, $w_C = w_{C'}$. Therefore, there is a single group element $w \in W$ which conjugates each generator of S to the corresponding generator of S'. W is thus strongly rigid.

8. Conclusion: implications

One of the reasons strongly rigid Coxeter groups are nice is the fact that their automorphism groups are easy to describe. Suppose (W, S) is a Coxeter system. We call an element $\alpha \in Aut(W)$ a *diagram automorphism* if α is induced by a labeled-graph automorphism of the diagram corresponding to (W, S). That is, there is an invertible map $\beta : \mathcal{V} \to \mathcal{V}$ so that $[xy]$ is an edge labeled n if and only if $[\beta(x)\beta(y)]$ is an edge labeled n and so that $\alpha(x) = \beta(x)$ for every generator x in S. Denote by $Diag(W, S)$ the group of diagram automorphisms corresponding to the system (W, S), where the S is omitted in case W is rigid.

If (W, S) is strongly rigid, then $Aut(W)$ has a very simple form: $Aut(W) \cong Inn(W) \times Diag(W)$.

The results of this paper therefore allow us to describe immediately the automorphism group of a large number of Coxeter groups.

Moreover, the "method of regular circuits" described in this paper can be applied to any even rigid Coxeter system (W, S). Given an automorphism α of W, $(W, \alpha(S))$ is another system for the group W. When W is not strongly rigid, there is no single element of W which conjugates S to $\alpha(S)$. However, the arguments given in this paper still produce a single conjugating element for each regular circuit, and by piecing together circuits which have enough "overlap," one can show that often many circuits share a conjugating element.

Therefore, even when a given even, reflection independent system (W, S) is not strongly rigid, the fact that it is reflection independent (and therefore rigid) can still be used to describe the form of an element of $Aut(W)$ in terms of the generators S. One needs to explain how the conjugating elements for different "components" of regular circuits may be chosen. A semidirect product formula, much like that which appears in case W is strongly rigid, will obtain in the more general setting. These points will be expanded upon in a later paper.

Finally, methods similar to those used in this paper turn out to be useful in other settings, too. We call a Coxeter system (W, S) a *large-type* system provided every exponent m_{ij} associated with the presentation corresponding to (W, S) is greater than 2. By applying the same "method of circuits" to the simple circuits which appear in diagrams corresponding to such systems, one can demonstrate a great deal of rigidity in the diagram $\mathcal{V}$. In this manner one can recover a number of the results recently proven by Mühlherr and Weidmann in [15]. There is some indication that the method of circuits may be able to slightly generalize this work, by considering diagrams which have no finite parabolic subgroups of rank 3 (the so-called *2-dimensional* groups), a class which contains the large-type groups.

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