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THE EXTRARESOLVABILITY HIERARCHY †

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Abstract. A space $X = (X, \mathcal{T})$ is *extraresolvable* [*strongly extraresolvable*, resp.] if there is a family $\mathcal{D}$ of $(\Delta(X))^+$-many dense subsets of X such that distinct $D_0, D_1 \in \mathcal{D}$ have $D_0 \cap D_1$ nowhere dense in X [have $|D_0 \cap D_1| < nwd(X)$, resp.]. Here, $\Delta(X) = \min\{|U| : \emptyset \neq U \in \mathcal{T}\}$, and $nwd(X) = \min\{|A| : A \subseteq X, int_X\, cl_X\, A \neq \emptyset\}$.

Solving in ZFC a generalized version of a specific problem (the case $\kappa = \omega$) posed by W. W. Comfort and S. García-Ferreira, in *Topology Proceedings*, Volume 23, Spring, the authors show for all $\kappa \geq \omega$ the existence of (many) Tychonoff spaces X with $|X| = nwd(X) = \Delta(X) = \kappa$ which are extraresolvable and $\Delta(X)$-resolvable, but not strongly extraresolvable. The arguments rest on a variant of a technique developed earlier by the second-listed author for refining (expanding) Tychonoff topologies.

1. Introduction and Historical Perspective

A space X is said to be *resolvable* [κ-*resolvable*, resp.] if X admits two [κ-many, resp.] pairwise disjoint dense subsets. (The terms were introduced by E. Hewitt [18] and J. G. Ceder [5], resp.) It is

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obvious, denoting by $\Delta(X)$ the cardinal number

$$\Delta(X) = \min\{|U| : U \neq \emptyset, U \text{ is open in } X\},$$

that if X is κ-resolvable, then $\kappa \leq \Delta(X)$. A space which is in fact $\Delta(X)$-resolvable is said to be *maximally resolvable*; it was shown by Ceder [5], generalizing earlier work of Hewitt [18], that every metric space X without isolated points, and every locally compact space X without isolated points, is maximally resolvable.

By any reasonable standard, the empty set may legitimately be called "small." Decades following the appearance of the works cited above, V. I. Malykhin [24] showed that if the requirement (on a family $\mathcal{D}$ of dense subsets of a space X) that the sets be pairwise disjoint is relaxed to a less restrictive definition of "small," then the limitation $|\mathcal{D}| \leq \Delta(X)$ might itself be relaxed. (For example, it might be required only that every pair of elements of $\mathcal{D}$ has an intersection which is finite, or countable, or of first category in X, or, if X comes equipped with a measure, of measure zero. Other definitions or interpretations of "small" may occur to the reader.) In any case, Malykhin explicitly proposed the following definition.

Definition 1.1. [24] A space X is *extraresolvable* if there is a family $\mathcal{D}$ of dense subsets of X such that $|\mathcal{D}| \geq (\Delta(X))^+$ and every two elements of $\mathcal{D}$ have an intersection which is nowhere dense in X.

(In addition, Malykhin [23] elsewhere has defined and studied generalizations of the concept of resolvability in a different direction. For example, the dense sets which constitute a partition of the given space might be required to be Baire sets, or Borel sets, or to be otherwise restricted. We mention [23] for the reader's interest only; the dense sets we define and consider in this paper are not required to satisfy additional topological constraints.)

While investigating the concept of extraresolvability and establishing the existence of several classes of extraresolvable spaces, W. W. Comfort and S. García-Ferreira introduced a stronger property, as follows.

Definition 1.2. [9] A space X is *strongly extraresolvable* if there is a family $\mathcal{D}$ of dense subsets of X such that

(a) $|\mathcal{D}| \geq (\Delta(X))^+$, and
(b) distinct $D_0, D_1 \in \mathcal{D}$ satisfy $|D_0 \cap D_1| < nwd(X)$.

Here $nwd(X)$, the *nowhere density number* of X, is the least cardinal of the form $|A|$ with $A \subseteq X$ and A not nowhere dense in X. It is easily seen that $nwd(X)$ coincides with the so-called *open density* number of X, sometimes denoted $od(X)$ (see [10]); this by definition is the least cardinal of the form $d(U)$ with $\emptyset \neq U \subseteq X$, U open in X.

It is clear for any space X that $|X| \geq \Delta(X) \geq nwd(X)$. Thus, if X satisfies $|X| = nwd(X) = \kappa$, then $\Delta(X) = \kappa$. In our search for spaces X with various (extra)resolvability properties, we arrange where possible in addition that $|X| = nwd(X)$. This challenge makes inoperable the naive device of finding, for example, small spaces X' with $\Delta(X') = nwd(X')$ and reverting then simply to $X := X' \times \kappa$; see Theorem 2.1 below for a brief comparison (taken from [8]) of the resolvability properties and the cardinal functions Δ and nwd of a space X' with those of $X = X' \times \kappa$.

The foregoing paragraphs suggest these questions.

Question 1.3. (1) *Is every ω-resolvable space maximally resolvable?*

(2) *Is every maximally resolvable space (strongly) extraresolvable?*

(3) *Is every extraresolvable space strongly extraresolvable?*

Discussion 1.4. Continuing this historical overview, we review the status of these three questions.

Question 1.3(1) dates from 1967 [6]. Examples responding in the negative have been given by A. G. El′kin [13], Malykhin [22], F. W. Eckertson [12], and W. Hu [19] and [20], but each of these examples has been considered to be mildly unsatisfactory in that each is either non-Tychonoff or is a "consistent example," i.e., a space defined when ZFC is augmented by some additional axiom(s). The existence of a Tychonoff example in ZFC has recently been given by I. Juhász, L. Soukup, and Z. Szentmiklóssy [21].

Concerning 1.3(2) and 1.3(3), it is noted in [8] that if X is an arbitrary space which is extraresolvable [maximally resolvable, resp.] and A is a discrete space with $|A| > (\Delta(X))^+$, then $Y := X \times A$ is an extraresolvable [maximally resolvable, resp.] space which is not strongly extraresolvable. (With X and A chosen to be topological groups, Y itself becomes a topological group which witnesses the failure of the implications suggested in 1.3(2) and 1.3(3).)

That simple construction sheds no light on questions 1.3(2) and 1.3(3) for countable spaces. The question of the existence of a countable extraresolvable space which is not strongly extraresolvable, left unsettled in [8], was posed explicitly by the authors there. Subsequently, García-Ferreira and R. A. González-Silva [16] established the existence of such a space. (We comment on the relation between their construction and ours in Remark 3.11.)

Examples responding to Question 1.3(2) in the negative exist in profusion. For example, the real line $\mathbb{R}$ in its usual topology is maximally resolvable [5], but it is easily seen that $\mathbb{R}$ admits no dense family $\mathcal{D}$ of cardinality $\mathfrak{c}^+$ whose pairwise intersections are nowhere dense. (Each member D of a dense family $\mathcal{D} \subseteq \mathcal{P}(\mathbb{R})$ contains a countable dense subset C_D, and since $|[\mathbb{R}]^\omega| = \mathfrak{c}$ the map $D \to C_D$ from $\mathcal{D}$ to $[\mathbb{R}]^\omega$ cannot be injective if $|\mathcal{D}| > \mathfrak{c}$.) Similar reasoning is advanced in [8, 2.4] to show that an infinite compact Hausdorff topological group, while necessarily maximally resolvable, cannot be extraresolvable; this result is strengthened in [17, 1.5]: Every space X such that $\Delta(X) = 2^{w(X)}$ is maximally resolvable but not extraresolvable.

In all of the (uncountable) examples cited above, however, the relation $|X| = nwd(X)$ fails. Our contribution to questions 1.3(2) and 1.3(3) is given in Theorem 3.10: For every $\kappa \geq \omega$ there is a space $X = X(\kappa)$, simultaneously maximally resolvable and extraresolvable, such that $|X| = nwd(X) = \kappa$ and X is not strongly extraresolvable.

When the results of this paper were presented at the Lubbock, Texas, meeting in March 2003, we raised explicitly Question 1.3(2) above in this form: Is there a maximally resolvable Tychonoff space X with $|X| = nwd(X)$ such that X is not extraresolvable? In response shortly thereafter, Michael Hrušak indicated in conversation and e-mail the availability of such a countable space in any model satisfying $\mathfrak{i} = \mathfrak{c}$; see, for example, [25] or [2] for the definition of the "small cardinal" $\mathfrak{i}$ and its relation to other cardinals in the Cichoń diagram. Subsequently, Juhász, Soukup and Szentmiklóssy [21] developed new arguments which prove in ZFC for every $\kappa \geq \omega$ the existence of a maximally resolvable, 0-dimensional c.c.c. space X with $|X| = \Delta(X) = \kappa$ such that X is not extraresolvable.

Notation 1.5. We use η, ξ, and α to denote ordinals, and κ and λ for cardinals (usually infinite). Of the popular equivalent notations $\omega = \aleph_0 = \mathbb{N}$, we use the first; and similarly with $\omega^+ = \aleph_1 = \omega_1$.

For X a set and κ a cardinal, we write $[X]^\kappa = \{A \subseteq X : |A| = \kappa\}$. The symbols $[X]^{\leq\kappa}$ and $[X]^{<\kappa}$ are defined analogously.

Definition 1.6. A family of dense subsets of a space X is called simply a *dense family.* A dense family which witnesses the maximal resolvability of a space X is called a *maximally resolvable family.* Similarly, we use the expressions *extraresolvable dense family* and *strongly extraresolvable dense family.* Note that a maximally resolvable family $\mathcal{D}$ on X satisfies $|\mathcal{D}| = \Delta(X)$, while a (strongly) extraresolvable family $\mathcal{D}$ satisfies $|\mathcal{D}| \geq (\Delta(X))^+$.

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2. Some Not Strongly Extraresolvable Spaces.

In order to find for $\kappa \geq \omega$ an extraresolvable, not strongly extraresolvable Tychonoff space of cardinality κ, the following observations from [8, 4.1] are helpful. In that paper and here a cardinal number, when treated as a topological space, is understood to carry the discrete topology.

Theorem 2.1. *Let $\kappa \geq \omega$ and let X be a space. Then*

(a) *$\Delta(X) = \Delta(X \times \kappa)$ and $nwd(X) = nwd(X \times \kappa)$;*

(b) *X is extraresolvable if and only if $X \times \kappa$ is extraresolvable;*

(c) *if X is not strongly extraresolvable then $X \times \kappa$ is not strongly extraresolvable; and*

(d) *if $\kappa > (\Delta(X))^+$ then $X \times \kappa$ is not strongly extraresolvable.*

Corollary 2.2. *If there is a Tychonoff space X such that $|X| = \lambda$ and X is extraresolvable and not strongly extraresolvable, then for every $\kappa \geq \lambda$ the space $X(\kappa) := X \times \kappa$ is a Tychonoff space such that $|X(\kappa)| = \kappa$, $X(\kappa)$ is extraresolvable, and $X(\kappa)$ is not strongly extraresolvable.*

Of course, the various spaces Y of Theorem 2.1 and Corollary 2.2 will fail in general to satisfy the additional condition $|Y| = nwd(Y)$.

The facts that the space $\mathbb{Q}$ is maximally resolvable, is extraresolvable, and is strongly extraresolvable were proved in [5], in [17], and in [8, 2.3], respectively. (The extraresolvability of $\mathbb{Q}$ was achieved in [17] as a consequence of a general result concerning "countable nowhere dense tightness" [see [4]].)

Specialized to $\mathbb{Q}$, the argument of [7, 2.2] provides a strongly extraresolvable family $\mathcal{D} \subseteq \mathcal{P}(\mathbb{Q})$ such that $|\mathcal{D}| = \omega^+$; and the argument of [17, 2.2] provides an extraresolvable family $\mathcal{D} \subseteq \mathcal{P}(\mathbb{Q})$ such that $|\mathcal{D}| = \mathfrak{c}$. Here, in Theorem 2.3, we arrange both features simultaneously.

As usual, we say that a space is *dense-in-itself* if it has no isolated points. A π-*base* for a space X is a family $\mathcal{B}$ of nonempty open subsets such that each nonempty open $U \subseteq X$ contains an element of $\mathcal{B}$. The π-*weight* of X, denoted $\pi w(X)$, is $\min\{|\mathcal{B}| : \mathcal{B}$ is a π-base for $X\}$.

Theorem 2.3. *Let X be a countable dense-in-itself Hausdorff space with a countable π-base. Then there is a dense family $\mathcal{D} = \{D_\eta : \eta < \mathfrak{c}\} \subseteq \mathcal{P}(X)$ such that $|D_{\eta_0} \cap D_{\eta_1}| < \omega$ whenever $\eta_0 < \eta_1 < \mathfrak{c}$.*

Proof: Let $\{B_n : n < \omega\}$ be a π-base and, using the so-called Disjoint Refinement Lemma (see, for example, [11, 7.5]), choose pairwise disjoint sets $A_n \in [B_n]^\omega$; write $A_n = \{a_{n,k} : k < \omega\}$ (faithfully indexed).

Now let $\{f_\eta : \eta < \mathfrak{c}\}$ be a set of $\mathfrak{c}$-many injections from ω into ω such that if $\eta_0 < \eta_1 < \mathfrak{c}$ then $|\{n < \omega : f_{\eta_0}(n) = f_{\eta_1}(n)\}| < \omega$. (To find such functions f_η it is enough to take a family $\{N_\eta : \eta < \mathfrak{c}\}$ of infinite subsets of ω such that if $\eta_0 < \eta_1 < \mathfrak{c}$ then $|N_{\eta_0} \cap N_{\eta_1}| < \omega$, and to choose for $\eta < \mathfrak{c}$ an injection f_η from ω into N_η; then for $\eta_0 < \eta_1 < \mathfrak{c}$ the set where f_{η_0} and f_{η_1} agree is mapped injectively into the finite set $N_{\eta_0} \cap N_{\eta_1}$, and hence is itself finite.) Then with $D_\eta := \{a_{n,f_\eta(n)} : n < \omega\}$, the family $\mathcal{D} = \{D_\eta : \eta < \mathfrak{c}\}$ is as required. $\square$

Remark 2.4. A space X as in Theorem 2.3 satisfies $|X| = nwd(X) = \pi w(X) = \omega$. Evidently the same argument shows more generally that if a space X satisfies $\pi w(X) \leq \Delta(X) = nwd(X) = \kappa$ and if there is family $\mathcal{N} \subseteq [\kappa]^\kappa$ such that $|\mathcal{N}| = \lambda$ and the intersection of any two elements of $\mathcal{N}$ has cardinality less than κ, then X admits a dense family $\mathcal{D} \subseteq \mathcal{P}(X)$ such that $|\mathcal{D}| = \lambda$ and the intersection of any two elements of $\mathcal{D}$ has cardinality less than κ. Always such $\mathcal{N}$ exists with $|\mathcal{N}| = \kappa^+$; therefore, as shown in [8, 2.3], every space X such that $\omega \leq \pi w(X) \leq \Delta(X) = nwd(X)$ is strongly extraresolvable.

The space $\mathbb{Q} \times \omega$, since it is homeomorphic to $\mathbb{Q}$ itself, is extraresolvable (and strongly extraresolvable); from Theorem 2.1(b) and 2.1(d), it then follows that for each cardinal $\kappa > \omega^+$ the space $\mathbb{Q} \times \kappa$ is extraresolvable but not strongly extraresolvable.

The behavior of the "intermediate space" $\mathbb{Q} \times \omega^+$ is determined by the following result.

Theorem 2.5. *The space $\mathbb{Q} \times \omega^+$ is strongly extraresolvable.*

Proof: According to the special case $\alpha = 0$ of Theorem 1 of [14], there is a family $\mathcal{F} = \{f_\eta : \eta < \omega^+\}$ of functions from ω^+ to ω such that no two distinct elements of $\mathcal{F}$ agree on an infinite set. (In symbols: $\eta_0 < \eta_1 < \omega^+ \Rightarrow |\{\xi < \omega^+ : f_{\eta_0}(\xi) = f_{\eta_1}(\xi)\}| < \omega$.) (This is the case $\kappa = \omega$ of a more general result exposed by N. H. Williams [26, 1.2.5]; in that exposition, the notation $\delta(\mathcal{F}) < \kappa$ should be replaced by $\delta(\mathcal{F}) \leq \kappa$.)

Now fix a countable dense partition $\{E(n) : n < \omega\}$ of $\mathbb{Q}$. Each $E(n)$ is homeomorphic to $\mathbb{Q}$, so (using $\omega^+ \leq \mathfrak{c}$) there is, by Theorem 2.3, a dense family $\{E(n, \eta) : \eta < \omega^+\} \subseteq \mathcal{P}(E(n))$ such that $|E(n, \eta_0) \cap E(n, \eta_1)| < \omega$ whenever $\eta_0 < \eta_1 < \omega^+$. We note that if $n_0 < n_1 < \omega$ then every two sets of the form $E(n_0, \eta_0)$, $E(n_1, \eta_1)$ (with η_i arbitrary, $\eta_i < \omega^+$) satisfy

$$E(n_0, \eta_0) \cap E(n_1, \eta_1) \subseteq E(n_0) \cap E(n_1) = \emptyset.$$

Now for $\eta < \omega^+$, $\xi < \omega^+$, define $D_\eta(\xi) := E(f_\eta(\xi), \eta)$, and set $D_\eta := \cup\{D_\eta(\xi) \times \{\xi\} : \xi < \omega^+\}$. We show that $\{D_\eta : \eta < \omega^+\}$ is a strongly extraresolvable family for $\mathbb{Q} \times \omega^+$. Note first that for $\eta_0 < \eta_1 < \omega^+$ the relation $(D_{\eta_0} \cap D_{\eta_1}) \cap (\mathbb{Q} \times \{\xi\}) \neq \emptyset$ can hold (at most) only for those $\xi < \omega^+$ such that $f_{\eta_0}(\xi) = f_{\eta_1}(\xi)$. There are finitely many such $\xi < \omega^+$, and for each such ξ, say with $f_{\eta_0}(\xi) = f_{\eta_1}(\xi) = n < \omega$, we have

$$\begin{aligned} (D_{\eta_0} \cap D_{\eta_1}) \cap (\mathbb{Q} \times \{\xi\}) &= (D_{\eta_0}(\xi) \cap D_{\eta_1}(\xi)) \times \{\xi\} \\ &= (E(f_{\eta_0}(\xi), \eta_0) \cap E(f_{\eta_1}(\xi), \eta_1)) \times \{\xi\} \\ &= (E(n, \eta_0) \cap E(n, \eta_1)) \times \{\xi\}, \end{aligned}$$

a finite set. $\square$

Remark 2.6. (a) We have included a proof of Theorem 2.5 here for the following reason. Corollary 2.2 reduces the question of the existence of an extraresolvable, not strongly extraresolvable Tychonoff space (of arbitrary cardinality $\lambda \geq \kappa$) to the case $\lambda = \kappa$. If the extraresolvable space $\mathbb{Q} \times \omega^+$ were not strongly extraresolvable, then our construction below (of such spaces X of pre-assigned cardinality, further with $|X| = nwd(X)$) would lose some of its interest and urgency. We note that Theorem 2.5 has already been achieved by García-Ferreira and González-Silva [16, 2.3] as a consequence of a more general result proved by recursive methods.

(b) It is well known (see, for example, [26] or [11, 12.19]) that for $\kappa \geq \omega$ there is a family of κ^+-many functions from κ to κ, every two of which agree only on a set of cardinality less than κ. The proof proceeds by an appeal to Zorn's Lemma: clearly there are such families of cardinality κ, and it is easily shown that such a family (of cardinality κ) cannot be maximal. It is interesting to note, in contrast, that the existence of an uncountable family of functions from ω^+ to ω as in the first paragraph of the proof of Theorem 2.5 is necessarily of a different flavor. Indeed, if $f_n : \omega^+ \to \omega$ is defined for $n < \omega$ by the rule $f_n(\xi) = n$ (all $\xi < \omega^+$) then the countable family $\mathcal{F} = \{f_n : n < \omega\}$ is already maximal with respect to the property: distinct elements of $\mathcal{F}$ agree (at most) finitely often.

(c) An analysis of the proof of Theorem 2.5 indicates its basic components: (1) the existence of ω^+-many functions from ω^+ to ω, each pair agreeing at most finitely often, and (2) the fact that $\mathbb{Q}$ is strongly extraresolvable; and hence, (3) there is a family $\mathcal{D} \subseteq \mathcal{P}(\omega \times \omega^+)$ of size ω^+ such that each $D \in \mathcal{D}$ and $\xi < \omega^+$ has $|D \cap (\omega \times \{\xi\})| = \omega$, but distinct $D, E \in \mathcal{D}$ satisfy $|D \cap E| < \omega$. Evidently, the proof of Theorem 2.5 yields the following statement.

Corollary 2.7. *For $\kappa \geq \omega$ the following three conditions are equivalent:*

(1) *There exists a family of κ-many functions from ω^+ to ω, each pair agreeing at most finitely often;*

(2) *$\mathbb{Q} \times \omega^+$ admits a strongly extraresolvable family of cardinality κ;*

(3) *there is a family $\mathcal{D} \subseteq \mathcal{P}(\omega \times \omega^+)$ of size κ such that each $D \in \mathcal{D}$ and $\xi < \omega^+$ satisfy $|D \cap (\omega \times \{\xi\})| = \omega$, but distinct $D, E \in \mathcal{D}$ satisfy $|D \cap E| < \omega$.*

Discussion 2.8. When the conditions of Corollary 2.7 are satisfied, the set ω^+ admits κ-many subsets of cardinality ω^+, each pair with finite intersection. It is interesting to inquire, though logically inessential to our work, whether the cardinal $\kappa = \omega^+$ is best possible. One may ask, more specifically: Is there such an almost disjoint family $\mathcal{S} \subseteq \mathcal{P}(\omega^+)$ such that $|\mathcal{S}| = \mathfrak{c}$? $|\mathcal{S}| = \aleph_2$? Of course these questions are settled by CH, affirmatively for $|\mathcal{S}| = \mathfrak{c} = \omega^+$ by the result cited from [14] and [26] and negatively for $|\mathcal{S}| = \aleph_2 = \mathfrak{c}^+$ (since each infinite subset of $\mathfrak{c}$ contains a countably infinite subset, and $|[\mathfrak{c}]^\omega| = \mathfrak{c}$). In contrast, J. E. Baumgartner [3, §6] has shown that the existence of an almost disjoint family $\mathcal{S} \subseteq \mathcal{P}(\omega^+)$ with $|\mathcal{S}| = \aleph_2$ is independent of the system ZFC + $\neg$CH. (We are indebted to István Juhász for reference in this context to [3].)

3. The $\mathcal{KID}$ Expansion.

Now we turn to our answers to questions 1.3(2) and (3). The argument depends on the availability of the $\mathcal{KID}$ expansion [19], [20], defined on suitable spaces as follows. (Here and later, the word *clopen* means open-and-closed.)

Definition 3.1. Let κ be an infinite cardinal, and let $X = (X, \mathcal{T})$ be a space. Let $\mathcal{D} = \{D_\eta : \eta < \kappa\} \subseteq \mathcal{P}(X)$ and $\mathcal{K} = \{K_\alpha : \alpha < 2^\kappa\} \subseteq \mathcal{P}(X)$, and let $\mathcal{I} = \{A_\alpha : \alpha < 2^\kappa\} \subseteq \mathcal{P}(\kappa)$ be a κ-independent family on κ (in the sense that if $\mathcal{F}_0, \mathcal{F}_1 \in [\mathcal{I}]^{<\omega}$ with $\mathcal{F}_0 \cap \mathcal{F}_1 = \emptyset$ then $|\cap\{A : A \in \mathcal{F}_0\} \cap \cap\{\kappa \backslash A : A \in \mathcal{F}_1\}| = \kappa$). For $A_\alpha \in \mathcal{I}$, let $U_\alpha := (\cup\{D_\eta : \eta \in A_\alpha\}) \backslash K_\alpha$.

Then $\mathcal{T}_{\mathcal{KID}}$, the $\mathcal{KID}$ *expansion* of $\mathcal{T}$, is the smallest topology on X which contains $\mathcal{T}$ and which contains also each set U_α ($\alpha < 2^\kappa$) as a clopen subset.

Notation 3.2. Given κ, X, $\mathcal{K}$, $\mathcal{I}$, and $\mathcal{D}$ as above, for $S \subseteq \kappa$ we write $X(S) := \cup\{D_\eta : \eta \in S\}$.

It is clear that if $S = A_\alpha \in \mathcal{I}$ and $K_\alpha = \emptyset$, then $X(S) = U_\alpha \subseteq X$.

Remark 3.3. (a) Concerning the hypothesized enumeration in Definition 3.1 of the families $\mathcal{K}$, $\mathcal{I}$, and $\mathcal{D}$, it is understood that the indexing $\mathcal{I} = \{A_\alpha : \alpha < 2^\kappa\}$ is faithful; in general, no such restriction is imposed upon the indexings $\mathcal{K} = \{K_\alpha : \alpha < 2^\kappa\}$ and $\mathcal{D} = \{D_\eta : \eta < \kappa\}$.

(b) That every infinite cardinal κ admits a κ-independent family $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ with $|\mathcal{I}| = 2^\kappa$ is well known (see [10] for several relevant references). In what follows, we need the fact that $\mathcal{I}$ may be chosen so that $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$ with $\mathcal{I}_0 \cap \mathcal{I}_1 = \emptyset$, $|\mathcal{I}_i| = 2^\kappa$, and so that if $S_0, S_1 \in [\kappa]^{<\kappa}$ with $S_0 \cap S_1 = \emptyset$ then there is $A \in \mathcal{I}_1$ such that either $S_0 \subseteq A \subseteq \kappa \backslash S_1$ or $S_1 \subseteq A \subseteq \kappa \backslash S_0$. (Such a family $\mathcal{I}_1$ we here call *small-set separating*.) The existence of such families $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1 \subseteq \mathcal{P}(\kappa)$ is given by the following argument, a variant of a trick devised by Eckertson [12]; see also our work [10]. Begin with any κ-independent family $\mathcal{J}$ on κ such that $|\mathcal{J}| = 2^\kappa$, and write $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1$ with $|\mathcal{J}_i| = 2^\kappa$ and $\mathcal{J}_0 \cap \mathcal{J}_1 = \emptyset$, say with $\mathcal{J}_1 = \{A_\alpha : \alpha < 2^\kappa\}$. Then using $|[\kappa]^{<\kappa}| \leq 2^\kappa$, let $\{\langle S_\alpha, T_\alpha \rangle : \alpha < 2^\kappa\}$ list all pairs of disjoint members of $[\kappa]^{<\kappa}$ with each such pair appearing infinitely often, and define $\mathcal{I}_0 := \mathcal{J}_0$ and $\mathcal{I}_1 := \{(A_\alpha \backslash S_\alpha) \cup T_\alpha : \alpha < 2^\kappa\}$.

(c) The preceding presentation of the $\mathcal{KID}$ expansion $\mathcal{T}_{\mathcal{KID}}$ of $\mathcal{T}$ is based upon the exposition given in Chapter 3 of [19], a work prepared by the present second-listed co-author under the guidance of the first-listed co-author. This method of expansion, which has variants in several contexts, has been exploited subsequently in [20].

(d) If $(X, \mathcal{T})$ is a Tychonoff space then for each triple $(\mathcal{K}, \mathcal{I}, \mathcal{D})$ of families as above, the expansion $\mathcal{T}_{\mathcal{KID}}$ is also Tychonoff. Indeed, $\mathcal{T}_{\mathcal{KID}}$ is the topology induced on X by the point-separating set of functions $C(X, \mathcal{T}) \cup \{\chi_\alpha : \alpha < 2^\kappa\}$, with χ_α the characteristic function of the set $U_\alpha \subseteq X$.

Our goal is to show that for suitably restricted Tychonoff spaces $(X, \mathcal{T})$ with $|X| = nwd(X) = \kappa$, there are families $\mathcal{K}, \mathcal{D} \subseteq \mathcal{P}(X)$ and $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ as in Definition 3.1 such that $\mathcal{T}_{\mathcal{KID}}$ is maximally resolvable and extraresolvable but not strongly extraresolvable, with $nwd(X, \mathcal{T}_{\mathcal{KID}}) = \kappa$. Theorem 3.6 shows the passage from $(X, \mathcal{T})$ to $(X, \mathcal{T}_{\mathcal{KID}})$; then, in Theorem 3.7 and Lemma 3.9, we show the existence of $(X, \mathcal{T})$ and of $\mathcal{K}$, $\mathcal{I}$, and $\mathcal{D}$ as required. In this connection, see also [7, 4.2(b)].

Definition 3.4. A space X is *open-hereditarily irresolvable* if each nonempty open subset of X is irresolvable; such a space X is called an OHI space.

Definition 3.5. A subset S of κ is *dense for* an independent family $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ if from $\mathcal{F}_0, \mathcal{F}_1 \in [\mathcal{I}]^{<\omega}$ with $\mathcal{F}_0 \cap \mathcal{F}_1 = \emptyset$ it follows that

$$|S \cap (\cap\{A : A \in \mathcal{F}_0\}) \cap (\cap\{\kappa \backslash A : A \in \mathcal{F}_1\})| \geq \omega.$$

Theorem 3.6. *Let κ be an infinite cardinal, and let $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1 = \{A_\alpha : \alpha < 2^\kappa\}$ be chosen so that $\mathcal{I}$ is κ-independent and $|\mathcal{I}_0| = |\mathcal{I}_1| = 2^\kappa$ with $\mathcal{I}_1$ small-set separating.*

Let $X = (X, \mathcal{T})$ be a Tychonoff space such that $|X| = \kappa$ with a maximally resolvable family $\mathcal{D} = \{D_\eta : \eta < \kappa\}$ such that $X = \cup\mathcal{K}$ and with each D_η an OHI space. Let $\mathcal{K} = \{K_\alpha : \alpha < 2^\kappa\}$ (repetitions allowed) be the set of all subsets K of X such that $K \cap D_\eta$ is nowhere dense in D_η for each $\eta < \kappa$; and further with $K_\alpha = \emptyset$, when $A_\alpha \in \mathcal{I}_1$.

Then the following conditions hold:

(a) *$(X, \mathcal{T}_{\mathcal{KID}})$ is a Tychonoff space with $nwd(X, \mathcal{T}_{\mathcal{KID}}) \geq nwd(X, \mathcal{T})$;*

(b) *if $S \subseteq \kappa$ and S is dense for $\mathcal{I}$, then $X(S)$ is dense in $(X, \mathcal{T}_{\mathcal{KID}})$;*

(c) *if $S \in [\kappa]^{<\kappa}$, then $X(S)$ is nowhere dense in $(X, \mathcal{T}_{\mathcal{KID}})$; and*

(d) *if $nwd(X, \mathcal{T}) = \kappa$, then $(X, \mathcal{T}_{\mathcal{KID}})$ is not strongly extraresolvable.*

Proof: (a) That $\mathcal{T}_{\mathcal{KID}}$ is a Tychonoff topology on the set X is noted in Remark 3.3(d). Now let W be a nonempty $\mathcal{T}_{\mathcal{KID}}$-basic set, say $W = U \cap \cap_{\alpha\in F} V_\alpha$ with $\emptyset \neq U \in \mathcal{T}$, with $F \in [2^\kappa]^{<\omega}$ and with, for $\alpha \in F$, either

$$V_\alpha = U_\alpha := \cup\{D_\eta : \eta \in A_\alpha\}\backslash K_\alpha \text{ or } V_\alpha = X\backslash U_\alpha.$$

Since $\mathcal{I}$ is a κ-independent family on κ, the set
$N(W) := \cap\{A_\alpha : \alpha \in F, V_\alpha = U_\alpha\} \cap \cap\{\kappa\backslash A_\alpha : \alpha \in F, V_\alpha = X\backslash U_\alpha\}$ satisfies $|N(W)| = \kappa$, and for each $\eta \in N(W)$ we have

$$D_\eta \backslash \cup_{\alpha\in F} K_\alpha \subseteq \cap_{\alpha\in F} V_\alpha. \qquad (1)$$

The union of finitely many nowhere dense sets is nowhere dense, so the left hand side of (1) is $\mathcal{T}$-dense in D_η and from (1) it follows that $W = U \cap \cap_{\alpha\in F} V_\alpha$ is $\mathcal{T}$-dense in U. Any $\mathcal{T}_{\mathcal{KID}}$-dense subset of W, being $\mathcal{T}$-dense in W, is then $\mathcal{T}$-dense in U, and (a) follows.

(b) Essentially the same argument applies. Indeed, for each nonempty $\mathcal{T}_{\mathcal{KID}}$-basic open set $W = U \cap \cap_{\alpha\in F} V_\alpha$ as above, we have, choosing $\eta \in S \cap N(W)$, the relations

$$X(S) \cap W \supseteq (U \cap D_\eta) \backslash \cup_{\alpha \in F} K_\alpha \neq \emptyset.$$

(c) The set $\kappa \backslash S$ is dense for $\mathcal{I}$, and hence the set $X(\kappa \backslash S) = X \backslash X(S)$ is dense in $(X, \mathcal{T}_{\mathcal{KID}})$ by (b). Further, each set of the form $X(S)$ with $S \in [\kappa]^{<\kappa}$ is closed in $(X, \mathcal{T}_{\mathcal{KID}})$. To see this, let $x \in X \backslash X(S)$ with $S \in [\kappa]^{<\kappa}$, say $x \in D_\eta$ with $\eta \in \kappa \backslash S$. Choose $T \in [\kappa]^{<\kappa}$ such that $\eta \in T \subseteq \kappa \backslash S$. (For example, take $T = \{\eta\}$.) There is $\alpha \in \mathcal{I}_1$ such that either $S \subseteq A_\alpha \subseteq \kappa \backslash T$ or $T \subseteq A_\alpha \subseteq \kappa \backslash S$, and since $K_\alpha = \emptyset$ we have $X(A_\alpha) = U_\alpha$; then U_α is a $\mathcal{T}_{\mathcal{KID}}$-clopen set satisfying either $X(S) \subseteq U_\alpha$ with $x \notin U_\alpha$ or $x \in X(T) \subseteq U_\alpha \subseteq X \backslash X(S)$.

(d) From $|X| = nwd(X, \mathcal{T}) = \kappa$ and (c) we have $nwd(X, \mathcal{T}_{\mathcal{KID}}) = \kappa$, so if (d) fails there is a $\mathcal{T}_{\mathcal{KID}}$-strongly extraresolvable family $\mathcal{E} = \{E_\xi : \xi < \kappa^+\}$. (Topological references in the rest of this proof are to the original topology $\mathcal{T}$ on X [and the topology which $\mathcal{T}$ induces on the sets $D_\eta \subseteq X$].) We claim that there is $\xi < \kappa^+$ such that $int_{D_\eta}(E_\xi \cap D_\eta) = \emptyset$ for all $\eta < \kappa$. (Actually, this holds simultaneously for all but κ-many $\xi < \kappa^+$, but verification for a single ξ will suffice for our purposes.) If the claim fails then for each $\xi < \kappa^+$ there is $\eta(\xi) < \kappa$ such that $int_{D_{\eta(\xi)}}(E_\xi \cap D_{\eta(\xi)}) \neq \emptyset$, so there are $A \in [\kappa^+]^{\kappa^+}$ and $\eta < \kappa$ such that $int_{D_\eta}(E_\xi \cap D_\eta) \neq \emptyset$ for all $\xi \in A$. For $\xi_0 < \xi_1 < \kappa^+$ we have

$$\begin{gathered} int_{D_\eta}(E_{\xi_0} \cap D_\eta) \cap int_{D_\eta}(E_{\xi_1} \cap D_\eta) = \\ int_{D_\eta}(E_{\xi_0} \cap E_{\xi_1} \cap D_\eta) \subseteq E_{\xi_0} \cap E_{\xi_1}. \end{gathered}$$

Since $|E_{\xi_0} \cap E_{\xi_1}| < nwd(X, \mathcal{T})$ while D_η is dense, we conclude that the family $\{int_{D_\eta}(E_\xi \cap D_\eta) : \xi \in A\}$ is a family of κ^+-many of pairwise disjoint nonempty (open) subsets of the set D_η, contradicting the fact that $D_\eta \subseteq X$ with $|X| = \kappa$. This contradiction shows the existence of $\xi < \kappa^+$ as desired. Now (for the only time in this proof), we use the hypothesis that the spaces D_η are OHI. From the condition $int_{D_\eta}(E_\xi \cap D_\eta) = \emptyset$ (all $\eta < \kappa$) it follows for each $\eta < \kappa$ that $E_\xi \cap D_\eta$ is nowhere dense in D_η. Then according to the definition of the family $\mathcal{K}$ there is $\alpha < 2^\kappa$ such that $E_\xi = K_\alpha$, but then (reverting now to the topology $\mathcal{T}_{\mathcal{KID}}$) we see that U_α is a nonempty $\mathcal{T}_{\mathcal{KID}}$-open set disjoint from the $\mathcal{T}_{\mathcal{KID}}$-dense set $E_\xi = K_\alpha$. This contradiction completes the proof. □

Theorem 3.6 effects a transition from suitably restricted Tychonoff spaces $(X, \mathcal{T})$ to a larger topology $\mathcal{T}_{\mathcal{KID}}$. In order to ensure that this theorem is non-vacuous, we show now in Theorem 3.7 that spaces $(X, \mathcal{T})$ with the hypotheses of Theorem 3.6 do exist. (For related results which construct—by other methods—spaces with similar strong irresolvability properties, the interested reader might consult [15] and [1].) Then in Lemma 3.9, we show how to find a κ-independent family $\mathcal{I}$ on κ with additional properties sufficient to ensure that the associated expansion $(X, \mathcal{T}_{\mathcal{KID}})$ is both maximally resolvable and extraresolvable.

Theorem 3.7. *Let $\kappa \geq \omega$. There is a Tychonoff space $(X, \mathcal{T})$ such that $|X| = nwd(X) = \kappa$, and X admits a maximally resolvable family $\mathcal{D} = \{D_\eta : \eta < \kappa\}$ such that $X = \cup\mathcal{D}$ and each D_η satisfies $|D_\eta| = \kappa$ and is an OHI space.*

Proof: We showed in [10, 5.4] that there is a dense subset E of the space $K := \{0,1\}^{2^\kappa}$ such that $|E| = nwd(E) = \kappa$ and E is even *hereditarily irresolvable* in the sense that no nonempty subset of E is resolvable. The subgroup $\langle E\rangle$ of K generated by E satisfies $|\langle E\rangle| = \kappa$ and hence $|K/\langle E\rangle| = 2^{2^\kappa}$, so there is a family $\mathcal{E} = \{E_\eta : \eta < \kappa\} \subseteq \mathcal{P}(K)$ of κ-many pairwise disjoint translates of E, each satisfying $|E_\eta| = nwd(E_\eta) = \kappa$. The sets $Y := \cup\mathcal{E}$ and $X := Y$ are then as required, except that in some models of set theory for some κ it may occur that

$$nwd(Y) \leq d(Y) = d(K) = \log(2^\kappa) < \kappa.$$

In order to address this difficulty we will replace the points y_ξ ($\xi < \kappa$) of Y by suitably chosen points x_ξ, and E_η by $D_\eta := \{x_\xi : y_\xi \in E_\eta\}$. Here are the details. We write $2^\kappa = \kappa \cup (2^\kappa \backslash \kappa)$ and $K = \{0,1\}^{2^\kappa} = \{0,1\}^\kappa \times \{0,1\}^{2^\kappa \backslash \kappa}$, identifying $\{0,1\}^\kappa$ with $\{0,1\}^\kappa \times \{0\}_{2^\kappa \backslash \kappa} \subseteq K$. We assume that the indexing $Y = \{y_\xi : \xi < \kappa\}$ is faithful, indeed that the projection from $Y \subseteq K$ into $\{0,1\}^\kappa \subseteq K$ is injective—that is, for $\xi_0 < \xi_1 < \kappa$ there is $\alpha < \kappa$ such that $y_{\xi_0}(\alpha) \neq y_{\xi_1}(\alpha)$. Let $[Y]^{<\kappa} = \{Y_\alpha : \alpha < 2^\kappa\}$ with each set listed 2^κ-many times and with $Y_\alpha = \emptyset$ when $\alpha < \kappa$, and define $x_\xi \in K$ by

$$x_\xi(\alpha) = y_\xi(\alpha) \text{ if } y_\xi \notin Y_\alpha, x_\xi(\alpha) = 0, \text{otherwise.}$$

Evidently, the surjection $Y \twoheadrightarrow X := \{x_\xi : \xi < \kappa\}$ given by $y_\xi \to x_\xi$ is bijective, so it suffices to show

(i) each D_η is dense in K,
(ii) $nwd(X) = \kappa$, and
(iii) each D_η is an OHI space.

For each of these, let $U = \bigcap_{\alpha \in F_0} \pi_\alpha^{-1}(\{0\}) \cap \bigcap_{\alpha \in F_1} \pi_\alpha^{-1}(\{1\})$ be a nonempty basic open subset of K; here, $F_i \in [2^\kappa]^{<\omega}$, $F_0 \cap F_1 = \emptyset$.

(i) Since E_η is dense in K and $nwd(E_\eta) = \kappa$ we have $|E_\eta \cap U| = \kappa$. For each $\alpha \in F_0 \cup F_1$ there are fewer than κ-many $\xi < \kappa$ such that $x_\xi(\alpha) \neq y_\xi(\alpha)$, so $D_\eta \cap U$ contains each point x_ξ such that $y_\xi \in E_\eta \cap U$, with fewer than κ-many exceptions.

(ii) Here we let $S = \{x_\xi : \xi \in A\}$ with $|A| = |S| < \kappa$, and we show that the relation $U \subseteq \overline{S}^K$ fails. Let $T := \{y_\xi : \xi \in A\} \subseteq Y$. There is $\alpha < 2^\kappa$ such that $\alpha \notin F_0 \cup F_1$ and $T = Y_\alpha$, and since each $\alpha \in A$ satisfies $y_\xi \in Y_\alpha$ (and hence $x_\xi(\alpha) = 0$), we have $S \cap \pi_\alpha^{-1}(\{1\}) = \emptyset$; *a fortiori* $S \cap U \cap \pi_\alpha^{-1}(\{1\}) = \emptyset$, and for $x \in U \cap \pi_\alpha^{-1}(\{1\}) \neq \emptyset$ we have $x \notin \overline{S}^K$.

(iii) From (i) and (ii), it follows that $nwd(D_\eta) = \kappa$. Suppose that H_0 and H_1 are complementary dense subsets of $D_\eta \cap U$, necessarily then with $|H_i| = \kappa$, and set $H_i' := \{y_\xi : x_\xi \in H_i\} \subseteq E_\eta$ $(i = 0, 1)$. We will show that each of the disjoint sets $E_\eta \cap U \cap H_i'$ is dense in $E_\eta \cap U$, contrary to the fact that E_η is an OHI space. If V is open in K and $V \cap U \neq \emptyset$, then, as in (i), $E_\eta \cap V \cap U$ contains each point y_ξ such that $x_\xi \in D_\eta \cap V \cap U$, with fewer than κ-many exceptions. Then from $|D_\eta \cap V \cap U \cap H_i| = \kappa$ follows $|E_\eta \cap V \cap U \cap H_i'| = \kappa$, and the assertion is proved. □

The special κ-independent family we find in Lemma 3.9 will be defined on the carefully chosen set G of cardinality κ given in the next lemma.

Lemma 3.8. *Let $\kappa \geq \omega$ and let $K = \{0,1\}^{2^\kappa}$. Then K contains a dense, maximally resolvable subgroup G such that $|G| = nwd(G) = \kappa$.*

Proof: We write $K = K_0 \times K_1$ with $K_0 = \{0,1\}^{2^\kappa \setminus \kappa}$ and $K_1 = \{0,1\}^\kappa$. As indicated in the proof of Theorem 3.7, there is by [10] a dense set $E \subseteq K_0$ such that $|E| = nwd(E) = \kappa$. As before we write $Y := \cup \mathcal{E}$, with $\mathcal{E}$ a set of κ-many pairwise disjoint translates (in K_0) of E; now set $G_0 := [Y]$, the subgroup of K_0 generated by

Y. Clearly, $|G_0| = \Delta(G_0) = \kappa$, and G_0, the union of copies of the κ-resolvable space Y, is itself κ-resolvable (cf. [7, 2.1]).

Next, let $G_1 := \{x \in K_1 : |\{\xi < \kappa : x_\xi \neq 0\}| < \omega\}$. Then it is easy to see that G_1, the so-called σ-product in K_1, is a totally bounded group such that $|G_1| = nwd(G_1) = \Delta(G_1) = \kappa$.

The group $G := G_0 \times G_1 \subseteq K_0 \times K_1 = K$ is as required, since $|G| = \Delta(G) = \kappa$ and $nwd(G) = \kappa$; for this last relation it is enough to note that if D is dense in a basic open subset $U_0 \times U_1$ of $K_0 \times K_1$ ($U_i \subseteq K_i$) then $\pi_1[D]$ is dense in U_1 and we have

$$\kappa = |U_0 \times U_1| \geq |D| \geq |\pi_1[D]| \geq d(U_1) \geq nwd(K_1) = \kappa. \qquad \square$$

Lemma 3.9. *Let $\kappa \geq \omega$. There are families $\mathcal{I}, \mathcal{S}_{mr}, \mathcal{S}_{er} \subseteq \mathcal{P}(\kappa)$ such that*

(i) *$\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$ is κ-independent on κ, $\mathcal{I}_0 \cap \mathcal{I}_1 = \emptyset$, $|\mathcal{I}_i| = 2^\kappa$, and $\mathcal{I}_1$ is small-set separating;*

(ii) *$\mathcal{S}_{mr}$ is a family of κ-many pairwise disjoint subsets of κ, with each $S \in \mathcal{S}_{mr}$ dense for $\mathcal{I}$; and*

(iii) *$\mathcal{S}_{er} = \{S_\eta : \eta < \kappa^+\}$ is a family of κ^+-many subsets of κ, each dense for $\mathcal{I}$, with $|S_{\eta_0} \cap S_{\eta_1}| < \kappa$ whenever $\eta_0 < \eta_1 < \kappa$.*

Proof: Let G be a dense, maximally resolvable subgroup of the group $K := \{0,1\}^{2^\kappa}$ such that $|G| = nwd(G) = \kappa$, as given by Lemma 3.8. Let $\mathcal{J} = \{\pi_\alpha^{-1}(\{0\}) \cap G : \alpha < 2^\kappa\}$, note from $nwd(G) = \kappa$ that $\mathcal{J}$ is κ-independent on G, and replace $\mathcal{J}$ by the family $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$ according to the protocol described in Remark 3.3(b); then $\mathcal{I}$ satisfies (i). We claim for $S \subseteq \kappa = G$ that S is dense for $\mathcal{I}$ if and only if S is dense for $\mathcal{J}$. Indeed, if S is dense for $\mathcal{J}$ then for disjoint $F_0, F_1 \in [2^\kappa]^{<\omega}$ we have

$$|S \cap \cap_{\eta \in F_0} \pi_\eta^{-1}(\{0\}) \cap \cap_{\eta \in F_1} \pi_\eta^{-1}(\{1\})| \geq \omega \tag{1}$$

and hence

$$|S \cap \cap_{\eta \in F_0} \pi_\eta^{-1}(\{0\}) \cap \cap_{\eta \in F_1} \pi_\eta^{-1}(\{1\})| = \kappa \tag{2}$$

(since $nwd(G) = \kappa$). Since each element of $\mathcal{I}$ differs from an element of $\mathcal{J}$ by a set of cardinality $< \kappa$, relation (2) shows that S is dense for $\mathcal{I}$. Conversely if (1) fails for some disjoint $F_0, F_1 \in [2^\kappa]^{<\omega}$ then with $\mathcal{F}_0, \mathcal{F}_1$ the corresponding disjoint elements of $[\mathcal{I}]^{<\omega}$ and with

$$T := \cap\{A : A \in \mathcal{F}_0\} \cap \cap\{\kappa \backslash A : A \in \mathcal{F}_1\},$$

we have $S \cap T \in [\kappa]^{<\kappa}$. In the enumeration of Remark 3.3(b) there is $\alpha < 2^\kappa$ such that $\alpha \notin F_0 \cup F_1$ and

$$\langle \emptyset, S \cap T \rangle = \langle S_\alpha, T_\alpha \rangle \in [\kappa]^{<\kappa} \times [\kappa]^{<\kappa}.$$

For this α with $A_\alpha \in \mathcal{J}$, we have $A_\alpha \backslash T_\alpha \in \mathcal{I}_1$ and

$$S \cap (A_\alpha \backslash T_\alpha) \cap T = \emptyset;$$

Thus, S is not dense for $\mathcal{I}$. The claim is proved.

The maximally resolvable space G is a totally bounded topological group such that $|G| = d(G)$; hence, by [9, 2.4], G is strongly extraresolvable. For the required families $\mathcal{S}_{mr}$ in (ii) and $\mathcal{S}_{er}$ in (iii), we may choose any families of dense subsets of G which witness, respectively, the maximal resolvability and the strong extraresolvability of G. □

With the details of our arguments established, our principal theorem may now be quickly proved.

Theorem 3.10. *Let $\kappa \geq \omega$. There is a Tychonoff space X such that $|X| = nwd(X) = \kappa$, and X is both maximally resolvable and extraresolvable but not strongly extraresolvable.*

Proof: We begin with a Tychonoff space $(X, \mathcal{T})$, as given by Theorem 3.7, and with a κ-independent family $\mathcal{I}$ on κ with dense subfamilies $\mathcal{S}_{mr}$ and $\mathcal{S}_{er}$, as given by Lemma 3.9. According to Theorem 3.6(a) and (d), the $\mathcal{KID}$ modification $(X, \mathcal{T}_{\mathcal{KID}})$ satisfies $nwd(X, \mathcal{T}_{\mathcal{KID}}) = \kappa$ and is not strongly extraresolvable; and according to Theorem 3.6(b) and (c), the families $\{X(S) : S \in \mathcal{S}_{mr}\}$ and $\{X(S) : S \in \mathcal{S}_{er}\}$ witness, respectively, the maximal resolvability and the extraresolvability of $(X, \mathcal{T}_{\mathcal{KID}})$. □

Remark 3.11. (Added April 2003) After our techniques had been developed (see [19], [10], [20]), but before this paper was submitted for publication, we learned that in the case $\kappa = \omega$ an affirmative answer to the question, "Is there in ZFC an extraresolvable Tychonoff space which is not strongly extraresolvable?" has already been achieved in [16]. We comment on the relation between these solutions. As we see it, each has an advantage over the other. Our argument is a trifle complex and detailed, but it dispatches the countable case as a special instance of an unrestricted phenomenon. In contrast, the construction of [16] is direct and *ad hoc*; it takes a familiar extremally disconnected space $(\mathbf{Seq}, \mathcal{T}_{\mathcal{A}})$ which is well-established in the literature and it demonstrates by a pleasing, direct argument for that space a new, unexpected topological property.

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