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**A NOTE ON GENERALIZED FARKAS
ALTERNATIVES**

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ABSTRACT. We show that the Farkas lemma can be generalized to easily established properties of best approximation operators onto closed convex cones in Hilbert space. Consequently, other theorems of the alternative follow from this easily stated and geometrically obvious “norm alternative.”

1. INTRODUCTION

The purpose of this paper is to provide a simply stated, simply proved theorem of the alternative whose setting is real Hilbert space. We refer to it as the “norm alternative.” We provide several equivalent forms and show that our norm alternative implies the Farkas lemma in Euclidean n -space and, hence, is a generalized version of other theorems of the alternative, namely, theorems of Slater, Motzkin, Tucker, Gordon, Gale, Stiemke, Broyden, and Dax. It is not our intention to redo or survey what others have done; however, we offer a fairly extensive list of references for those interested in further reading. In light of the difficulty of most stand-alone proofs of the Farkas lemma, our generalizations and easy proofs should be of interest. In particular, our norm alternative (which is pictorially obvious) stresses the simple geometric structure underlying the Farkas lemma. In [2], C. G. Broyden compared theorems of the alternative with cities situated on a high plateau;

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travel between them is not too difficult, the hard part is the ascent from the plains below.

The Farkas lemma was the first theorem of the alternative and these theorems have proved to be quite useful in many applications, particularly in establishing duality theorems in linear programming. In “On the Development of Optimization Theory” [28], A. Prékopa gives a nice historical account of Farkas’s famous paper of 1902 [12] and of Farkas’s contributions to the theory of linear inequalities and to physics, in particular to the principles of mechanical equilibrium. Typically, a linear programming problem consists of a system of linear inequalities in n variables and a linear form in n variables for which one wishes to find a maximum or a minimum value among the solutions of the system of inequalities. An important tool for solving linear programming problems is the Duality Theorem, which says that each minimum problem (called the primal problem) has a dual maximum problem (called the dual problem) and furthermore, an optimal solution for one is an optimal solution for the other. A good introductory reference with proofs and examples of these and related ideas is [14, Ch. 1].

Theorems of the alternative usually involve two systems of linear inequalities and/or equalities and the assertion that one or the other has a solution, but not both. As mentioned above, theorems of the alternative have historically been a main tool in proving duality theorems in linear programming problems. In 1975, L. McLinden [23] noted that, in fact, every pair of dual problems defines a theorem of the alternative. O. L. Mangasarian [22] first collected many of these theorems and provided proofs of their logical equivalences (see also [29]). More recently, Broyden [3] and A. Dax [10] have added theorems of their own to this collection.

2. MAIN RESULTS

We will show that the Farkas lemma can be realized (in a more general form) as an easily established property of closed convex cones and best approximation operators in a real Hilbert space, and in one form is a simple, intuitively clear norm inequality.

Our generalization is not the first generalization of the Farkas lemma to an infinite dimensional setting (see [6, 17, 18, 20, 26, 27]). Furthermore, others have noted connections between theorems of

the alternative, convex cones, projection maps, norms, and hyperplane separation theorems (see [10, 20, 27]). In fact, most textbook proofs of the Farkas lemma take a geometric approach and first prove the existence of a hyperplane separating a convex cone and a point not in it (see for example [5, 13, 14, 16]). Some recent “simple proofs” of the Farkas lemma can be found in [2, 8, 21]. In [8], Dax gives a simple algebraic proof that avoids the use of the hyperplane separation theorem. To our knowledge, no one has noted that the simple norm inequality which we provide in this paper is a generalized form of the Farkas lemma. As we will see later, our norm inequality is an immediate consequence of the geometric fact that projection onto a convex cone is orthogonal. Convex cones have historically played a role in solutions of systems of inequalities and, in particular, to methods for finding these solutions. We say more on this subject a little later after we have defined cones, polar cones, and dual cones.

The usual setting for the Farkas lemma and other theorems of the alternative:

Let \mathbb{E}^n denote Euclidean n -space with the usual norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. If $v \in \mathbb{E}^n$, we write $v \geq 0$ to mean that each coordinate of v is non-negative. We also view vectors in \mathbb{E}^n as column matrices and we form the $n \times m$ matrix $A = [v_1 v_2 \dots v_m]$ by augmenting the vectors v_1, v_2, \dots, v_m as columns of A . Let A^T denote the transpose of a matrix A .

Lemma 2.1. The Farkas lemma. *Given vectors v_1, v_2, \dots, v_m and v in \mathbb{E}^n , let $A = [v_1 v_2 \dots v_m]$. Then either*

- (i) $A \cdot x = v$ has a solution with $x \geq 0$, or
- (ii) $y^T \cdot A \geq 0$ and $y^T \cdot v < 0$ has a solution y in \mathbb{E}^n .

Of course (i) can be viewed as a system of n linear equations in m variables and (ii) as a system of $m + 1$ linear inequalities in n variables. The “or” in the Farkas lemma is exclusive.

The real Hilbert Space setting:

A real *Hilbert Space* is a complete inner product space over the field of real numbers. Hereafter, we let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. The set C in H is said to be *convex* if $tx + (1 - t)y \in C$ whenever $x, y \in C$ and $0 \leq t \leq 1$. For each closed, convex set C in H , we can associate a surjective function $p: H \rightarrow C$ such that for $x \in H$, $p(x)$ is the nearest point of C to x . These functions have been extensively studied and referred to in a variety of ways (see for example [1, 7, 19]). If C is a subspace of H , p is usually called a *projection operator* and it is well-known that the algebraic properties of idempotence, symmetry, and linearity characterize p as a projection onto a subspace of H . For other types of closed, convex sets, p has been called a proximity map, a nearest point map, a best approximation operator, and an orthogonal projection map. Such a map p is clearly always idempotent, but need not always be symmetric or linear.

A nonempty subset C of H is a *convex cone* if it is closed under addition and closed under multiplication by positive scalars. We will also assume that the $\mathbf{0}$ vector is an element of all cones under consideration in this paper. Linear subspaces are convex cones and convex cones are clearly convex. For a closed convex cone, we will refer to the associated nearest point mapping p as the projection map onto C . J. M. Ingram and the author showed in [19, Theorem 2] that p is characterized by the following properties when C is a closed, convex cone. For $x, y \in H$,

- (P1) $p^2(x) = p(x)$,
- (P2) $p(\alpha x) = \alpha p(x)$ for $\alpha > 0$, and
- (P3) $p(x+y) = p(x)+p(y)$ if and only if $\langle p(x), y \rangle = \langle p(x), p(y) \rangle = \langle x, p(y) \rangle$.

We assume hereafter that C is a closed, convex cone in H , $p: H \rightarrow C$ is the projection map of H onto C , and v is a vector in H . Or equivalently by [19, Theorem 2], we assume that p is a function with closed image C satisfying properties (P1), (P2), and (P3) above.

Two notions of importance related to cones are the polar cone of C and the dual cone of C . Unfortunately, the terms have been interchanged by some authors. We define the *polar cone*, C^0 , as the set $\{x \in H \mid \langle x, y \rangle \leq 0 \text{ for all } y \in C\}$, and the *dual cone*, C^* , as the set

$\{x \in H \mid \langle x, y \rangle \geq 0 \text{ for all } y \in C\}$. Cones and/or polar/dual cones have been used to prove theorems of the alternative, to characterize the existence of solutions of systems of equations/inequalities, and to develop methods for finding these solutions. For an early reference see [15]. In 1962, J. J. Moreau [24, 25] proved that each point x of H has a unique decomposition/representation as the sum of orthogonal vectors $p(x)$ and $x - p(x)$, which are, respectively, the nearest points of the cone C and the polar cone C^0 to x . Ingram and the author used this result and other properties of convex cones to establish the characterization of projections on closed convex cones mentioned above. Recently, others have used Moreau's representation theorem to solve certain optimization problems related to systems of equations/inequalities, (see [4, 9, 27]). For example, in [9], Dax considers the problem of finding a smallest correction vector for an inconsistent system of linear inequalities $A \cdot x \geq b$, extending the usual least squares problem. He shows that the solution of the problem determines the components (via Moreau's representation theorem) of the "data" vector b . Our purpose, however, is not to characterize solutions associated with the Farkas lemma or to develop methods for finding them; rather, it is to show, in a straightforward simple manner, that the Farkas lemma and its many applications are "hidden" within the norm inequality $\|p(x)\| \leq \|x\|$ for $x \in H$. That is, $\|p(x)\| \leq \|x\|$ implies the Farkas lemma.

We state four generalized forms of the Farkas lemma (referred to as GF1, GF2, GF3, and GF4). Each of these forms of the Farkas lemma is true (has a stand-alone proof); therefore, in a sense, they are all equivalent. However, we wish to show that there are natural, easily proved implications between them, establishing the logical equivalence of GF2, GF3, and GF4 and showing that the usual Farkas lemma is a special case of GF1 which follows from GF2. Namely,

$$GF4 \Leftrightarrow GF3 \Leftrightarrow GF2 \Rightarrow GF1 \Rightarrow \text{the usual Farkas alternative.}$$

GF1: *Either*

- (i) $v \in C$ or
- (ii) *there exists a $y \in H$ such that $\langle v, y \rangle < 0$ and $\langle x, y \rangle \geq 0$ for all $x \in C$.*

GF2: *Either*

- (i) $v \in C$ or
- (ii) $\langle v, v - p(v) \rangle > 0$; equivalently $\|v\|^2 > \langle v, p(v) \rangle$.

GF3: *Either*

- (i) v is a fixed point of p or
- (ii) $\langle v, v - p(v) \rangle > 0$.

GF4: $\|p(v)\| \leq \|v\|$.

First, we point out that the usual Farkas lemma is a special case of GF1. We recall the setting for the usual Farkas lemma and let v_1, v_2, \dots, v_m, v , and A be as they were given there. Let C be the non-negative span of the vectors v_1, v_2, \dots, v_m in \mathbb{E}^n . That is, $C = \{A \cdot x \mid x \geq 0\}$. It is well-known that C is a closed, convex polyhedral cone in \mathbb{E}^n (see [14, pp. 52-55]). Thus, condition (i) that $A \cdot x = v$ has a solution for $x \geq 0$ is clearly equivalent to v belonging to C . For condition (ii), we observe that $y^T \cdot A \geq 0$ is equivalent to having each inner product $\langle y, v_i \rangle \geq 0$ and similarly $y^T \cdot v < 0$ is equivalent to $\langle y, v \rangle < 0$. Thus, condition (ii) is a special case of condition (ii) in GF1.

Now we will show the equivalences of GF2, GF3, and GF4 and that GF2 implies GF1. Hence, we establish that the Farkas lemma follows from the intuitively obvious inequality $\|p(v)\| \leq \|v\|$. We need two properties of the projection map p (see Lemma 2.2 below) not yet mentioned. Only the first property is needed to establish our “norm alternative (inequality),” but we will need the second property to establish some of the above implications. The second property in Lemma 2.2 is sufficient to establish the separating hyperplane theorem. Dax and V. P. Sreedharan [10] have established stronger, more specific relationships between the Minimum Norm Duality (MND) Theorem and theorems of the alternative. The MND Theorem states that the minimum distance from a point z to a convex set C is equal to the maximum distance from z to hyperplanes separating z and C . Specifically, they show that with proper choice of a convex set C , the theorems of Farkas, Gale, and Gordon, and a theorem of theirs can be recast as steepest descent problems, resembling the original primal problem and a dual least norm problem, resembling the original dual system (see [10, Table 1]). The MND Theorem is valid in real normed vector spaces.

We point out that Lemma 2.2, as a result of the algebraic characterization of p mentioned earlier, can be viewed either geometrically (thinking of p as projection onto a closed convex cone) or algebraically (thinking of p as a non-linear map satisfying the properties (P1), (P2), and (P3)). Proofs of statements similar to Lemma 2.2 can be found in [11, Proposition 1.12.4] or [7, Lemma].

Lemma 2.2. *For $x \in H$ and $y \in C$,*

(a) $\langle x - p(x), p(x) \rangle = 0$ and

(b) $\langle x - p(x), y \rangle \leq 0$.

Proof: (a) Applying property (P2) of p , we get $p(x+x) = p(2x) = 2p(x) = p(x) + p(x)$. By property (P3), $\langle x, p(x) \rangle = \langle p(x), p(x) \rangle$. Therefore, $\langle x - p(x), p(x) \rangle = 0$.

(b) By part (a) and a little inner product manipulation, we get that $\|x\|^2 = \|x - p(x)\|^2 + \|p(x)\|^2$. Since $p(x)$ is the nearest point of C to x , we have that $\|x\|^2 \leq \|x - y\|^2 + \|p(x)\|^2$. It follows that $0 \leq -2\langle x, y \rangle + \langle y, y \rangle + \langle p(x), p(x) \rangle$. Hence,

$$\begin{aligned} 0 &\leq 2\langle p(x), y \rangle - 2\langle x, y \rangle + (\langle y, y \rangle - 2\langle p(x), y \rangle + \langle p(x), p(x) \rangle) \\ &= -2\langle x - p(x), y \rangle + \|y - p(x)\|^2. \end{aligned}$$

This inequality is true for all $x \in H$ and $y \in C$. Since C is a convex cone, for $t \geq 0$, $p(x) + ty \in C$. So, in particular, the inequality holds for x and $p(x) + ty$. Thus,

$$\begin{aligned} 0 &\leq -2\langle x - p(x), p(x) + ty \rangle + \|ty\|^2 \\ &= -2(\langle x - p(x), p(x) \rangle + t\langle x - p(x), y \rangle) + t^2\|y\|^2. \end{aligned}$$

And by part (a), we have that

$$0 \leq -2t\langle x - p(x), y \rangle + t^2\|y\|^2.$$

The expression on the right side of the inequality is a quadratic expression in the variable t whose graph in the plane passes through the origin. Thus, if $\langle x - p(x), y \rangle > 0$, the inequality would not be true for positive values of t close to zero. It follows that $\langle x - p(x), y \rangle \leq 0$. \square

We note that GF2 and GF3 are almost restatements of each other and hence, immediately equivalent. Observe that $v \in C$ if and only if v is a fixed point of p . This is clear since C is the image of p and $p(v)$ is the nearest point of C to v .

Now we show the implications $GF2 \Rightarrow GF1$ and $GF2 \Leftrightarrow GF4$.

Theorem 2.3. $GF2 \Rightarrow GF1$

Proof: Suppose (ii) in GF1 is not the case. Then the vector $y = p(v) - v$ is not a solution of (ii). So, either $\langle v, p(v) - v \rangle \geq 0$ or there exists an $x \in C$ such that $\langle x, p(v) - v \rangle < 0$. By Lemma 2.2(b), the latter is not the case. So, $\langle v, p(v) - v \rangle \geq 0$ and $\langle v, v - p(v) \rangle \leq 0$. Thus, (ii) is not the case in GF2. Hence, by GF2, $v \in C$, and we are done. \square

Theorem 2.4. $GF4 \Leftrightarrow GF2$

Proof: \Rightarrow : Suppose (ii) is not the case in GF2. Then $\|v\|^2 \leq \langle v, p(v) \rangle$. Now if $v = \mathbf{0}$, then $v \in C$, and we are done. So, assume that $v \neq \mathbf{0}$. We get that

$$\|v\|^2 \leq \langle v, p(v) \rangle \leq \|v\| \cdot \|p(v)\|,$$

and thus,

$$\|v\| \leq \|p(v)\|.$$

By GF4, $\|v\| \geq \|p(v)\|$. Therefore, we have that $\|v\| = \|p(v)\|$. By Lemma 2.2(a), we get that $\|v\|^2 = \|v - p(v)\|^2 + \|p(v)\|^2$. So, $\|v - p(v)\| = 0$ and thus $v = p(v)$. Therefore, $v \in C$.

\Leftarrow : Suppose that $\|p(v)\| \neq \|v\|$. Then $p(v) \neq v$; so, $v \notin C$. Therefore, by GF2, $\langle v, p(v) \rangle < \|v\|^2$. By Lemma 2.2(a), $\|p(v)\|^2 = \langle v, p(v) \rangle$. So, $\|p(v)\| < \|v\|$. \square

Lastly, we provide a simple stand-alone proof of GF4. Note that the proof only uses the orthogonality of the map p .

Proof of GF4. As above, by Lemma 2.2(a), we have that $\|v\|^2 = \|v - p(v)\|^2 + \|p(v)\|^2$. It follows that $\|p(v)\| \leq \|v\|$. \square

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