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GENERALIZED BALANCED PAIR ALGORITHM[1]

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Abstract. We present here a more general version of the balanced pair algorithm. This version works in the reducible case and terminates more often than the standard algorithm. We present examples to illustrate this point. Lastly, we discuss the features which lead to balanced pair algorithms not terminating and state several conjectures.

1. Introduction

The balanced pair algorithm was first introduced by A. N. Livshits [8, 9] for checking the pure discrete spectrum for the $\mathbb{Z}$-action of a substitution. A version of this algorithm was presented by V. F. Sirvent and B. Solomyak [12] for irreducible substitutions. For substitutions of Pisot type, Sirvent and Solomyak also give an explicit relationship between this algorithm and an overlap algorithm used in [13] for checking the pure discrete spectrum of the $\mathbb{R}$-action on the substitution tiling space. Recent results of A. Clark and L. Sadun [3] give conditions for the conjugacy of the $\mathbb{R}$-actions

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on substitution tiling spaces which otherwise do not differ combinatorially or topologically. Their results give an immediate relation between the $\mathbb{Z}$-action on the sequence space and the $\mathbb{R}$-action on the tiling space. The conditions given in [3] also allow us a way to generalize the balanced pair algorithm to reducible substitutions. The procedure for doing this is described in section 3 below.

The purpose of extending this algorithm is two-fold. First, it creates a way in which the algorithm will terminate more often. When the balanced pair algorithm terminates has been much studied. M. Hollander [5] (see also [6]) has shown that the balanced pair algorithm will terminate for all 2-symbols Pisot substitutions. This fact, along with the solution to the coincidence conjecture for 2-symbols [2], has shown that all 2-symbol Pisot substitutions have pure discrete spectrum. This has not yet been shown for an arbitrary number of symbols. In fact, it is not yet known whether the balanced pair algorithm terminates for all Pisot substitutions.

The second reason for extending the algorithm to the reducible case is related to collaring or rewriting substitutions to obtain new (yet conjugate) substitutions. Collaring or rewriting procedures generally increase the number of symbols which turns irreducible substitutions into reducible ones. It would be beneficial to know that such procedures did not change the potential for a balanced pair algorithm to terminate. In particular, there are rewriting procedures which automatically produce coincidences. Thus, the question of pure point spectrum (and even the coincidence conjecture) relies entirely on whether these reducible systems terminate. We explore these and other questions in section 4.

2. Preliminaries

Let $\mathcal{A} = \{1, 2, \ldots, n\}$ be a finite alphabet and $\mathcal{A}^*$ denote the collection of finite nonempty words with letters in $\mathcal{A}$. A *substitution* is a map $\varphi : \mathcal{A} \to \mathcal{A}^*$. It extends naturally to $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ and $\varphi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by concatenation.

We associate to each word w a population vector

$$\mathbf{p}(w) = (p_1(w), p_2(w), \ldots, p_n(w))$$

which assigns to each $p_i(w)$ the number of appearances of the letter i in the word w. To a substitution φ there is an associated transition matrix $A_\varphi = (a_{ij})_{i\in\mathcal{A}, j\in\mathcal{A}}$ in which $a_{ij} = p_i(\varphi(j))$. Note that

$A_\varphi(\mathbf{p}(w)) = \mathbf{p}(\varphi(w))$. We say a substitution φ is *primitive* if $\varphi^m(i)$ contains j for all $i, j \in \mathcal{A}$ and sufficiently large m. Equivalently, φ is primitive if and only if the matrix A_φ is eventually positive (there exists m such that the entries of A_φ^m are strictly positive). This condition implies that A_φ has an eigenvalue λ_φ larger in modulus than its remaining eigenvalues called the Perron-Frobenius eigenvalue of A_φ (and φ).

We form a space Ω_φ as the set of *allowed bi-infinite words* for φ. That is, $\mathbf{u} \in \Omega_\varphi$ if and only if for each finite subword w of $\mathbf{u}$, there are $i \in \mathcal{A}$ and $n \in \mathbb{N}$ such that w is a subword of $\varphi^n(i)$. We give Ω_φ the subspace topology (of $A^{\mathbb{Z}}$ with the product topology) and denote the natural shift homeomorphism by σ. Then (Ω_φ, σ) is a topological dynamical system which is minimal and uniquely ergodic provided the substitution is primitive. We will often use the fact that, due to unique ergodicity, any allowable word w appears in any $\mathbf{u} \in \Omega_\varphi$ with a well-defined and bounded positive frequency.

The spectral type of the measure-preserving transformation $(\Omega_\varphi, \sigma, \mu)$ is, by definition, the spectral type of the unitary operator $U_\varphi : f(\cdot) \mapsto f(\sigma\cdot)$ on $L^2(\Omega_\varphi, \mu)$. We say $(\Omega_\varphi, \sigma, \mu)$ has *pure discrete spectrum* if and only if there is a basis for $L^2(\Omega_\varphi, \mu)$ consisting of eigenfunctions for U_φ.

Additionally, to each substitution φ and any $\mathbf{u} \in \Omega_\varphi$ we can form a tiling of the real line by intervals. For each letter $i \in \mathcal{A}$ we assign a closed interval of length l_i. We refer to the vector $\mathbf{L} = (l_1, \ldots, l_n)$ as the length vector. We then "lay" copies of these closed intervals down on the real line (so that they do not overlap on their interiors) according to the order prescribed by $\mathbf{u} = \ldots u_{-1}.u_0 u_1 \ldots$, with the placement of the origin given by the decimal point. There is a natural translation acting Γ^t on a tiling T which forms new tilings by simply moving the origin by a distance $t \in \mathbb{R}$ to the right. For a fixed $\mathbf{u}$, we form a compact metric space by taking the completion of the space of the translates of $\mathbf{u}$. Here the completion is taken with respect to the metric which considers two tilings to be close if they agree on a large neighborhood about the origin after a small translation. If the substitution is primitive, then this space is independent of $\mathbf{u}$, is minimal and uniquely ergodic. We refer to the $\mathbb{R}$-action on $(\Omega_\varphi, \Gamma^t, \mathbf{L})$ as the *tiling dynamical system*. It is topologically conjugate to the suspension flow over the $\mathbb{Z}$-action (Ω_φ, σ), with the height function equal to l_i over the cylinder i.

The system $(\Omega_\varphi, \Gamma^t, \mathbf{L}, \mu)$ is then said to have *pure point spectrum* if there is a basis of $L^2(\Omega_\varphi, \Gamma^t, \mathbf{L}, \mu)$ consisting of eigenfunctions for the $\mathbb{R}$-action.

A primitive substitution is of *Pisot type* if all of its non-Perron-Frobenius eigenvalues are strictly between 0 and 1 in magnitude. Sirvent and Solomyak [12] have shown, for a substitution of Pisot type, that if the $\mathbb{R}$-action has pure discrete spectrum then so does the $\mathbb{Z}$-action. A recent result of Clark and Sadun [3] implies that this can be strengthened to an "if and only if" statement.

In fact, the result of Clark and Sadun has more general applications than just the Pisot case and we briefly describe some aspects of their results. First, however, we should give a few observations in order that we may put their results in context.

We denote as $\mathbf{L}_\lambda$ the left eigenvector associated to the Perron-Frobenius eigenvalue λ of the transition matrix A_φ. The choice of $\mathbf{L}_\lambda$ as a length vector causes any tiling which is (combinatorially) fixed under φ to be *self-similar*. The similarity map is expansion by λ and the tile-substitution is the geometric version of ϕ. On the other hand, the length vector $\mathbf{L}_1 = (1, 1, \ldots, 1)$ is a natural choice since the group action of the time-1 return map for the tiling dynamical system is conjugate to the shift homeomorphism of the substitution dynamical system. Other length changes, though producing tiling dynamical systems conjugate to the suspension of the shift, will produce no such group action. It is easy to see that the $\mathbb{R}$-action of the tiling dynamical system with constant length vector $\mathbf{L}_1$ has pure discrete spectrum if and only if the $\mathbb{Z}$-action does. Therefore, questions about the relation of the $\mathbb{R}$-action with length vector L to the $\mathbb{Z}$-action can be rephrased into questions about the conjugacy of the $\mathbb{R}$-actions of the systems using $\mathbf{L}$ and $\mathbf{L}_1$, respectively.

Clark and Sadun [3] give explicit conditions for the $\mathbb{R}$-action associated to one length vector to be conjugate (up to an overall rescaling) to that of another via a homeomorphism which preserves the combinatorics (i.e., is homotopic to the identity). Assume the length vectors $\mathbf{L}$ and $\mathbf{L}'$ have been rescaled to agree in the Perron-Frobenius direction. Then, the conjugacy occurs if and only if $\mathbf{L}A_\varphi^k - \mathbf{L}'A_\varphi^n \to 0$ as $n \to \infty$. Note that every length vector can be written as a linear combination of vectors living in the Perron-Frobenius eigenspace, the small eigenspaces (those associated to the eigenvalues of magnitude less than one) and the large eigenspaces (those associated to eigenvalues of magnitude greater than or equal to one, though we don't include the P.F.-eigenspace here). In particular, after the rescaling, $\mathbf{L} - \mathbf{L}'$ must avoid the large eigenspaces for the conjugacy to occur. If the substitution is of Pisot-type, then all systems length changes produce conjugate systems and we arrive at the result above.

We point out a few special kinds of substitutions we will be considering. A substitution φ is said to be *irreducible* if the characteristic polynomial of its transition matrix is irreducible. In particular, $\mathbf{L}_\lambda$ of an irreducible substitution is such that none of its entries are rationally related to one another. Lastly, a substitution φ is said to have *constant length* if the number of letters appearing in $\varphi(i)$ is the same for all $i \in \mathcal{A}$.

3. Balanced Pair Algorithms

We now describe the balanced pair algorithm in a variety of circumstances. We begin with the case that the substitution is irreducible. This case was studied extensively in [12] as an adaptation of the algorithm of [9]. The extension of this algorithm to a specific class of reducible substitutions containing "letter equivalences" can then easily be described. We end this section with a description of how one extends this procedure for generic reducible cases.

3.1. Irreducible case

A pair of allowable words u and v is called *balanced* if each member of the pair has the same population vector. We write $\left|\begin{smallmatrix} u \\ v \end{smallmatrix}\right|$ if u and v are balanced. Note that if $\left|\begin{smallmatrix} u \\ v \end{smallmatrix}\right|$ is balanced, then so is $\left|\begin{smallmatrix} \varphi(u) \\ \varphi(v) \end{smallmatrix}\right|$.

Let the right infinite sequence $\mathbf{u} = u_0 u_1 \ldots$ be fixed under the substitution and let w be a non-empty prefix of $\mathbf{u}$. Since w appears in $\mathbf{u}$ with positive frequency, $\mathbf{u}$ can be written as $\mathbf{u} = wX_1wX_2\ldots$ We may then speak of splitting $\left|\begin{smallmatrix} \mathbf{u} \\ \sigma^{|w|}\mathbf{u} \end{smallmatrix}\right|$ into balanced pairs in the following way:

$$\begin{vmatrix} \mathbf{u} \\ \sigma^{|w|}\mathbf{u} \end{vmatrix} = \begin{vmatrix} wX_1 \\ X_1 w \end{vmatrix} \begin{vmatrix} wX_2 \\ X_2 w \end{vmatrix} \cdots .$$

Since appearances of w in $\mathbf{u}$ are bounded, there are only finitely many different balanced pairs encountered in the process above. We may further reduce each of these to form a finite set of irreducible balanced pairs which we will refer to as $I_1(w)$.

We may now inductively define, for $n > 1$:

$$I_n(w) = \left\{ \begin{vmatrix} u \\ v \end{vmatrix} : \begin{vmatrix} u \\ v \end{vmatrix} \text{ appears as an irreducible balanced pair} \right.$$
$$\left. \text{in the reduction of} \begin{vmatrix} \varphi(x) \\ \varphi(y) \end{vmatrix}, \text{ for some } \begin{vmatrix} x \\ y \end{vmatrix} \in I_{n-1} \right\}.$$

Let $I(w) = \bigcup_{n=1}^{\infty} I_n(w)$. If $I(w)$ is finite, then we say that the balanced pair algorithm associated to a prefix w, or bpa-w, terminates. Below, we will state how this algorithm is used to determine pure discrete spectrum, though our main interest here is in determining precisely when the algorithm terminates. We illustrate the computation of $I(w)$ for a simple example.

Example 3.1 (*Fibonacci substitution*)**.** Consider the substitution given by:

$$\begin{array}{ccc} 1 & \to & 112 \\ 2 & \to & 12 \end{array}$$

Then $\mathbf{u} = 11211212112\ldots$ and we take the prefix $w = 1$. Then it is easy to see that:

$$I_1 = \left\{ \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \begin{vmatrix} 12 \\ 21 \end{vmatrix} \right\}.$$

Now,

$$\begin{vmatrix} 1 \\ 1 \end{vmatrix} \to \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{vmatrix} 2 \\ 2 \end{vmatrix} \text{ and } \begin{vmatrix} 12 \\ 21 \end{vmatrix} \to \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{vmatrix} 12 \\ 21 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{vmatrix} 2 \\ 2 \end{vmatrix}$$

so that $I_2 = \left\{ \left|\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right|, \left|\begin{smallmatrix} 12 \\ 21 \end{smallmatrix}\right|, \left|\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right| \right\}$ and further, $I_2 = I_3 = \ldots = I(w)$. Thus, the algorithm terminates.

3.2 Substitutions involving letter equivalences

The balanced pair algorithm as originally described by Livshits [8, 9] has an additional feature. Consider two letters $i, j \in \mathcal{A}$ to be equivalent if $\varphi^n(i)$ and $\varphi^n(j)$ have the same number or symbols for all $n \in \mathbb{N}$. Then, a pair of words is balanced if both contain the same number of symbols from each equivalence class. The algorithm runs in the usual way, starting with an initial list of balanced pairs, substituting and reducing. It stops if no new balanced pairs are produced. In particular, in the constant length case all letters are equivalent so all irreducible balanced pairs are just pairs of symbols. Thus, the algorithm always terminates in this case. This way, the algorithm includes F. M. Dekking's criterion [4] in the constant length case. For any irreducible case, no two symbols are equivalent and thus the algorithm runs just as before.

Example 3.2 (*Substitution of constant length)***.** Consider the substitution given by:

$$\begin{array}{ccc} 1 & \to & 112 \\ 2 & \to & 122 \end{array}$$

which is a constant length substitution. If we ignore the equivalence relation of Livshits, then the troubling balanced pair is $\left|\begin{smallmatrix} 12 \\ 21 \end{smallmatrix}\right|$. Iterating this pair, we see:

$$\left|\begin{matrix} 12 \\ 21 \end{matrix}\right| \to \left|\begin{matrix} 1 \\ 1 \end{matrix}\right| \left|\begin{matrix} 1212 \\ 2211 \end{matrix}\right| \left|\begin{matrix} 2 \\ 2 \end{matrix}\right|$$

$$\left|\begin{matrix} 1212 \\ 2211 \end{matrix}\right| \to \left|\begin{matrix} 1 \\ 1 \end{matrix}\right| \left|\begin{matrix} 1212211212 \\ 2212211211 \end{matrix}\right| \left|\begin{matrix} 2 \\ 2 \end{matrix}\right|.$$

In particular, this process generates new balanced pairs of the form $\left|\begin{smallmatrix} 1z2 \\ 2z1 \end{smallmatrix}\right|$ for longer and longer words z.

Once we take the equivalence relation into account, however, all letters are equivalent and therefore the above process terminates.

Example 3.3 (*A non-constant length substitution)***.** Consider the substitution given by:

$$\begin{array}{ccc} 1 & \to & 31 \\ 2 & \to & 412 \\ 3 & \to & 312 \\ 4 & \to & 412 \end{array}$$

If we were to again ignore the equivalence relation of Livshits, then there is a potential problem with balanced pairs that "match up" the letter 3 with 4. To see this, note that the right eigenvectors of the transition matrix in some sense describe the frequencies in which letters appear. Here, there is one large eigenvalue other than the Perron-Frobenius eigenvalue. The eigenvalue is 1 and it has right eigenvector $(0, 0, 1, -1)^T$. We can thus view a balanced pair of the form $\left|\begin{smallmatrix} 3\dots \\ 4\dots \end{smallmatrix}\right|$ as initially having an abundance of a 3 and a lack of a 4 on top. Since this corresponds to the right eigenvector of 1, this difference persists under substitution. Note that we say this is a "potential" problem as this association to a large eigenvalue in itself will not force the bpa-w to not terminate. In this case, however, we show that this does in fact occur. After shifting the fixed word $\mathbf{u} = 312\ldots$ by the prefix $w = 31$, we see the balanced pair $\left|\begin{smallmatrix}31412\\41231\end{smallmatrix}\right|$. Iterating this pair, we see:

$$\begin{vmatrix}31412\\41231\end{vmatrix} \to \begin{vmatrix}3123141231412\\4123141231231\end{vmatrix}.$$

Thus, we generate new balanced pairs of the form $\left|\begin{smallmatrix}3z412\\4z231\end{smallmatrix}\right|$ for longer and longer words z. The equivalence relation tells us, however, that $2, 3$, and 4 are actually equivalent letters and thus, the above process terminates as $\left|\begin{smallmatrix}3\\4\end{smallmatrix}\right|$ and $\left|\begin{smallmatrix}4\\2\end{smallmatrix}\right|\left|\begin{smallmatrix}12\\31\end{smallmatrix}\right|$ are balanced. Note that the left Perron-Frobenius eigenvector for this system is $(1, \lambda, \lambda, \lambda)$ so that $2, 3$, and 4 all have the same length in the self-similar system.

3.3 Reducible case

The example above gives some hint as to how to describe the balanced pair algorithm in the generic reducible case. For a system whose lengths of tiles have rational relations which persist under substitution, it may be the case that entire words should be identified even when no individual letters are. We therefore introduce an equivalence relation which exploits these rational relations. Consider two words v and w to be equivalent if:

$$\mathbf{L}(\mathbf{p}(\varphi^n(v)) - \mathbf{p}(\varphi^n(w))) = 0, \forall n \in \mathbb{N},$$

or equivalently, if

$$\mathbf{L}(A_\varphi^n \mathbf{p}(v) - A_\varphi^n \mathbf{p}(w)) = 0, \forall n \in \mathbb{N}.$$

We write $v \sim_{\mathbf{L}} w$ to emphasize the dependence of the equivalence relation on choice of $\mathbf{L}$. In the case in which we use the left Perron-Frobenius eigenvector $\mathbf{L}_\lambda$ for our lengths, the equivalent pairs correspond simply to the *geometric balanced pairs* of [12]. In fact, we will show below that in this case, the algorithm is equivalent to the overlap algorithm and for a general $\mathbf{L}$ is meant to bridge the gap between the balanced pair and overlap algorithms in the reducible case.

The algorithm once again runs in the usual way, with a pair of words considered balanced if they are equivalent under our relationship above. We denote this algorithm by $bpa(w, L)$ and the set of balanced (or equivalent) pairs by $I(w, \mathbf{L})$ to emphasize that there is now an additional dependence on $\mathbf{L}$.

Remark 3.4. Our equivalence condition above that $\mathbf{L}(\mathbf{p}(\varphi^n(v)) - \mathbf{p}(\varphi^n(w))) = 0, \forall n \in \mathbb{N}$, could be weakened to allow $\mathbf{L}(\mathbf{p}(\varphi^n(v)) - \mathbf{p}(\varphi^n(w))) \to 0$ as $n \to \infty$, which essentially allows inconsequential length changes in the small eigenspaces mentioned above.

Example 3.5 (*A newly balanced substitution)*. Consider the substitution given by:

$$\begin{array}{lcl} 1 & \to & 112 \\ 2 & \to & 2321 \\ 3 & \to & 12. \end{array}$$

The eigenvalues of the transition matrix are $\frac{3\pm\sqrt{13}}{2}$ and 1 while $\mathbf{L}_\lambda = (1, \frac{-1+\sqrt{13}}{2}, \frac{5-\sqrt{13}}{2})$. The right eigenvector associated to the eigenvalue 1 is $(2, -1, -1)^T$. This indicates that there is a potential problem with balanced pairs which "match up" the word 11 with 23. Such a matching occurs in the balanced pair $\left|\begin{smallmatrix} 11223 \\ 23211 \end{smallmatrix}\right|$, which was found by performing the balanced pair algorithm after shifting the fixed word by two spaces. Iterating this balanced pair, we generate balanced pairs of the form $\left|\begin{smallmatrix} 11z23 \\ 23z11 \end{smallmatrix}\right|$ for longer and longer words z.

Fortunately, $\mathbf{L}_\lambda \cdot (2, -1, -1) = 0$ so that 11 and 23 are equivalent words and hence $\left|\begin{smallmatrix} 11 \\ 23 \end{smallmatrix}\right|$ is balanced and the above algorithm will terminate. Also note that $\mathbf{L}_1$ is also perpendicular to $(2, -1, -1)$ so that the algorithm will also terminate for what we consider to be the other interesting case. This is immediate from the fact that these two systems are conjugate by [3]. On the other hand, $\mathbf{L} = (1, 1, 2)$ for example is not conjugate to these and furthermore the algorithm does not terminate in this case.

Proposition 3.6. Let $\mathcal{E}(L) = \{(u, v) : u \sim_{\mathbf{L}} v\}$. Then, for any length vector $\mathbf{L}$, $\mathcal{E}(L) \subseteq \mathcal{E}(\mathbf{L}_\lambda)$

Proof: Let $u \sim_{\mathbf{L}} v$. Let $\mathbf{z} = \mathbf{p}(u) - \mathbf{p}(v)$. Since $\mathbf{L}$ has only positive terms, it must have a component in the $\mathbf{L}_\lambda$ direction. Let $\mathbf{L} = a_\lambda \mathbf{L}_\lambda + \sum_{j=1}^{l} b_j B_j + \sum_{j=1}^{s} a_j A_j$, where B_j is a left eigenvector for eigenvalue $\beta_j \geq 1$ and A_j is a left eigenvector for eigenvalue $\alpha_j < 1$ for each j. Then,

$$0 = \mathbf{L} \cdot A^n \mathbf{z} = (\lambda^n a_\lambda \mathbf{L}_\lambda + \sum_{j=1}^{l} \beta_j^n b_j B_j + \sum_{j=1}^{s} \alpha_j^n a_j A_j) \cdot \mathbf{z}.$$

If $\mathbf{L}_\lambda \cdot \mathbf{z} \neq 0$, then the above implies $\lambda^n \approx \sum_{j=1}^{l} C_j \beta_j^n$ for large enough n and some constants C_j. This is a contradiction since the Perron-Frobenius eigenvalue dominates all others. □

Corollary 3.7. *For any length vector* $\mathbf{L}$*, if the bpa*$(w, \mathbf{L})$ *terminates, then the bpa*$(w, \mathbf{L}_\lambda)$ *terminates.*

Let $\mathcal{L}$ denote the span of the right eigenvectors with eigenvalues greater than or equal to 1 in magnitude but strictly less than the Perron-Frobenius eigenvalue. Similarly, let $\mathcal{S}$ denote the span of the right eigenvectors with eigenvalues strictly less than 1 in magnitude. We say that (u, v) lies in a vector space $\mathcal{P}$ if $(\mathbf{p}(u) - \mathbf{p}(v)) \in \mathcal{P}$.

Remark 3.8. Then, the set of equivalence words lies entirely in these spaces since they form precisely what is perpendicular to $\mathbf{L}_\lambda$. In our examples above, choosing an $\mathbf{L}$ which missed identifying equivalent pairs in $\mathcal{L}$ would have led to the bpa$(w, \mathbf{L})$ not terminating. Contrast this with Example 3.3 in which choosing an $\mathbf{L}$ which neglects to identify 2 and 4 will have no effect on whether the algorithm terminates. The vector associated to the pairing of 2 and 4, namely $(0, 1, 0, -1)$, lives in the zero-eigenspace and thus differences in frequencies should be quickly dispelled. Our contention here is that generally the equivalence relations which live in $\mathcal{S}$ should not affect the algorithm whereas those in $\mathcal{L}$ affect it greatly.

Corollary 3.9. $u \sim_{\mathbf{L}} v$ *implies* $\mathbf{p}(u) - \mathbf{p}(v) \in \mathcal{L} \oplus \mathcal{S}$*. The converse is true if* $\mathbf{L} = \mathbf{L}_\lambda$*.*

A balanced (equivalent) pair $\left|\begin{smallmatrix} i \\ i \end{smallmatrix}\right|$, for $i \in \mathcal{A}$, is called a *coincidence*. We say that a balanced pair $\left|\begin{smallmatrix} u \\ v \end{smallmatrix}\right|$ *leads to a coincidence* if there exists m such that the reduction of $\left|\begin{smallmatrix} \varphi^n(u) \\ \varphi^n(v) \end{smallmatrix}\right|$ contains a coincidence. Notice that coincidences lead to coincidences since $\left|\begin{smallmatrix} \varphi(i) \\ \varphi(i) \end{smallmatrix}\right|$ has nothing but coincidences in its reduction.

Theorem 3.10. *Let* φ *be a primitive substitution such that* $\mathbf{u} = u_0 u_1 \ldots$ *is a right infinite fixed word and let* $\mathbf{L}$ *be a length vector.*

(a) *If for some prefix* w *the bpa*$(w, \mathbf{L})$ *terminates and every equivalent pair in* $I(w, \mathbf{L})$ *leads to a coincidence, then* $(\Omega_\varphi, \Gamma^t, \mathbf{L}_\lambda)$ *has pure discrete spectrum.*

(b) *If the bpa*$(w, \mathbf{L})$ *terminates for some prefix* $w = u_0 \dots u_m$ *such that* $u_{m+1} = u_0$*, and* $(\Omega_\varphi, \Gamma^t, \mathbf{L}_\lambda)$ *has pure discrete spectrum, then every balanced pair in* $I(w, \mathbf{L})$ *leads to a coincidence.*

Before beginning the proof of Theorem 3.10, we make the following observations regarding the densities of coincident pairs. Let $\mathbf{u} = u_0 u_1 \dots$ be a fixed right-sided sequence. Let z be a prefix of $\mathbf{u}$ and $\mathbf{u} - \mathbf{L} \cdot \mathbf{p}(z) = v_0 v_1 \dots$. Let $D(z) = \{u_i : u_i = v_j \text{ some } j \text{ with } u_0 \dots u_{i-1} \sim_{\mathbf{L}} v_0 \dots v_{j-1}\}$. Suppose we define a density function, $dens_{\mathbf{L}}(D(z)) = \lim_{k \to \infty} \frac{\mathbf{L} \cdot \mathbf{p}(D(z) \cap u_0 \dots u_k)}{\mathbf{L} \cdot \mathbf{p}(u_0 \dots u_k)}$, if the limit exists. The existence of this limit follows from the unique ergodicity of $(\Omega_\varphi, \Gamma^t, \mathbf{L})$. Notice that for $\mathbf{L} = \mathbf{L}_1$, this definition of density agrees with that used in [12] for the (irreducible) balanced pair algorithm and for $\mathbf{L} = \mathbf{L}_\lambda$ it agrees with that of [13] for the overlap algorithm. Now, by Proposition 3.6, coincident pairs for $\mathcal{E}(\mathbf{L})$ are also coincident pairs for $\mathcal{E}(L_\lambda)$.

Remark 3.11. The proof of this theorem differs from that of the irreducible case only in the way in which we define density and the set of irreducible balanced pairs. We therefore only include a sketch of the proof that follows closely a sketch provided in [12] for the irreducible case. The full details of that case have been worked out in [5]. A theorem of this sort was proved by [9], though coincidences and the balanced pair algorithm go back to [4] and [10], respectively. Part (a) is largely contained in [11].

Proof: Let w be a prefix of the fixed word $\mathbf{u}$. Denote by $D(z)$ the density defined above. We will be interested in $dens_{\mathbf{L}}(D(\varphi^l(w)))$ as $l \to \infty$. Let

$$\mathbf{u}^{(l)} = \mu_1^{(l)} \mu_2^{(l)} \dots$$

be the reduction of $\left|\begin{smallmatrix} \mathbf{u} \\ \sigma^{p_l}\mathbf{u} \end{smallmatrix}\right|$ into irreducible equivalent pairs. For an equivalent pair $\beta = \left|\begin{smallmatrix} u \\ v \end{smallmatrix}\right|$ let $|\beta| = \mathbf{L} \cdot \mathbf{p}(u)$ and $\delta(\beta) = \{u_i \in u : u_i = v_j \text{ some } j \text{ with } u_0 \dots u_{i-1} \sim_{\mathbf{L}} v_0 \dots v_{j-1}\}$. Then

$$dens_{\mathbf{L}}(D(\varphi^l(w))) = \lim_{N \to \infty} \frac{\sum_{j=1}^{N} \mathbf{L} \cdot \mathbf{p}(\delta(\mu_j^{(l)}))}{\sum_{j=1}^{N} \mathbf{L} \cdot \mathbf{p}(\mu_j^{(l)})}.$$

Now consider the substitution $\hat{\varphi}$ on the set of irreducible balanced pairs $I(w, \mathbf{L})$. By definition, clearly $\mathbf{u}^{(l)} \in I(w, \mathbf{L})^{\mathbb{N}}$ and $\mathbf{u}^{(l)} =$

$\hat{\varphi}^l(\mathbf{u}^{(0)})$, where $\mathbf{u}^{(0)}$ is the reduction of $\left|{}_{\sigma^{|w|}\mathbf{u}}^{\mathbf{u}}\right|$ into irreducible equivalent pairs.

There is a directed graph $\mathcal{G}(\hat{\varphi})$ associated with the substitution $\hat{\varphi}$. Its vertices are labelled by the members of $I(w, \mathbf{L})$, and for every vertex β there are directed edges from β into of the letters of $\hat{\varphi}(\beta)$ with multiplicities.

(a) Let $\beta \in I(w, \mathbf{L})$ be an irreducible equivalent pair which is not a coincidence. By assumption there is a path in the graph $\mathcal{G}(\hat{\varphi})$ leading from β to a coincidence. Since all the edges from coincidences lead to coincidences, a standard argument shows that the frequency of the symbol β in $\mathbf{u}^{(l)} = \hat{\varphi}^l(\mathbf{u}^{(0)})$ goes to zero geometrically fast as $l \to \infty$. Since $I(w, \mathbf{L})$ is finite and $1 - \delta(\beta) > 0$ if and only if β is a non-coincidence, it follows that

$$1 - dens(D_{p_l}) \leq const \cdot \gamma^l$$

for some $\gamma \in (0, 1)$. This implies that $\mathbf{u}$ is mean-almost periodic so we can conclude (similar to [11, VI.25]) that φ has pure discrete spectrum. We note that the argument only relied on $I(w, \mathbf{L})$ being finite and that the equivalence relation does not create false coincidences.

(b) Let $w = u_o ... u_m$ be such that $u_{m+1} = u_0$ and $p_l = \mathbf{L} \cdot \mathbf{p}(\varphi^l(w))$. It follows from [7] (see also [13, Theorem 4.3]) that $\lim_{l\to\infty} e^{2\pi i \lambda p_l} = 1$ for any eigenvalue $e^{2\pi i \lambda}$ of the dynamical system $(\Omega_\varphi, \Gamma^t, \mathbf{L}, \mu)$. If the spectrum is pure discrete, then the eigenfunctions span a dense subset of $L^2(\Omega_\varphi)$ so that $\lim_{l\to\infty} ||U_\varphi^{p_l} f - f||_2 = 0$ for every $f \in L^2(\Omega_\varphi)$. Taking f to be the characteristic function of the cylinder set corresponding to $i \in \mathcal{A}$ with heights prescribed by $\mathbf{L}$, we obtain (see [11, just after Lemma VI.26]) $\lim_{l\to\infty} dens_{\mathbf{L}}(D(\varphi^l(w))) = 1$. On the other hand, suppose that there is an irreducible equivalent pair in $I(w, \mathbf{L})$ which does not lead to a coincidence. Then there exists an irreducible component $\mathcal{G}_0$ of the graph $\mathcal{G}(\hat{\varphi})$ which contains no coincidences. There exists l_0 such that for every $l \geq l_0$ elements of the component $\mathcal{G}_0$ occur in $\mathbf{u}^{(l)}$ with positive frequency. (Note that different elements of $\mathcal{G}_0$ may occur for different l.) Further, it can be shown that this frequency is bounded away from zero as $l \to \infty$. Since $1 - \delta(\beta) > 0$ for all $\beta \in \mathcal{G}_0$, it follows that $dens_{\mathbf{L}}(D(\varphi^l(w))) \not\to 1$, which is a contradiction. □

We also note the following relationship between the balanced pair algorithm and the overlap algorithm in the case that the tiling space is self-similar.

Proposition 3.12. Let φ be a primitive substitution such that $\varphi(1)$ begins with 1. Then the overlap-algorithm associated to $x = x(w)$ terminates with half-coincidences if and only if $\mathrm{bpa}(w, \mathbf{L}_\lambda)$ terminates.

Proof: Assume the $\mathrm{bpa}(w, \mathbf{L}_\lambda)$ terminates. Then the distance between half-coincidences (endpoints of our equivalent pairs) arising from looking at $(T, T-\lambda^n x)$ is bounded, where $x = x(w) = \mathbf{L}\cdot\mathbf{p}(w)$. Theorem 5.6 of [12] implies that the overlap algorithm associated to $x(w)$ terminates with half-coincidences.

Assume the overlap-algorithm associated to $x = \mathbf{L} \cdot \mathbf{p}(w)$ terminates with half-coincidences. Again by Theorem 5.6 of [12], the distance between half-coincidences arising from $(T, T - \lambda^n x)$ is bounded. Hence, $\mathrm{bpa}(w, \mathbf{L}_\lambda)$ terminates. □

Example 3.13 (*A non-terminating example)***.** We now give an example which will not terminate even with the extended equivalence relations presented here. This example is a rewriting of the Morse-Thue systems in which we have forced coincidences. We will use $\mathbf{L}_\lambda$ as our length vector so that we are conjugate to the original Morse-Thue system and hence do not have pure discrete spectrum. This example therefore cannot terminate for any version of the balanced pair algorithm.

The substitution is given by:

$$\begin{array}{ccl} 1 & \to & 1234 \\ 2 & \to & 124 \\ 3 & \to & 13234 \\ 4 & \to & 1324 \end{array}$$

Here, the P.F. eigenvalue is 4 with left eigenvector $(3, 2, 4, 3)$. (Note that this vector also corresponds to the $\mathbb{Z}$-action of the original Morse-Thue system.) The other important eigenvalue (the remaining two are zero) is 1 with right eigenvector $(1, 1, -2, 1)^T$. Thus, 124 is equivalent to 33; however, these words do not cluster close enough to each other to aid in terminating the balanced pair algorithm. Notice also the system generated by the length vector $(1, 1, 1, 1)$ is not conjugate to the system generated by $\mathbf{L}_\lambda$. But by

Corollary 3.7, the balanced pair algorithm will also not terminate for this length vector. In fact, using techniques from [3] to directly compute its spectrum, one can see that it will not have pure discrete spectrum either.

4. Open problems and conjectures

The motivation for studying the effects of the new equivalence relation on the balanced pair algorithm was not primarily to understand reducible substitutions, but mainly to aid in analyzing substitutions which have been collared or rewritten. We use the example below as a test case for a general Pisot substitution. Beginning with a Pisot substitution, we will rewrite it into an equivalent substitution that always begins with the same letter. In this way we force every balanced pair to lead to a coincidence. It remains to show that the new substitution will terminate in order to show that all Pisot substitutions have pure discrete spectrum. A difficulty arises in that rewriting increases the size of the alphabet and can therefore add additional eigenvalues of 0 and ± 1. The zeros do not concern us, but the roots of unity might. Considering the $\mathbb{Z}$-action on our original substitution by changing tiles to unit lengths produces an integer vector in the rewritten substitution which will not have a component in the eigenspace of the roots of unity. Further, these eigenspaces generate equivalent words so that it is our hope that the equivalence classes will always force the algorithm to terminate.

Example 4.1 (*A rewritten Pisot substitution).* Consider the substitution φ given by $a \to abb$ and $b \to ba$. Using the rewriting procedure of [1], we square φ and generate a substitution on $1 = abbb, 2 = ab, 3 = aabbb$ and $4 = aabb$ by:

$$\begin{array}{rcl} 1 & \to & 122334 \\ 2 & \to & 1224 \\ 3 & \to & 12322334 \\ 4 & \to & 1232234 \end{array}$$

The eigenvalues of the transition matrix of this substitution are $0, 3 \pm 2\sqrt{2}$ and 1. Since the original substitution is Pisot, it is insensitive to length changes in a and b. The new substitution will also be insensitive to any length change which is consistent with length changes in a and b. For example, setting $a = b = 1$ generates the length vector $\mathbf{L} = (3, 2, 5, 4)$ and will in fact miss the eigenspace associated to 1. (Note, however, that $\mathbf{L}_1 = (1,1,1,1)$ does not.) The right eigenvalue of 1, $v_1 = (1,1,-2,1)^T$, is perpendicular to $\mathbf{L}$ so that $abd \sim_{\mathbf{L}} cc$. Because $\mathbf{L}_1$ is not perpendicular to v_1, this makes the $\text{bpa}(w,\mathbf{L})$ difficult to run. Some tedious calculations reveal that $\text{bpa}(w,\mathbf{L})$, where one must keep an eye out for the equivalence relations, will terminate. This example produces 30 different irreducible balanced pairs, the longest one of which contains a word of length 11. We suspect that $\text{bpa}(w,\mathbf{L}_1)$ will not terminate, but this has also not yet been shown.

Generally, if the balanced pair algorithm terminates for a Pisot substitution, [12] gives us that the distance between half-coincidences is bounded. For the rewritten substitution, the new tiles are compositions of smaller old tiles and these half-coincidences may occur "internally." We suspect this cannot happen and that the algorithm must terminate for the new substitution as well. More precisely we have:

Conjecture 4.2. Let φ be a Pisot substitution on n-letters. Let $\tilde{\varphi}$ be a rewriting of φ so that for some letters b, e in the rewritten alphabet, $\tilde{\varphi}(i) = b \dots e$ for all i. Assume that the balanced pair algorithm for the original Pisot substitution φ terminates. Then the balanced pair algorithm for the rewritten substitution $\tilde{\varphi}$ also terminates for length vector L_λ.

An immediate corollary of this would be that if the balanced pair algorithm of a Pisot type substitution terminates, then it must do so with coincidences.

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