Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 28, No. 1, 2004
Pages 179-187

ON λ'-SETS

ARNOLD W. MILLER

Abstract. A set $X \subseteq 2^\omega$ is a λ'-set iff for every countable set $Y \subseteq 2^\omega$ there exists a G_δ set G such that $(X \cup Y) \cap G = Y$. In this paper, we prove two forcing results about λ'-sets. First, we show that it is consistent that every λ'-set is a γ-set. Second, we show that it is independent whether or not every $(\dagger)$-λ'-set is a λ'-set.

1. λ'-sets and γ-sets

A set $X \subseteq 2^\omega$ is a λ'-set iff for all countable $A \subseteq 2^\omega$ there exists a G_δ set G such that

$$(X \cup A) \cap G = A.$$

An ω-cover of X is a countable set of open sets such that every finite subset of X is contained in an element of the cover. A γ-cover of X is a countable sequence of open subsets of X such that every element of X is in all but finitely many elements of the sequence.

Define X to be a γ**-set** iff any ω-cover of X contains a γ-cover of X.

In this section we answer a question of Gary Gruenhage who asked if there is always a λ'-set which is not a γ-set. We answer this in the negative.

2000 *Mathematics Subject Classification.* 03E17, 03E35.

Key words and phrases. continuum hypothesis, forcing, independence results, sets of reals.

Thanks to the Fields Institute for Research in Mathematical Sciences at the University of Toronto for its support during the time this paper was written and to Juris Steprans who directed the special program in set theory and analysis.

It is well known (see [4]) that MA(σ-centered) implies that every set of reals of cardinality less than the continuum is a γ-set. The standard model for MA(σ-centered) (see [8]) is obtained as follows:

Suppose that M is a countable standard model of ZFC+CH and we iterate σ-centered forcings of size ω_1 in M with a finite support iteration of length ω_2. In the final model M_{ω_2}, we have that MA(σ-centered) is true and the continuum is ω_2.

Theorem 1.1. *In the standard model for MA(σ-centered) every λ' set has cardinality $\leq \omega_1$, and (it follows from MA(σ-centered)) every set of size ω_1 is a γ-set. Hence, in this model, every λ'-set is a γ-set.*

Proof: We will use the following lemma in our proof.

Lemma 1.2. *Suppose that $\mathbb{P}$ is a σ-centered forcing such that*

$$|\vdash \tau \in 2^\omega.$$

Then there exists a countable set $A \subseteq 2^\omega$ in the ground model such that for every $p \in \mathbb{P}$ and open set $U \supseteq A$ coded in the ground model there exists $q \leq p$ such that $q|\vdash \tau \in U$.

Proof: To prove the lemma we will use the following claim.

Claim. Suppose $\Sigma \subseteq \mathbb{P}$ is a centered subset. Then there exists $x \in 2^\omega$ such that for every $p \in \Sigma$ and for every $n < \omega$ there exists $q \leq p$ such that

$$q|\vdash \check{x} \upharpoonright n = \tau \upharpoonright n.$$

Proof of Claim: Otherwise, by the compactness of 2^ω, there exists a finite set

$$\{p_m : m < N\} \subseteq \Sigma \text{ and } \{s_m : m < N\} \subseteq 2^{<\omega}$$

such that $\{[s_m] : m < N\}$ covers 2^ω and for each $m < N$ we have that

$$p_m|\vdash \tau \notin [s_m].$$

But this is a contradiction since there exists some $p \in \mathbb{P}$ below all of the p_m. This proves the Claim. □

Let $\mathbb{P} = \bigcup_{n<\omega} \Sigma_n$ be a sequence of centered sets. Then for each n there exists $x_n \in 2^\omega$ such that for every $p \in \Sigma_n$ and for every $m \in \omega$ there exists $q \leq p$ such that

$$q|\vdash \check{x}_n \upharpoonright m = \tau \upharpoonright m.$$

Now let $A = \{x_n : n < \omega\}$. This proves the Lemma. $\square$

Suppose $X \subseteq 2^\omega$ is a λ'-set in M_{ω_2}. For each $\alpha \leq \omega_2$ define

$$\mathbf{X}_\alpha = X \cap M_\alpha.$$

By a standard Lowenheim-Skolem argument, we can find $\alpha < \omega_2$ such that

(1) $X_\alpha \in M_\alpha$ and
(2) for every countable $A \subseteq 2^\omega$ which is in M_α there exists a G_δ-set G coded in M_α such that

$$(X_{\omega_2} \cup A) \cap G = A.$$

We claim that $X = X_{\omega_2} = X_\alpha$ and hence has cardinality $\leq \omega_1$. Suppose that τ is any term for an element of 2^ω in M_{ω_2}. Since τ is added at some latter stage β with $\alpha \leq \beta < \omega_2$ and the iteration of σ-centered forcings of length $< \omega_2$ is σ-centered, it follows that τ is added by a σ-centered forcing over M_α. Let $A \subseteq 2^\omega$ be the countable set given by Lemma 1.2. By the Lemma it follows that τ must be an element of any G_δ set coded in M_α which contains A. Using item (2) above, we see that τ must be in A if it is in X_{ω_2}. Therefore, $X_{\omega_2} \setminus X_\alpha = \emptyset$, and the proof of Theorem 1.1 is complete. $\square$

Remark. This argument is similar to the proof that there are no λ'-sets of size ω_2 in R. Laver's model; (see A. W. Miller [12]).

Remark. A set of reals X is a λ-set iff every countable subset of X is a relative G_δ. In ZFC we must always have a λ-set which is not a γ-set. To see this let

$$X = \{f_\alpha \in \omega^\omega : \alpha < \mathfrak{b}\}$$

be well-ordered by eventual dominance and unbounded. F. Rothberger [15] (or see Miller [11]) showed that X is a λ-set. However, X is not a γ-set as is witnessed by the sequences of ω-covers

$$\mathcal{U}_m = \{U_n^m : n \in \omega\} \text{ where } U_n^m = \{f \in \omega^\omega : f(m) < n\}.$$

In fact, the set X is a λ'-set with respect to ω^ω.

Remark. A Hausdorff gap is an example of a λ' set of cardinality ω_1. γ-sets have strong measure zero and Laver [9] proved that it is consistent that every strong measure zero set is countable.

Suppose there exists $X, Y \subseteq 2^\omega$ such that $|X| = |Y|$ and X is a λ'-set and Y is not a γ-set. Then there exists Z which is a λ'-set and not a γ-set. To see this let $X = \{x_\alpha : \alpha < \kappa\}$ and $Y = \{y_\alpha : \alpha < \kappa\}$. Put $Z = \{(x_\alpha, y_\alpha) : \alpha < \kappa\}$. The first κ for which MA(σ-centered) fails is $\mathfrak{p}$ [1], and $\mathfrak{p}$ is also the size of the smallest non γ-set. Hence, any model where every λ'-set is γ-set and $\mathfrak{c} \leq \omega_2$ must satisfy MA(σ-centered) and $\mathfrak{c} = \omega_2$.

Remark. G. Gruenhage and P. Szeptycki [7] were interested in obtaining a set of reals $X \subseteq 2^\omega$ which is γ-set and not a λ'-set because of the following two topological games.

Let X be a topological space and $x \in X$ and define the following games:

$G_{\mathcal{O},P}(X, x)$: On round n, player $\mathcal{O}$ chooses an open neighborhood U_n of x and player $\mathcal{P}$ chooses a point $p_n \in U_n$. Player $\mathcal{O}$ wins iff the sequence p_n converges to x.

$G^{fin}_{\mathcal{O},P}(X, x)$: The same except we allow player $\mathcal{P}$ to choose a finite set of points $P_n \subseteq U_n$ on his move and $\mathcal{O}$ wins iff $\cup_{n<\omega} P_n$ converges to x.

It is not hard to check that player $\mathcal{O}$ has a winning strategy in $G_{\mathcal{O},P}(X, x)$ iff player $\mathcal{O}$ has a winning strategy in $G^{fin}_{\mathcal{O},P}(X, x)$. Also, if player $\mathcal{P}$ has a winning strategy in $G_{\mathcal{O},P}(X, x)$, then it is a winning strategy in $G^{fin}_{\mathcal{O},P}(X, x)$.

Given $X \subseteq 2^\omega$, consider the topology on $2^{<\omega} \cup \infty$ generated by

(1) $\{\sigma\}$ for each $\sigma \in 2^{<\omega}$ and
(2) $\{\infty\} \cup (2^{<\omega} \setminus \{x \restriction n : n < \omega\})$ for each $x \in X$.

Let X_F denote this countable topological space. Gruenhage [5], P. Nyikos [14], P. L. Sharma [16], and Gruenhage and Szeptycki [7] can be combined to show that:

X is not a γ-set iff player $\mathcal{P}$ has a winning strategy in $G^{f}_{\mathcal{O},P}(X_F, \infty)$.
If X is a λ'-set, then $\mathcal{P}$ has no winning strategy in $G_{\mathcal{O},P}(X_F, \infty)$.

Hence, if there is a set X which is a λ'-set and not a γ-set, then $\mathcal{P}$ has a winning strategy in $G^{f}_{\mathcal{O},P}(X_F, \infty)$ but not in $G_{\mathcal{O},P}(X_F, \infty)$.

D. K. Ma [10] has a clearer proof of the connection between γ-sets and such games.

A. Dow's [2] results imply that in Laver's model [9]:

X is a λ'-set iff $\mathcal{P}$ has no winning strategy in $G_{\mathcal{O},P}(X_F,\infty)$.

But it is also consistent that they are not the same. In F. Galvin and Miller [3], it is shown that assuming MA(σ-centered) there is a γ-set X which is concentrated on a countable subset of itself. Hence, $\mathcal{P}$ has no winning strategy in $G^f_{\mathcal{O},P}(X_F,\infty)$, hence none in $G_{\mathcal{O},P}(X_F,\infty)$, but X is not a λ'-set.

Question 1.3. *Is it consistent with ZFC that for every $X \subseteq 2^\omega$ that*

$\mathcal{P}$ has no winning strategy in $G_{\mathcal{O},P}(X_F,\infty)$

iff

$\mathcal{P}$ has no winning strategy in $G^f_{\mathcal{O},P}(X_F,\infty)$?

After the first version of this paper was written, Gruenhage [6] constructed an example of a countable space (in ZFC using a gap construction) which distinguishes the two games.

2. (†)-λ'-SET

In this section we answer Problem 2.12 from A. Nowik and T. Weiss [13], which asks basically whether it is true that every (†)-λ'-set is a λ'-set.

Definition. For any $a \in [\omega]^\omega$, let $a = \{a_0, a_1, \ldots\}$ be its increasing enumeration, then for any $f \in \omega^\omega$, let

$$G_f = \{a \in [\omega]^\omega \subseteq 2^\omega \ : \ \forall n \ \exists m > n \ \ a_n < f(n)\}.$$

Definition. A set $X \subseteq 2^\omega$ is a **(†)-λ'-set** iff for every $f \in \omega^\omega$ we have $X \cap G_f$ is a λ'-set.

Theorem 2.1. *Suppose that the continuum hypothesis is true or even just $\mathfrak{b} = \mathfrak{d}$. Then there exists a (†)-λ'-set which is not a λ'-set.*

Theorem 2.2. *In the Cohen real model (Cohen's original model for not CH), every (†)-λ'-set is a λ'-set.*

Proof of Theorem 2.1: Assume CH. Let $\{f_\alpha \in \omega^\omega : \alpha < \omega_1\}$ be a scale. That is, for $\alpha < \beta$ we have that $f_\alpha <^* f_\beta$ and for all $g \in \omega^\omega$ there exists $\alpha < \omega_1$ such that $g <^* f_\alpha$. We may also assume that the f_α are strictly increasing. Let $X \subseteq [\omega]^\omega$ be the set of ranges of the elements of the scale. Then for any $g \in \omega^\omega$ we have that $G_g \cap X$ is countable and hence a λ'-set. On the other hand, X is not a λ'-set because of the countable set $[\omega]^{<\omega}$. If $U \subseteq P(\omega)$ is an open set containing $[\omega]^{<\omega}$, then $K = P(\omega) \setminus U$ is a compact subset of $[\omega]^\omega$. If we identify $[\omega]^\omega$ with the strictly increasing elements of ω^ω (via the homeomorphism $a \mapsto \{a_0, a_1, \ldots\}$), then there exists $f \in \omega^\omega$ such that for all $g \in K$ we have $\forall n \;\, g(n) < f(n)$. It follows that for all but countably many α we have that the range$(f_\alpha) \in U$.

The proof using $\mathfrak{b} = \mathfrak{d}$ is similar. Start with a scale indexed by $\mathfrak{b}$ and note that any set $Y \subseteq P(\omega)$ of size less than $\mathfrak{b}$ is a λ'-set (this is due to Rothberger; see the proof of Lemma 2.4). □

Proof of Theorem 2.2: Assume that M is a countable transitive standard model of ZFC+CH.

For any $\alpha \leq \omega_2^M$, let $\mathbb{P}_\alpha$ be the finite partial functions from α into 2. We claim that for any G, a $\mathbb{P}_{\omega_2}$-generic filter over M, in the model $M[G]$ every (†)-λ'-set is a λ'-set. In order to prove this claim, we first prove Lemma 2.3 and Lemma 2.4.

Lemma 2.3. *Suppose N is a countable standard model of ZFC+CH, $\mathbb{P}$ is a countable poset in N, and*

$$N \models X \subseteq \omega^\omega \textit{ is unbounded in } \leq^* .$$

Then for any G which is $\mathbb{P}$-generic over N, we have that

$$N[G] \models X \textit{ is unbounded in } \leq^* .$$

Proof: Let $\{g_\alpha : \alpha < \omega_1^N\}$ be a scale in N. Working in N choose $f_\alpha \in X$ so that

$$\exists^\infty n \;\; f_\alpha(n) > g_\alpha(n).$$

Note that for every $g \in \omega^\omega \cap N$ there exists $\alpha < \omega_1$ such that

$$\forall \beta > \alpha \;\; \exists^\infty n \;\; f_\beta(n) > g(n).$$

Suppose by way of contradiction that for some $g \in N[G] \cap \omega^\omega$ and all $\alpha < \omega_1$ we have that $f_\alpha \leq^* g$. Then for some $\Sigma \in [\omega_1]^{\omega_1}$ and $n < \omega$ we have that

$$\forall m > n \;\; \forall \alpha \in \Sigma \;\; f_\alpha(m) \leq g(m).$$

Let $q \in G$ force this fact. Now since $\mathbb{P}$ is a countable poset, there exists some $p \in G$ with $p \leq q$ such that

$$\Gamma = \{\alpha < \omega_1 : p | \!\vdash \alpha \in \dot{\Sigma}\}$$

is uncountable (and by definability of forcing it is in N). But note that $\{f_\alpha : \alpha \in \Gamma\}$ is unbounded and so for some $m > n$ the set $\{f_\alpha(m) : \alpha \in \Gamma\}$ is unbounded in ω.

Let $r \leq p$ decide $g(m)$; i.e., for some $k < \omega$, suppose

$$r \Vdash \dot{g}(m) = k.$$

Choose $\alpha \in \Gamma$ such that $f_\alpha(m) > k$; then r forces a contradiction and the Lemma is proved. □

Lemma 2.4. *Suppose N is a countable standard model of ZFC+CH, $\mathbb{P}$ is a countable poset in N, and*

$$N \models Y \subseteq 2^\omega \text{ is not a } \lambda' \text{ - set.}$$

Then for G $\mathbb{P}$-generic over N we have that

$$N[G] \models Y \text{ is not a } \lambda' \text{ - set.}$$

Proof: Let $D \subseteq 2^\omega$ be countable in N and witness that Y is not a λ'-set, i.e., there is no G_δ set $\bigcap_n U_n$ coded in N with

$$\bigcap_n U_n \cap (Y \cup D) = D.$$

Working in N, let $D = \{x_n : n < \omega\}$ and let $Z = Y \setminus D$ and for each $z \in Z$, define $f_z \in \omega^\omega$ such that $f_z(n)$ is the least m such that $x_n \restriction m \neq z \restriction m$. Now the family $X = \{f_z : z \in Z\}$ must be unbounded in $\leq^*$ in N. Suppose not, then there exists $g \in \omega^\omega \cap N$ which eventually dominates each element of X. It follows that if we let

$$U_n = \bigcup_{m<\omega} [x_m \restriction \max\{n, g(m)\}]$$

then

$$(\bigcap_{n<\omega} U_n) \cap (Y \cup D) = D,$$

which is a contradiction.

It follows from Lemma 2.3 that X is unbounded in $N[G]$. We claim that D cannot be G_δ in $Y \cup D$ in the model $N[G]$. Suppose it is, and let $\bigcap_{n<\omega} U_n$ be a G_δ in $N[G]$ such that

$$\bigcap_{n<\omega} U_n \cap (Y \cup D) = D.$$

For each n, let $g_n \in \omega^\omega$ be such that for every m we have that

$$[x_m \restriction g_n(m)] \subseteq U_n.$$

Now for any $z \in Z$ there exists an n such that $z \notin U_n$. But this means that $f_z(m) \leq g_n(m)$ for every m since otherwise

$$x_m \upharpoonright g_n(m) = z \upharpoonright g_n(m),$$

and then $z \in U_n$. This proves the Lemma. $\square$

Now we complete the proof of Theorem 2.2. Suppose that $X \subseteq 2^\omega$ is in $M[G]$ where G is $\mathbb{P}_{\omega_2}$-generic over M and

$$M[G] \models X \text{ is not a } \lambda'\text{-set.}$$

By Lowenheim-Skolem arguments, there exists $\alpha < \omega_2$ such that

$$X_\alpha =^{def} X \cap M[G_\alpha], \;\; X_\alpha \in M[G_\alpha], \text{ and } M[G_\alpha] \models X_\alpha \text{ is not a } \lambda'\text{-set.}$$

Since being a λ'-set only depends on codes for G_δ-sets and reals are added by countable suborders of $\mathbb{P}_{[\alpha,\omega_2)}$, it follows from Lemma 2.4 that

$$M[G] \models X_\alpha \text{ is not a } \lambda'\text{-set.}$$

But if $f \in \omega^\omega \in M[G]$ is $\omega^{<\omega}$-generic over $M[G_\alpha]$, then $X_\alpha \subseteq G_f$. It follows that

$$M[G] \models X \text{ is not } (\dagger)\text{-}\lambda'\text{-set}$$

as was to be proved. $\square$

References

[1] M. G. Bell, *On the combinatorial principle $P(\mathfrak{c})$*, Fund. Math. **114** (1981), no. 2, 149–157.

[2] A. Dow, *Two Classes of Fréchet-Urysohn Spaces*, Proc. Amer. Math. Soc. **108** (1990), no. 1, 241–247.

[3] F. Galvin and A. W. Miller, *γ-sets and other singular sets of real numbers*, Topology Appl. **17** (1984), no. 2, 145–155.

[4] J. Gerlits and Zs. Nagy, *Some properties of $C(X)$, I*, Topology Appl. **14** (1982), no. 2, 151–161.

[5] G. Gruenhage, *Infinite games and generalizations of first-countable spaces*, General Topology and Appl. **6** (1976), no. 3, 339–352.

[6] G. Gruenhage, *The story of a topological game*, eprint 1-2003. To appear in Rocky Mountain Journal of Mathematics.

[7] G. Gruenhage and P. Szeptycki, *Fréchet-Urysohn for finite sets*, eprint 12-2002. To appear in Topology and its Applications.

[8] K. Kunen and F. D. Tall, *Between Martin's axiom and Souslin's hypothesis*, Fund. Math. **102** (1979), no. 3, 173–181.

[9] R. Laver, *On the consistency of Borel's conjecture*, Acta Math. **137** (1976), no. 3-4, 151–169.

[10] D. K. Ma, *The Cantor tree, the γ-property, and Baire function spaces*, Proc. Amer. Math. Soc. **119** (1993), no. 3, 903–913.

[11] A. W. Miller, *Special subsets of the real line*, in Handbook of Set-Theoretic Topology, Ed. K. Kunen and J. E. Vaughan. Amsterdam: North-Holland, 1984. pp. 201–233.

[12] A. W. Miller, *Special sets of reals*, in Set Theory of the Reals (Ramat Gan, 1991), Israel Math. Conf. Proc., 6. Ramat Gan: Bar-Ilan Univ. Press, 1993. pp. 415–431

[13] A. Nowik and T. Weiss, *Some remarks on totally imperfect sets.* To appear.

[14] P. Nyikos, *The Cantor tree and the Fréchet-Urysohn property*, in Papers on General Topology and Related Category Theory and Topological Algebra (New York, 1985/1987), Ann. New York Acad. Sci., 552. New York: New York Acad. Sci., 1989. pp. 109–123.

[15] F. Rothberger, *Sur les familles indénombrables de suites de nombres naturels et les problemes concernant la propriété C*, (French). Proc. Cambridge Philos. Soc. **37** (1941), 109–126.

[16] P. L. Sharma, *Some characterizations of W-spaces and w-spaces*, General Topology Appl. **9** (1978), no. 3, 289–293.

University of Wisconsin-Madison; Department of Mathematics; Van Vleck Hall; 480 Lincoln Drive; Madison, Wisconsin 53706-1388

http://www.math.wisc.edu/~miller

E-mail address: miller@math.wisc.edu