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INCOMPRESSIBILITY OF TORUS TRANSVERSE TO VECTOR FIELDS

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ABSTRACT. We give sufficient conditions for a torus T embedded in a closed orientable 3-manifold M to be *incompressible*; namely, the homomorphism $\pi_1(T) \rightarrow \pi_1(M)$ induced by the inclusion is injective. The motivation is the well known fact that T is incompressible (and M is irreducible) if T is transverse to an Anosov vector field [3], [4]. Here we still assume that T is transverse to a vector field X , but we don't assume that X is Anosov. Instead, we assume that X exhibits a unique orbit O which does not intersect T . If, in addition, O is hyperbolic and *not* null homotopic in M , then T is incompressible and M is irreducible.

1. INTRODUCTION

In this paper, we give sufficient conditions for a torus T embedded on a closed 3-manifold M to be *incompressible*; namely, the homomorphism $\pi_1(T) \rightarrow \pi_1(M)$ induced by the inclusion is injective.

We have two motivations for this investigation. The first one is the importance of an incompressible torus in 3-manifold topology expressed in the Jaco-Shalen-Johannson Theory [9], [10]. The second one is the known fact that T is incompressible (and M is irreducible) if T is transverse to an Anosov vector field X (see [3], [4]).

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Remember, a 3-manifold is *irreducible* if any embedded sphere on it bounds a 3-ball; and a vector field is Anosov if it exhibits contracting and expanding invariant directions which (together with the flow's direction) forms a continuous tangent bundle decomposition. In this paper we still assume that T is transverse to a vector field X , but we don't assume that X is Anosov. Under some additional conditions we prove that T is incompressible and M is irreducible. Recall that a closed orbit of a C^1 vector field is *hyperbolic* if its associated Poincaré map has no modulus one eigenvalues [15].

Theorem 1.1. *Let T be an embedded torus on a closed orientable 3-manifold M . Suppose that*

- (1) *T is transverse to a C^1 vector field X in M ;*
- (2) *there is a unique orbit O of X which does not intersect T ;*
- (3) *O is hyperbolic;*
- (4) *O is not null homotopic in M .*

Then, T is incompressible and M is irreducible.

An example of an embedded torus satisfying (1)-(4) above can be found in [2]. The proof of Theorem 1.1 is as follows. Let M, T, X be as in the statement. In section 2, we prove that T is incompressible if M is irreducible. More than this, it is proved that if the union of the orbits of X which do not intersect T is connected and M is irreducible, then T is incompressible. In section 3, we prove that M is irreducible and therefore Theorem 1.1 follows. Note that the theory of codimension one foliations plays no role here since the vector fields under consideration are not Anosov. Instead, we use some well known properties of 3-manifolds [6], [7] and a lemma in [11]. It is possible that similar arguments and [14] can be used to prove that a manifold as in Theorem 1.1 is a BL-manifold (as defined in [1]) provided the closed orbit O has positive eigenvalues.

The author would like to thank the referee by the reference [6] for Lemma 3.2 and the shorter proof of Lemma 3.4, and for suggesting Example 3.

2. TRANSVERSE TORUS ON IRREDUCIBLE 3-MANIFOLDS

In this section, we give a simple sufficient condition for a torus transverse to a C^1 vector field on a closed irreducible 3-manifold to

be incompressible. Let us introduce some definitions and notations. Hereafter, X_t denotes the flow of a C^1 vector field X on a closed 3-manifold M . Denote by $\Omega(X)$ the nonwandering set of X . Recall that $x \in \Omega(X)$ iff for every $T > 0$ and every neighborhood U of x there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. If $x \in M$, we denote by $O_X(x) = \{X_t(x) : t \in \mathbf{R}\}$ the X -orbit of x . In addition, we denote by $\omega_X(x)$ the set of $y \in M$ such that $y = \lim_{n \rightarrow \infty} X_{t_n}(x)$ for some sequence $t_n \rightarrow \infty$. We denote $\alpha_X(x) = \omega_{-X}(x)$, where $-X$ is the time-reversed flow of X ($\omega_X(x)$ and $\alpha_X(x)$ are called the ω -limit set and the α -limit set of x , respectively).

If A is a subset of a 3-manifold N we denote by $\text{int}(A)$ and $\text{Cl}(A)$ the interior and the closure of A in N , respectively. Let S be a surface embedded in N . We say that S is *2-sided* if there is an embedding $h : S \times [-1, 1] \rightarrow N$ such that $h(x, 0) = x$ (see [7, p. 14]). A surface S *separates* N if $N \setminus S$ is not connected. We say that S is *properly embedded* in N if S is embedded in M and $\partial N \cap S = \partial S$. A solid torus will be a compact 3-manifold homeomorphic to $(\text{two-disk}) \times S^1$. Clearly, a torus bounding a solid torus separates N . A surface S is *transverse to a vector field X in N* if $X(x) \notin T_x S$, for every $x \in S$. Note that a surface transverse to a C^1 vector field is necessarily 2-sided. The result of this section is the following.

Theorem 2.1. *Let T be a torus transverse to a C^1 vector field X on a closed irreducible 3-manifold M . If the union of the orbits of X which does not intersect T is connected in M , then T is incompressible.*

The proof will be consequence of the following two lemmas.

Lemma 2.2. *If M is a closed irreducible 3-manifold and T is a 2-sided embedded torus in M , then T either is incompressible or separates M .*

Proof: Assume that T is not incompressible. As M is irreducible and T is 2-sided it follows that T either bounds a solid torus or is contained in a 3-ball B in M (see [6]). In the first case, T separates M and we are done. In the second case, it is easy to prove that T also separates M . The proof goes as follows. Capping B with a 3-ball B' we obtain a manifold H which is diffeomorphic to S^3 and contains T . It follows from the Solid Torus Theorem ([12, p. 107])

that T separates $H \approx S^3$. Denote by H_1, H_2 the two connected components of $H \setminus T$. Since $T \subset \text{Int}(B)$ we have that $B' \cap T = \emptyset$. Hence, we can assume that $B' \subset \text{int}(H_1)$ and then $\partial(H_1 \setminus B') = T \cup \partial B$ because $\partial B' = \partial B$. In addition, $\partial(M \setminus B) = \partial B$. So, we can glue $H_1 \setminus B'$ and $M \setminus B$ along ∂B in order to obtain a connected 3-manifold A with boundary $\partial A = T$. Similarly H_2 is connected and satisfies $\partial H_2 = T$. Gluing A and H_2 along T in a suitable way yields M . From this it follows that $M \setminus T$ has two components, A and H_2 . So, T separates M in this case as well. The lemma is proved. \square

If S is a surface transverse to a C^1 vector field X , we denote by σ_S the union of the orbits of X which does not intersect S , i.e.,

$$\sigma_S = \{x \in M : O_X(x) \cap S = \emptyset\}.$$

Lemma 2.3. *Let T be a torus transverse to a C^1 vector field X on a closed 3-manifold M . If σ_T is connected, then T does not separate M .*

Proof: We can assume that $\sigma_T \neq \emptyset$ without loss of generality. Assume by contradiction that T separates M and denote by A, B the connected components of $M \setminus T$. We can assume that X points inward to A in $T = \partial A$ and outward to B in $S = \partial B$. As $M - T$ is not connected, it follows that a positive orbit with initial point in T cannot return to T . Similarly for the negative orbits. From this we conclude that $\Omega(X) \subset \sigma_T$, and so, $\omega_X(p) \cup \alpha_X(p) \subset \sigma_T$ for every $p \in M$ (this is true in particular when $p \in T$). Fix $p \in T$. On one hand, $\omega_X(p) \subset \text{int}(A)$ since X points inward to A at T . So $\sigma_T \cap \text{int}(A) \neq \emptyset$. On the other hand, $\alpha_X(p) \subset \text{int}(B)$ since X points outward to B at $T = \partial B$. So $\sigma_T \cap \text{int}(B) \neq \emptyset$. It would follow that σ_T is not connected, a contradiction. The result follows. \square

Proof of Theorem 2.1: Direct from lemmas 2.2 and 2.3. \square

Example 1. The conclusion of Theorem 2.1 fails if σ_T were not connected. Indeed, consider $M = S^3$ (which is irreducible) as the union of two solid tori ST_1, ST_2 glued along their boundary tori T . In each solid torus, we set a periodic orbit in a way that the flow goes from one solid torus to the another. The resulting vector field X has a transverse torus T such that σ_T is both non-empty and non-connected. The desired counterexample is then obtained because $M = S^3$ has no incompressible torus.

Example 2. It is easy to see that every torus bundle over S^1 supports a C^1 vector field X with a transverse torus T satisfying the hypothesis of Theorem 2.1. See Figure 1. Here σ_T is the non-hyperbolic singularity indicated in the figure. This example motivates the question whether, for every torus bundle over S^1 , there is a C^1 vector field X with a transverse torus T such that σ_T is a *non-trivial hyperbolic basic set*. The answer is positive by the following proposition which is interesting by itself. Recall that a compact invariant set Λ of a C^1 vector field X is *transitive* if it is the ω -limit set of one of its points, and *non-trivial* if it is not a single orbit of X . In addition, Λ is *hyperbolic* if the tangent bundle over Λ admits an invariant splitting $E^s \oplus E^X \oplus E^u$ such that E^s is contracted by X , E^u is expanded by X , and E^X is the direction of X . A hyperbolic set is *basic* if it is transitive and *isolated*, i.e., there is a compact neighborhood U of Λ such that $\Lambda = \cap_{t \in \mathbb{R}} X_t(U)$.

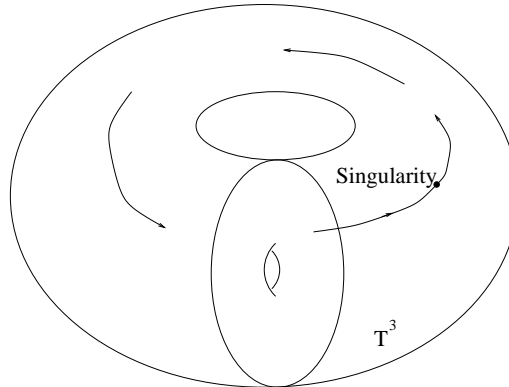


FIGURE 1.

Proposition 2.4. *Every torus bundle over S^1 exhibits a C^1 vector field X with a transverse torus T such that σ_T is a non-trivial hyperbolic basic set.*

Proof: Consider the classical Smale's diffeomorphism f_0 in the 2-sphere S^2 [15]. The non-wandering set of f_0 consists of an attracting fixed point o_1 , a repelling fixed point o_2 , and a hyperbolic horseshoe Λ_0 . The suspension of f_0 yields a C^1 vector field X^0 in

$S^2 \times S^1$ whose non-wandering set consists of an attracting closed orbit O_1 , a repelling closed orbit O_2 , and a non-trivial hyperbolic basic set Λ (the suspension of Λ_0). Remove from $S^2 \times S^1$ two solid tori neighborhoods ST_1 and ST_2 around O_1 and O_2 , respectively, so that X^0 points inward in ST_2 and outward in ST_1 . (This is similar to the construction of the Anomalous Anosov flow [5].) The resulting manifold M_1 is diffeomorphic to $T^2 \times I$ with the boundary tori of $T^2 \times I$ identified with the boundary tori of ST_1 and ST_2 respectively. The vector field X^0 induces a vector field X^1 in M_1 which is transverse to the boundary tori of M_1 . On one hand, every torus bundle over S^1 can be obtained by identifying the two boundary tori of M_1 . On the other hand, the vector field X^1 produces (on any of such identifications) a C^1 vector field X with a transverse torus T satisfying the conclusion of the proposition with $\sigma_T = \Lambda$. The proof follows. \square

Corollaire 0.1 in [2] shows that none of the vector fields X in the above proof can be chosen to be Anosov.

3. PROOF OF THEOREM 1.1

We start with the following standard definition. Let X be a C^1 vector field on a closed 3-manifold M . Recall that a periodic orbit O of X is *hyperbolic* if the Poincaré return map associated to O has no eigenvalues with modulus one. The proof of Theorem 1.1 is based on the following theorem which is the main result of this section.

Theorem 3.1. *Let M be a closed orientable 3-manifold. Suppose that M exhibits a C^1 vector field X with a transverse torus T satisfying the following properties:*

- (1) *There is a unique orbit O of X which does not intersect T .*
- (2) *O is hyperbolic and not null-homotopic in M .*

Then, M is irreducible.

We use three lemmas to prove Theorem 3.1. The first one is standard in 3-manifold topology [6] (or [7]).

Lemma 3.2. *Let M be a closed 3-manifold. Suppose that M exhibits an embedded non-separating torus T such that the manifold M_0 obtained by cutting open M along T is irreducible. Then, M is irreducible.*

Now, let O be a hyperbolic periodic orbit of a vector field X . We say that O is *saddle-type* if it is neither attracting nor repelling; (i.e., its associated Poincaré map has at least one eigenvalue with modulus > 1 and at least one eigenvalue with modulus < 1).

Lemma 3.3 ([11]). *Let M be a closed 3-manifold and let X be a C^1 vector field with a transverse torus T . Suppose that there is a unique orbit O of X which does not intersect T . If O is hyperbolic, then O is saddle-type.*

The next lemma uses the following notations and facts. Let O be a hyperbolic periodic orbit of a C^1 vector field X . It follows from the stable manifold theory [8] that the sets

$$W_X^s(O) = \{x \in M : X_t(x) \rightarrow O, t \rightarrow \infty\}$$

and

$$W_X^u(O) = \{x \in M : X_t(x) \rightarrow O, t \rightarrow -\infty\}$$

are C^1 submanifolds of M . If X is three-dimensional and O is saddle-type, then both $W_X^s(O)$ and $W_X^u(O)$ are two-dimensional.

Lemma 3.4. *Let M_0 be a compact 3-manifold whose boundary ∂M_0 consists of two tori T_1, T_2 . Suppose that there is a C^1 vector field Y transverse to ∂M_0 , exhibiting a unique orbit O which does not intersect ∂M_0 and satisfying:*

- (1) Y points inward in T_1 and outward in T_2 .
- (2) O is a hyperbolic saddle-type periodic orbit.
- (3) O is not null-homotopic in M_0 .

Then, M_0 is irreducible.

Proof: By hypothesis (1), the vector field Y induces a transition map $\Xi : \text{Dom}(\Xi) \subset T_1 \rightarrow T_2$, where $\text{Dom}(\Xi)$ denotes the domain of Ξ . Denote by $\text{Im}(\Xi)$ the image of Ξ .

Let us describe $\text{Dom}(\Xi)$ and $\text{Im}(\Xi)$. O is a saddle-type periodic orbit by hypothesis (2). Let $W_Y^s(O)$ and $W_Y^u(O)$ be the stable and unstable manifold of O , respectively. We shall assume that the eigenvalues of O are positive (the negative case is similar). In this case $W_Y^s(O)$ and $W_Y^u(O)$ are properly embedded annuli which for simplicity will be denoted by A^s and A^u , respectively. Let C_1^s, C_2^s and C_1^u, C_2^u be the boundary circles of A^s and A^u . If $x \in T_1 \setminus \text{Dom}(\Xi)$, then the positive orbit of x by X does not intersect T_2 and so $T \setminus \text{Dom}(\Xi) \subset W_X^s(O)$. This proves $T \setminus \text{Dom}(\Xi) \subset C_1^s \cup C_2^s$.

The reversed inclusion is obvious and then $T \setminus \text{Dom}(\Xi) = C_1^s \cup C_2^s$. Similarly, $T \setminus \text{Im}(\Xi) = C_1^u \cup C_2^u$. This finishes the description of $\text{Dom}(\Xi)$ and $\text{Im}(\Xi)$.

By the above description, we can fix two disjoint small annulus neighborhoods R_1^s, R_2^s in T_1 of C_1^s, C_2^s , respectively, and two disjoint small annulus neighborhoods R_1^u, R_2^u in T_2 of C_1^u, C_2^u , respectively, satisfying $T_1 \setminus (R_1^s \cup R_2^s) \subset \text{Dom}(\Xi)$ and

$$(3.1) \quad \Xi(T_1 \setminus (R_1^s \cup R_2^s)) = T_2 \setminus (R_1^u \cup R_2^u).$$

Due to hypothesis (3), the curves $\{C_1^s, C_2^s\}$ and $\{C_1^u, C_2^u\}$ cannot be null-homotopic in T_1 and T_2 , respectively. Then, by shrinking $\{R_1^s, R_2^s\}$ and $\{R_1^u, R_2^u\}$ respectively, the annuli $\{R_1^s, R_2^s\}$ and $\{R_1^u, R_2^u\}$ can be chosen to be non-null-homotopic in T_1 and T_2 , respectively. It follows that $\{R_1^s, R_2^s\}$ and $\{R_1^u, R_2^u\}$ are parallel in T_1 and T_2 , respectively.

Now, the union of the saturation of $R_1^s \cup R_2^s$ (by the flow of Y) and the annulus A^u is a solid torus ST with core O . This solid torus has eight annuli in the boundary, corresponding to four annuli $\{R_1^s, R_2^s, R_1^u, R_2^u\}$ in the boundary of M_0 and the other four in the interior. According to the hypothesis, the interior annuli are incompressible in the original manifold. The remaining pieces are annuli in M_0 being saturated by the flow. Each saturation is a solid torus. So there are 3 solid tori being glued along annuli on their boundaries. The glueing annuli are incompressible in the resulting manifold, so M_0 is irreducible. \square

Proof of Theorem 3.1: Let M, X, T , and O be as in the statement. Since there is a unique orbit O of X which does not intersect T we have that $\sigma_T = O$, and so σ_T is connected. It follows from Lemma 2.3 that T does not separate M . Then, to prove Theorem 3.1, it suffices by Lemma 3.2 to prove that the manifold M_0 obtained by cutting open M along T is irreducible. To prove that M_0 is irreducible, we observe that M_0 is a compact, connected 3-manifold whose boundary ∂M_0 consists of two tori, T_1, T_2 . Let Y be the vector field induced by X in M_0 . Clearly, O is the unique orbit of Y which does not intersect ∂M_0 . We claim that Y satisfies the properties (1)-(3) of Lemma 3.4. Indeed, it follows from the definition that Y is inward in T_1 and outward in T_2 , proving (1). In addition, O is hyperbolic (by hypothesis) saddle-type (by Lemma 3.3) and periodic since it is not null-homotopic. Hence,

(2) holds. Obviously, O is not null-homotopic in M_0 since it is not null-homotopic in M . Hence, (3) holds. We conclude by Lemma 3.4 that M_0 is irreducible and the proof follows. \square

Proof of Theorem 1.1: Let T, X, M, O be as in the statement. On one hand, T, X, M, O satisfy the hypothesis of Theorem 3.1 and so M is irreducible. On the other hand, since there is a unique orbit O which does not intersect T , we have $\sigma_T = O$, and so, σ_T is connected. As M is irreducible, it follows from Theorem 2.1 that T is incompressible and the proof follows. \square

Remark. The referee suggested the following example to show that Theorem 1.1 is false without hypothesis (4).

Example 3. Consider the notation in the proof of Lemma 3.4. Suppose C_1^s, C_2^s are null homotopic circles in T_1 and that the disk that C_1^s bounds in T_1 contains C_2^s . Then C_1^s, C_2^s split T_1 into 3 pieces: one is a disk bounded by C_2^s , another is an annulus from C_2^s to C_1^s , and the last is a torus minus a disk. Let E_1, E_2, E_3 be the corresponding 3 dimensional saturations by some flow (for instance, the one in Figure 2). E_1 is a ball, E_2 is a solid torus, and E_3 is (torus - disk) times interval. Gluing the boundary tori T_1, T_2 , we obtain a manifold M with a flow X such that the torus $T \approx T_1 \approx T_2$ is transverse to X . Let O be the periodic orbit of the flow described in the figure. Clearly, O is both hyperbolic and the

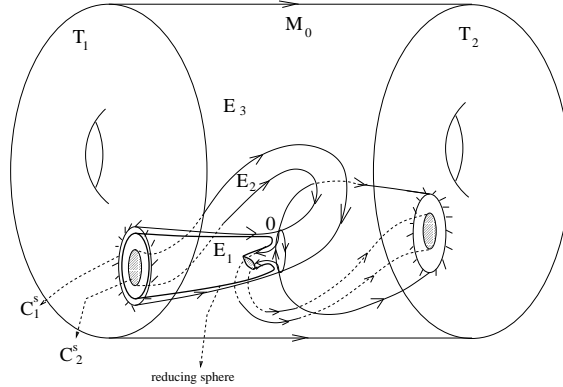


FIGURE 2.

unique orbit of X which does not intersect T . In addition, O is *null homotopic* in M . Although T is incompressible, we have that M is *not irreducible* since a reducing sphere can be constructed starting with an annulus in E_2 and capping off with two disks, each with a part in the solid torus and a subdisk in E_1 .

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