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SEQUENTIAL DECREASING WHITNEY PROPERTIES II

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ABSTRACT. Let X be a metric continuum and $C(X)$ the hyperspace of subcontinua of X. A topolological property $\mathcal{P}$ is said to be a *sequential decreasing Whitney property* provided that if μ is a Whitney map for $C(X)$, $\{t_n\}_{n=1}^{\infty}$ is a sequence in the interval $(t, 1)$ such that $t_n \to t$ and each fiber $\mu^{-1}(t_n)$ has property $\mathcal{P}$, then $\mu^{-1}(t)$ has property $\mathcal{P}$. In this paper we show that the following properties are sequential decreasing Whitney properties: atriodicity, containing no arc, irreducibility, indecomposability, hereditary indecomposability, and unicoherence.

1. INTRODUCTION

A *continuum* is a compact, connected metric space. Throughout this paper X will denote a continuum with a metric d. A continuum X is said to be:

(a) a *triod* if there is a subcontinuum N of X such that the complement of N in X is the union of three nonempty mutually separated sets,
(b) *atriodic* provided X does not contain triods,
(c) a *weak triod* if there are subcontinua X_1, X_2, and X_3 of X such that $X = \bigcup_{i=1}^{3} X_i$, $X_i \not\subset \bigcup_{j\neq i} X_j$ and $\bigcap_{i=1}^{3} X_i \neq \emptyset$,

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(d) *irreducible* provided that there exist two points p and $q \in X$ such that no proper subcontinuum of X contains p and q,
(e) *unicoherent* if $A \cap B$ is connected whenever A and B are subcontinua of X such that $A \cup B = X$,
(f) *indecomposable* provided that X is not the union of two proper subcontinua,
(g) *hereditarily indecomposable* provided that each of its nondegenerate subcontinua is indecomposable.

Let $C(X)$ be the hyperspace of all nonempty subcontinua of X, with the Hausdorff metric H. A *Whitney map* for $C(X)$ is a continuous function $\mu : C(X) \to [0,1]$ such that (i) $\mu(\{x\}) = 0$ for each $x \in X$, (ii) if $A, B \in C(X)$ and $A \subset B \neq A$, then $\mu(A) < \mu(B)$, and (iii) $\mu(X) = 1$. A *Whitney level* for $C(X)$ is a set of the form $\mu^{-1}(t)$ where μ is a Whitney map for $C(X)$ and $0 \leq t < 1$. Whitney levels are always continua [1, p. 1032].

A topological property $\mathcal{P}$ is said to be:

- a *sequential decreasing Whitney property*, provided that if μ is a Whitney map for $C(X)$, $\{t_n\}_{n=1}^{\infty}$ is a sequence in the interval $(t, 1)$ such that $t_n \to t$ and each Whitney level $\mu^{-1}(t_n)$ has property $\mathcal{P}$, then $\mu^{-1}(t)$ has property $\mathcal{P}$,
- a *sequential strong Whitney-reversible property*, provided that whenever X is a continuum such that there is a Whitney map μ for $C(X)$ and a sequence $\{t_n\}_{n=1}^{\infty}$ in $(0,1)$ such that $t_n \to 0$ and $\mu^{-1}(t_n)$ has property $\mathcal{P}$ for each n, then X has property $\mathcal{P}$.

Many authors have studied sequential strong Whitney-reversible properties; in Chapter VIII of [3], a detailed list of the known sequential strong Whitney-reversible properties is presented.

In [7], the author introduced the concept of sequential decreasing Whitney properties and proved that the following properties are of this type: the property of Kelley, local connectedness, and continuum chainability.

In particular, the following properties are known to be sequential strong Whitney-reversible properties:

(1) atriodicity (see [5, Theorem 14.49, p. 458]),
(2) containing no arc (see [5, Theorem 14.52, p. 460]),
(3) irreducibility (see [3, Theorem 49.3, p. 274]),
(4) indecomposability (see [5, Theorem 14.46 (1), p. 454]),

(5) hereditary indecomposability (see [5, Theorem 14.54 (1), p. 461]),
(6) unicoherence (see [5, Theorem 14.46 (2), p. 454]).

In this paper we prove that the properties described in (1)-(6), are sequential decreasing Whitney properties.

2. Preliminares

Suppose X is a continuum and μ is a Whitney map for $C(X)$. For $t \leq \mu(A)$, let $C(A,t) = (\mu \mid C(A))^{-1}(t)$, and for $t \geq \mu(A)$ let $C_A^t = \{B \in \mu^{-1}(t) : A \subset B\}$. Let $t_0 \in [0,1)$ and $\mathcal{A}$ be a nondegenerate subcontinuum of $\mu^{-1}(t_0)$. For each $t \in [t_0, \mu(\sigma(\mathcal{A})))$, let $X(\mathcal{A},t) = \{B \in \mu^{-1}(t) :$ there exists $\mathcal{B} \in C(\mathcal{A})$ such that $\sigma(\mathcal{B}) = B\}$. Since $C(A,t)$ is a Whitney level for $C(A)$, $C(A,t)$ is a subcontinuum of $\mu^{-1}(t)$. It known that C_A^t is a continuum when $A \in C(X)$ (see [9, Theorem 4.2]) and $X(\mathcal{A},t)$ is a continuum (see [7, Lemma 7, p. 300]). J. L. Kelley has shown (see [4, Lemma 1.1, p. 23]) that the function $\sigma : C(C(X)) \to C(X)$ defined by $\sigma(\mathcal{A}) = \bigcup\{A : A \in \mathcal{A}\}$ is continuous and surjective. So, for each $t \in [0,1]$, the function $\sigma_{t,\mu} : C(\mu^{-1}(t)) \to \mu^{-1}([t,1]$ defined by $\sigma_{t,\mu}(\mathcal{A}) = \sigma(\mathcal{A})$ is continuous. For $x \in X$ and $\epsilon > 0$, let $B_d(\epsilon,x) = \{y \in X : d(x,y) < \epsilon\}$. If X is a continuum and $B \subset X$, let $N(\epsilon,B) = \bigcup\{B_d(\epsilon,x) : x \in B\}$. We say that a subset $\mathcal{A}$ of $C(X)$ is an *anti-chain* if A, $B \in \mathcal{A}$ and $A \subset B$, implies that $A = B$. An *order arc* in $C(X)$ is an arc α in $C(X)$ such that if A, $B \in \alpha$, then $A \subset B$ or $B \subset A$. By Theorem 1.8 of [6], for each A, $B \in C(X)$ such that $A \subset B$ and $A \neq B$, there is an order arc joining A and B.

The following lemma is due to Kelley (see [5, Lemma 1.28, p. 76]).

Lemma 2.1. *Let X be a continuum and μ a Whitney map for $C(X)$. Then, for each $\epsilon > 0$, there exists $\delta > 0$ such that if K, $L \in C(X)$ satisfy $K \subset N(\delta, L)$ and $|\mu(L) - \mu(K)| < \delta$, then $H(K,L) < \epsilon$.*

3. Atriodicity

Theorem 3.1. *Let X be a continuum, μ be a Whitney map for $C(X)$, $t_0 \in [0,1)$ and $t \in (t_0, 1]$. If $\mathcal{A}$ is a subcontinuum of $\mu^{-1}(t_0)$, then $\bigcup_{A \in \mathcal{A}} C_A^t$ is a subcontinuum of $\mu^{-1}(t)$.*

Proof: Let $\mathcal{L} = \bigcup_{A\in\mathcal{A}} C_A^t$. It is easy to prove that $\mathcal{L}$ is closed in $\mu^{-1}(t)$. In order to see that $\mathcal{L}$ is connected, suppose the contrary. Then there exist two nonempty disjoint closed subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{L}$ such that $\mathcal{L} = \mathcal{F}_1 \cup \mathcal{F}_2$. For each $i = 1, 2$, let $\mathcal{W}_i = \{A \in \mathcal{A} : C_A^t \subset \mathcal{F}_i\}$. Observe that $\mathcal{W}_1$ and $\mathcal{W}_2$ are nonempty disjoint subsets of $\mathcal{A}$ and $\mathcal{A} = \mathcal{W}_1 \cup \mathcal{W}_2$. Now we show that they are closed. We only prove that $\mathcal{W}_1$ is closed since the argument for $\mathcal{W}_2$ is similar. In order to do this, let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{W}_1$ converging to an element A of $\mathcal{A}$. We need to prove that $A \in \mathcal{W}_1$. Since $\{C_{A_n}^t\}_{n=1}^{\infty}$ is a sequence of elements of $C(\mu^{-1}(t))$ (see [9, Theorem 4.2]), taking subsequences if necessary, we may assume that $\{C_{A_n}^t\}_{n=1}^{\infty}$ converges to an element $\mathcal{C}$ of $C(\mu^{-1}(t))$. Since $C_{A_n}^t \subset \mathcal{F}_1$, for each $n \geq 1$, we get $\mathcal{C} \subset \mathcal{F}_1$. Let $E \in \mathcal{C}$. Then there exists a sequence $\{E_n\}_{n=1}^{\infty}$ such that $\lim E_n = E$ and $E_n \in C_{A_n}^t$ for each $n \geq 1$. Notice that $A \subset E$. So $E \in C_A^t$. Thus, $\mathcal{C} \subset C_A^t$. Since C_A^t is connected, we conclude that $C_A^t \subset \mathcal{F}_1$. Hence, $A \in \mathcal{W}_1$.

Therefore, $\mathcal{A}$ is not connected, a contradiction. This completes the proof that $\mathcal{L}$ is a subcontinuum of $\mu^{-1}(t)$. □

Theorem 3.2. *Let X be a continuum, let μ be a Whitney map for $C(X)$, and let $t_0 \in [0, 1)$. If $\mu^{-1}(t_0)$ contains a triod, then there exists $s \in (t_0, 1)$ such that $\mu^{-1}(t)$ contains a weak triod for each $t \in (t_0, s]$.*

Proof: Let $\mathcal{M}$ be a triod contained in $\mu^{-1}(t_0)$. Then there is a subcontinuum $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{M} \backslash \mathcal{N}$ is the union of three nonempty mutually separated sets $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. For each $i \leq 3$, let $\mathcal{A}_i = \mathcal{N} \cup \mathcal{S}_i$. Each $\mathcal{A}_i$ is a subcontinuum of $\mu^{-1}(t_0)$. For each $i \leq 3$, choose an element $E_i \in \mathcal{S}_i$. Choose $\epsilon > 0$ such that $B_H(\epsilon, E_i) \cap \bigcup_{j\neq i} \mathcal{A}_j = \emptyset$ for each $i \leq 3$. Let δ be as in Lemma 2.1 for the number $\frac{\epsilon}{2}$. Choose $s \in (t_0, 1]$ such that $s - t_0 < \delta$ and $t \in (t_0, s]$. For each $i \leq 3$, let $\mathcal{T}_i = \bigcup_{A\in\mathcal{A}_i} C_A^t$. We will prove that $\mathcal{T} = \bigcup_{i=1}^{3} \mathcal{T}_i$ is a weak triod. Since $\bigcup_{A\in\mathcal{N}} C_A^t \subset \bigcap_{i=1}^{3} \mathcal{T}_i$, we see that $\bigcap_{i=1}^{3} \mathcal{T}_i$ is nonempty. For each $i \leq 3$, $\mathcal{T}_i$ is a subcontinuum of $\mu^{-1}(t)$ (by Theorem 3.1). Hence, $\mathcal{T}$ is a subcontinuum of $\mu^{-1}(t)$. Now, we prove that $\mathcal{T}_i \not\subset \bigcup_{j\neq i} \mathcal{T}_j$. It is enough to show that $C^t_{E_i} \cap \bigcup_{j\neq i} \mathcal{T}_j = \emptyset$. Let $E \in C^t_{E_i}$. Since $|\mu(E) - \mu(E_i)| < \delta$, by the choice of δ, we infer that $H(E, E_i) < \frac{\epsilon}{2}$. If $E \in \bigcup_{j\neq i} \mathcal{T}_j$, then there exists $A \in \bigcup_{j\neq i} \mathcal{A}_j$ such that $E \in C^t_A$. Since $|\mu(E) - \mu(A)| < \delta$, by the choice of δ, we conclude that $H(A, E) < \frac{\epsilon}{2}$. So $A \in B_H(\epsilon, E_i) \cap \bigcup_{j\neq i} \mathcal{A}_j$, a contradiction. Thus, $E \notin \bigcup_{j\neq i} \mathcal{T}_j$. Hence, since $C^t_{E_i} \subset \mathcal{T}_i$, we obtain that $\mathcal{T}_i \not\subset \bigcup_{j\neq i} \mathcal{T}_j$ for each $i \leq 3$. Therefore, $\mathcal{T}$ is a weak triod. □

Corollary 3.3. *The property of being atriodic is a sequential decreasing Whitney property.*

Proof: This corollary follows easily from Theorem 3.2 and the fact each weak triod contains a triod (see [10, Theorem 3, p. 443]). □

4. Containing no arc

Theorem 4.1. *The property of not containing arcs is a sequential decreasing Whitney property.*

Proof: Take a Whitney map μ for $C(X)$, a number $t \in [0, 1)$ and sequence $\{t_n\}_{n=1}^{\infty}$ in $(t, 1]$ such that $t_n \to t$ and $\mu^{-1}(t_n)$ contains no arc for each $n \geq 1$. Suppose that $\mu^{-1}(t)$ contains an arc α. Notice that $\mu(\sigma(\alpha)) > t$. We are going to prove that $C(\sigma(\alpha), t)$ is arcwise connected. Since $\mu(\sigma(\alpha)) > t$, we have that $C(\sigma(\alpha), t)$ is nondegenerate. Now, let A, $B \in C(\sigma(\alpha), t)$ and $A \neq B$. Then there exist A_1, $B_2 \in \alpha$ such that $A \cap A_1 \neq \emptyset \neq B \cap B_2$. We need to show that there exists an arc in $C(\sigma(\alpha), t)$ which joins A and B. If $A = A_1$ and $B = B_2$, then there exists a subarc α_1 of α which joins A and B. Thus, α_1 is an arc in $C(\sigma(\alpha), t)$ which joins A and B. Hence, we only need to prove that there exists an arc in $C(\sigma(\alpha), t)$ which joins A and A_1; similarly, the same happens with B and B_1. By Theorem 14.8.1 [5, p. 405], there exists an arc α_1 in $C(A \cup A_1, t)$ joining A and A_1. Since $A \cup A_1 \subset \sigma(\alpha)$, α_1 is an arc in $C(\sigma(\alpha), t)$. Hence, the element A can be joined with A_1 in $C(\sigma(\alpha), t)$. Therefore, $C(\sigma(\alpha), t)$ is arcwise connected. Let n be a positive integer such that $t_n \in (t, \mu(\sigma(\alpha)))$. Since $C(\sigma(\alpha), t)$ is a Whitney level for $C(\sigma(\alpha))$, by Proposition 2

[8, p. 151], we obtain that $C(\sigma(\alpha), t_n)$ is arcwise connected. Thus, we see that $C(\sigma(\alpha), t_n)$ is nondegenerate and $\mu^{-1}(t_n)$ contains an arc, a contradiction. This completes the proof of this theorem. □

5. Irreducibility

Theorem 5.1. *Irreducibility is a sequential decreasing Whitney property.*

Proof: Take a Whitney map μ for $C(X)$, a number $t \in [0,1)$ and a sequence $\{t_n\}_{n=1}^{\infty}$ in $(t,1]$ such that $t_n \to t$ and $\mu^{-1}(t_n)$ is irreducible for each $n \geq 1$. We prove that $\mu^{-1}(t)$ is irreducible. Suppose the contrary. Since X is irreducible (see Theorem 49.3, [3, p. 274]), there are p, $q \in X$ such that X is irreducible between p and q. Let A, $B \in \mu^{-1}(t)$ be such that $p \in A$ and $q \in B$. By the irreducibility of X, we have that $A \neq B$. Let $\mathcal{A} \in C(\mu^{-1}(t))$ be such that $\mathcal{A}$ is irreducible between A and B (see Exercise 4.35 (b) [6]). Since $\mu^{-1}(t)$ is not irrreducible, we infer that $\mathcal{A} \neq \mu^{-1}(t)$. Let $D \in \mu^{-1}(t) \setminus \mathcal{A}$ and take $\epsilon > 0$ in such a way $B_H(\epsilon, D) \cap \mathcal{A} = \emptyset$. Let δ be as in Lemma 2.1 for this ϵ. By Corollary 5.5 of [6], there exists $E \in C(X)$ such that $D \subset E \subset N(\delta, D)$ and $D \neq E$. Choose a positive integer m such that $t_m \in (t, \min\{\mu(E), \mu(\sigma(\mathcal{A})), t+\delta\})$. We need to prove that the set $X(\mathcal{A}, t_m)$ is a proper subcontinuum of $\mu^{-1}(t_m)$ and $\sigma(X(\mathcal{A}, t_m))$ is a proper subcontinuum of X which contains p and q. Clearly, $X(\mathcal{A}, t_n)$ is a subcontinuum of $\mu^{-1}(t)$ (see [7, Lemma 7, p. 300]). Using order arcs, it can be shown that there exists $F \in C(E)$ such that $F \in \mu^{-1}(t_m)$. If $F \in X(\mathcal{A}, t_m)$, then there exists $\mathcal{L} \in C(\mathcal{A})$ such that $\sigma(\mathcal{L}) = F$. Given $L \in \mathcal{L}$, $L \subset \sigma(\mathcal{L}) = F \subset E \subset N(\delta, D)$. By the choice of δ, we have that $H(L, D) < \epsilon$. Thus, $\mathcal{L} \subset B_H(\epsilon, D)$. So, since $\mathcal{L} \subset \mathcal{A}$, we conclude that $\mathcal{L} \subset B_H(\epsilon, D) \cap \mathcal{A}$. This contradicts the choice of ϵ. We have shown that $F \notin X(\mathcal{A}, t_m)$. Hence, $X(\mathcal{A}, t_m)$ is a proper subcontinuum of $\mu^{-1}(t_m)$. Since $\mu^{-1}(t_m)$ is irreducible, by Theorem 14.73.2 of [5], we have $\sigma(X(\mathcal{A}, t_m))$ is a proper subcontinuum of X.

On the other hand, since A, $B \in \mathcal{A}$, using order arcs in $C(\mathcal{A})$, it can be shown that there exist $\mathcal{H}_1$, $\mathcal{H}_2 \in C(\mathcal{A})$ such that $A \in \mathcal{H}_1$, $B \in \mathcal{H}_2$, and $\mu(\sigma(\mathcal{H}_i)) = t_m$ for each $i \leq 2$. Let $F_1 = \sigma(\mathcal{H}_1)$ and $F_2 = \sigma(\mathcal{H}_2)$. Notice that F_1, $F_2 \in X(\mathcal{A}, t_m)$. Then $A \subset F_1 \subset \sigma(X(\mathcal{A}, t_m))$ and $B \subset F_2 \subset \sigma(X(\mathcal{A}, t_m))$. Thus, p and $q \in \sigma(X(\mathcal{A}, t_m))$.

Therefore, X is not irreducible between p and q. This contradiction completes the proof that $\mu^{-1}(t)$ is irreducible. □

6. Indecomposability

Theorem 6.1. *Indecomposability is a sequential decreasing Whitney property.*

Proof: Take a Whitney map μ for $C(X)$, a number $t \in [0,1)$ and a sequence $\{t_n\}_{n=1}^{\infty}$ in $(t,1]$ such that $t_n \to t$. We are going to prove that if $\mu^{-1}(t)$ is decomposable, then there exists a positive integer m such that $\mu^{-1}(t_m)$ is decomposable. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be proper subcontinua of $\mu^{-1}(t)$ such that $\mu^{-1}(t) = \mathcal{A}_1 \cup \mathcal{A}_2$. Let $A_1 \in \mathcal{A}_1 \backslash \mathcal{A}_2$ and $A_2 \in \mathcal{A}_2 \backslash \mathcal{A}_1$. Then there exists $\epsilon > 0$ such that $B_H(\epsilon, A_1) \cap \mathcal{A}_2 = \emptyset = B_H(\epsilon, A_2) \cap \mathcal{A}_1$. Let δ be as in Lemma 2.1 for the number $\frac{\epsilon}{2}$ and take m a positive integer such that $t_m - t < \delta$. For each $i \in \{1,2\}$, let $\mathcal{G}_i = \bigcup_{A \in \mathcal{A}_i} C_A^{t_m}$. By Theorem 3.1, $\mathcal{G}_1$ and $\mathcal{G}_2$ are subcontinua of $\mu^{-1}(t_m)$. Given $E \in \mu^{-1}(t_m)$, using an order arc from a one point set to E, it is possible to find $A \in \mu^{-1}(t)$ such that $A \subset E$. Since $A \in \mathcal{A}_i$ for some $i \in \{1,2\}$, then $E \in \mathcal{G}_i$ for some $i \in \{1,2\}$. We have shown that $\mu^{-1}(t_m) = \mathcal{G}_1 \cup \mathcal{G}_2$. Fix an element $E_1 \in \mu^{-1}(t_m)$ such that $A_1 \subset E_1$. If $E_1 \in \mathcal{G}_2$, then there exists $A \in \mathcal{A}_2$ such that $A \subset E_1$. By the choice of δ, we have that $H(A, E_1)$, $H(A_1, E_1) < \frac{\epsilon}{2}$. Thus, $A \in B(\epsilon, A_1) \cap \mathcal{A}_2$. This contradicts the choice of ϵ and proves that $E_1 \notin \mathcal{G}_2$. Thus, $\mathcal{G}_2$ is a proper subcontinuum of $\mu^{-1}(t_m)$. Similarly, $\mathcal{G}_1$ is a proper suncontinuum of $\mu^{-1}(t_m)$. Hence, $\mu^{-1}(t_m)$ is decomposable. □

The proof of the next result is similar to that of Theorem 6.1.

Theorem 6.2. *Hereditary indecomposability is a sequential decreasing Whitney property.*

7. Unicoherence

Theorem 7.1. *Unicoherence is a sequential decreasing Whitney property.*

Proof: Take a Whitney map μ for $C(X)$, a number $t \in [0,1)$, and a sequence $\{t_n\}_{n=1}^{\infty}$ in $(t,1]$ such that $t_n \to t$ and $\mu^{-1}(t)$ is not unicoherent. We will show that there exists a positive integer m

such that $\mu^{-1}(t_m)$ is not unicoherent. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be subcontinua of $\mu^{-1}(t)$ such that $\mu^{-1}(t) = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2$ is not connected. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be disjoint nonempty compact sets of $\mu^{-1}(t)$ such that $\mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{F}_1 \cup \mathcal{F}_2$. Let $\epsilon > 0$ be such that $4\epsilon < H(F_1, F_2)$ for every $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$. For each $i \in \{1,2\}$, let $\mathcal{U}_i = \{F \in \mu^{-1}(t) :$ there exists $F_i \in \mathcal{F}_i$ such that $H(F, F_i) < \epsilon\}$. Let $\mathcal{B}_i = \mathcal{A}_i \setminus (\mathcal{U}_1 \cup \mathcal{U}_2)$. Since $\mathcal{A}_i$ is connected and $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$, $\mathcal{B}_i$ is nonempty. Let $\epsilon_1 > 0$ be such that $\epsilon_1 < \epsilon$ and $H(B_1, B_2) > 4\epsilon_1$ for every $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. Let $\delta > 0$ be as in Lemma 2.1 for this number ϵ_1. Let m be a positive integer such that $t_m - t < \delta$. For each $i \in \{1,2\}$, let $\mathcal{C}_i = \bigcup_{A \in \mathcal{A}_i} C_A^{t_m}$. By Theorem 3.1, $\mathcal{C}_1$ and $\mathcal{C}_2$ are subcontinua of $\mu^{-1}(t_m)$. Clearly, $\mu^{-1}(t_m) = \mathcal{C}_1 \cup \mathcal{C}_2$. Given $F \in \mathcal{C}_1 \cap \mathcal{C}_2$, there exist $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$ such that $A_1 \subset F$ and $A_2 \subset F$. By the choice of δ, $H(A_1, A_2) \leq H(A_1, F) + H(A_2, F) < 2\epsilon_1$. By the choice of ϵ_1, one of the sets A_1 or A_2 belongs to $\mathcal{U}_1 \cup \mathcal{U}_2$. This proves that, for each $E \in \mathcal{C}_1 \cap \mathcal{C}_2$, there exists $A \in \mathcal{U}_1 \cup \mathcal{U}_2$ such that $A \subset E$. For each $i \in \{1,2\}$, let $\mathcal{L}_i = \mathrm{Cl}_{\mu^{-1}(t)}(\mathcal{U}_i)$ and $\mathcal{G}_i = \{F \in \mu^{-1}(t_m) :$ there exists $A \in \mathcal{L}_i$ such that $A \subset F\}$. Thus, $\mathcal{C}_1 \cap \mathcal{C}_2 \subset \mathcal{G}_1 \cup \mathcal{G}_2$. Next, we see that $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$. Suppose that there exists $F \in \mathcal{G}_1 \cap \mathcal{G}_2$. Then there exist $E_1 \in \mathcal{L}_1$ and $E_2 \in \mathcal{L}_2$ such that $E_1, E_2 \subset F$. By the choice of δ, $H(E_1, E_2) < 2\epsilon_1$. By the definition of $\mathcal{U}_i$, there exists $G_i \in \mathcal{F}_i$ such that $H(E_i, G_i) < \epsilon$. Thus, $H(G_1, G_2) < 4\epsilon$. This contradicts the choice of ϵ and proves that $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$. Given $F_i \in \mathcal{F}_i$, there exists $D_i \in \mu^{-1}(t_m)$ such that $F_i \subset D_i$. Thus, $D_i \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{G}_i$. We have shown that $\mathcal{G}_1$ and $\mathcal{G}_2$ are disjoint subsets of $\mu^{-1}(t_m)$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 \subset \mathcal{G}_1 \cup \mathcal{G}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{G}_1 \neq \emptyset \neq \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{G}_2$. It is easy to prove that $\mathcal{G}_1$ and $\mathcal{G}_2$ are closed. This proves that $\mathcal{C}_1 \cap \mathcal{C}_2$ is disconnected. Therefore, $\mu^{-1}(t_m)$ is not unicoherent. □

The proof of the next result is similar to that of Theorem 7.1.

Theorem 7.2. *Hereditary unicoherence is a sequential decreasing Whitney property.*

8. Other properties

Lemma 8.1. *Let X be a continuum, μ a Whitney map for $C(X)$, and $t_0 \in [0,1)$. If $\sigma_{t_0,\mu}|\sigma_{t_0,\mu}^{-1}(\mu^{-1}([t_0,1)))$ is one-to-one, then for each $t \in (t_0,1)$, $\sigma_{t_0,\mu}^{-1}(\mu^{-1}(t))$ is a Whitney level for $C(\mu^{-1}(t_0))$.*

Proof: Let $t \in [t_0, 1)$ and let $\mathcal{M} = \sigma_{t_0,\mu}^{-1}(\mu^{-1}(t))$. We will prove that $\mathcal{M}$ has the following properties:

(1) $\mathcal{M} \subset C(\mu^{-1}(t_0)) \setminus (\{\mu^{-1}(t_0)\} \cup \{\{E\} : E \in \mu^{-1}(t_0)\}$,

(2) $\mathcal{M}$ is an anti-chain and

(3) $\mathcal{M}$ intersects every order arc α in $C(\mu^{-1}(t_0))$ such that α joins a one-point set and $\mu^{-1}(t_0)$.

We need to prove that $\mu^{-1}(t_0) \notin \mathcal{M}$ and $\mathcal{M} \cap \{\{E\} : E \in \mu^{-1}(t_0)\} = \emptyset$. If $\mu^{-1}(t_0) \in \mathcal{M}$, then $\sigma_{t_0,\mu}(\mu^{-1}(t_0)) = X \in \mu^{-1}(t)$, a contradiction. Now, suppose that $\mathcal{M} \cap \{\{E\} : E \in \mu^{-1}(t_0)\} \neq \emptyset$. Let $\mathcal{A} \in \mathcal{M} \cap \{\{E\} : E \in \mu^{-1}(t_0)\}$. Then $\mu(\sigma(\mathcal{A})) = t$ and $\mathcal{A} = \{E\}$ for some $E \in \mu^{-1}(t_0)$, a contradiction. This completes the proof of (1).

In order to see that $\mathcal{M}$ is an anti-chain, let $\mathcal{A}$, $\mathcal{B} \in \mathcal{M}$ be such that $\mathcal{A} \subset \mathcal{B}$. Since $\sigma_{t_0,\mu}(\mathcal{A}) \subset \sigma_{t_0,\mu}(\mathcal{B})$ and $\sigma_{t_0,\mu}(\mathcal{A})$, $\sigma_{t_0,\mu}(\mathcal{B}) \in \mu^{-1}(t)$, we infer that $\sigma_{t_0,\mu}(\mathcal{A}) = \sigma_{t_0,\mu}(\mathcal{B})$. Since $\sigma_{t_0,\mu}$ is one-to-one, we have that $\mathcal{A} = \mathcal{B}$. This ends the proof of (2).

We consider an order arc α in $C(\mu^{-1}(t_0))$ joining a one-point set, $\{E\}$, for some $E \in \mu^{-1}(t_0)$ and $\mu^{-1}(t_0)$. Using order arcs, it can be shown that there exists $\mathcal{A} \in \alpha$ such that $\mu(\sigma(\mathcal{A})) = t$. Thus, $\mathcal{A} \in \mathcal{M}$, and part (3) is proved.

Therefore, by Theorem 1.2 of [2], we conclude that $\mathcal{M}$ is a Whitney level for $C(\mu^{-1}(t_0))$. □

Theorem 8.2. *Let X be a continuum and μ be a Whitney map for $C(X)$. Assume that $\sigma_{t,\mu}|\sigma_{t,\mu}^{-1}(\mu^{-1}([t, 1)))$ is one-to-one for each $t \in [0, 1)$. Then, if $\mathcal{P}$ is a sequential strong Whitney-reversible property, $\mathcal{P}$ is also a sequential decreasing Whitney property.*

Proof: Take a Whitney map μ for $C(X)$, a number $t_0 \in [t_0, 1)$, and a sequence $\{t_n\}_{n=1}^{\infty}$ in $(t_0, 1]$ such that $t_n \to t_0$ and $\mu^{-1}(t_n)$ has $\mathcal{P}$ for each $n \geq 1$, where $\mathcal{P}$ is a sequential strong Whitney-reversible property. We are going to prove that $\mu^{-1}(t_0)$ has $\mathcal{P}$. Since $\sigma_{t_0,\mu}|\sigma_{t_0,\mu}^{-1}(\mu^{-1}([t_0, 1)))$ is one-to-one, by Lemma 8.1, for each $t \in (t_0, 1)$, we infer that $\sigma_{t_0,\mu}^{-1}(\mu^{-1}(t))$ is a Whitney level for $C(\mu^{-1}(t_0))$. Now, since $\sigma_{t_0,\mu}|\sigma_{t_0,\mu}^{-1}(\mu^{-1}([t_0, 1)))$ is one-to-one, it follows that $\sigma_{t_0,\mu}^{-1}(\mu^{-1}(t))$ is homeomorphic to $\mu^{-1}(t)$ for every $t \in (t_0, 1)$. It is easy to see that $\sigma_{t_0,\mu}^{-1}(\mu^{-1}(t_n)) \to \{\{E\} : E \in \mu^{-1}(t_0)\}$.

Therefore, since $\mathcal{P}$ is a sequential strong Whitney-reversible property, $\mu^{-1}(t_0)$ has $\mathcal{P}$, and the theorem is proved. □

A continuum X is said to have the *covering property*, written $X \in$ CP, provided that no proper subcontinuum of $\mu^{-1}(t)$ covers X, for any Whitney map μ for $C(X)$ and any $t \in [0,1]$. Notice that if $X \in$ CP and μ is a Whitney map for $C(X)$, then $\sigma_{t,\mu}$ is one-to-one. By Theorem 8.2, we can conclude that any sequential strong Whitney-reversible property is a sequential decreasing Whitney property in continua having property CP.

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