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Mail: Topology Proceedings
Department of Mathematics & Statistics
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SPRING TOPOLOGY AND DYNAMICS CONFERENCE 2004 REPORT[1]

The 2004 Spring Topology and Dynamics Conference was held March 25–27 at the University of Alabama at Birmingham. Seven plenary talks, twelve semi-plenary talks, and four parallel special sessions were held. In this issue, we will report on the areas of Continuum Theory, General/Set-Theoretic Topology, and Dynamical Systems, in which well-attended special sessions were held. A report on the Special Session in Geometric Topology/Geometric Group Theory will be combined with the report of the 2005 conference in a subsequent issue.

Continuum Theory

The Special Session in Continuum Theory attracted a broad spectrum of researchers from the United States and Mexico and included speakers from Canada and Poland.

As was the case in previous years, many strong results were presented in the intersection of continuum theory and dynamical systems. Indecomposable continua play a pivotal role in this area. On one hand, continua with no indecomposable subcontinua (such as dendrites, dendroids, λ-dendroids, hereditarily decomposable continua) frequently have dynamics similar to real numbers (including behavior of periodic and recurrent points, the Sarkovskii ordering, etc.). On the other hand, interesting dynamics often lead to indecomposable continua in such diverse contexts as Julia sets, positive entropy homeomorphisms, inverse limits, and even area preserving

[1]Beginning with this issue, the Problems Section will report on the annual Spring Topology and Dynamics Conference by publishing a survey of research in the areas represented at the conference. This will usually be a version of the "white paper" submitted to the National Science Foundation which has supported the conference nearly every year in which it has been held.

dynamical systems. Strong connections to dynamical systems give special importance to study of inverse limits of $[0,1]$ (chainable continua), and in particular, inverse limits of the form $([0,1], f)$ for f in the tent family. (An important question in this last area is whether two maps in the tent family with homeomorphic inverse limits must be identical.)

Indecomposable continua also have connections with general topology and analysis. There have been many recent results on the pseudo-arc and other hereditarily indecomposable continua and Bing maps. Additionally, an analytic approach to continuum theory (through the study of the descriptive complexity of various types of curves) seems to be very promising.

The study of hyperspaces is another very active area of research in continuum theory, with several talks devoted to this topic.

Questions on fixed (and periodic) points play a very important role in continuum theory and are also interesting from a dynamical systems point of view. Especially interesting is the fundamental question of whether every non-separating plane continuum has the fixed point property. (It is even not known if a continuous self map of such a continuum must have a periodic point.) Recently, two old fixed point problems (posed by R. H. Bing) were solved, and their solutions were presented at the conference.

Other selected topics and problems important to continuum theory that were discussed at the conference include the following:

I. Homogeneous Continua. Problems of classification of homogeneous curves: Is every homogeneous tree-like continuum a pseudo-arc? Is every hereditarily decomposable homogeneous continuum a simple closed curve? Is every homogeneous indecomposable nondegenerate continuum one-dimensional?

II. Span Zero and Chainability. Does span zero imply chainability? ("Yes" would yield the classification of homogeneous plane curves.) Is the confluent image of a chainable continuum chainable?

III. Fixed Points. Does every inverse limit of a fixed tree have the fixed point property? In particular, must any two maps of a simple triod (a tree) to itself have a coincidence point?

IV. 2-to-1 Maps. Is any tree-like continuum the 2-to-1 image of a continuum? Can the pseudo-arc map 2-to-1 onto any continuum?

Is any dendroid the at-most-2-to-1 image of any indecomposable continuum?

V. Embeddings into the Plane. There is no generally applicable method of determining if a curve can be embedded into the plane. The question of embeddability into the plane is central to the fixed point problem for plane non-separating continua. A related question asks, "Given a point p in a chainable continuum X, can X be embedded into the plane with p accessible?"

VI. Dendroids. Can every dendroid be retracted by a small retraction onto a tree?

Piotr Minc, Janusz Prajs, and Sergio Maciás are to be thanked for writing the above summary of the Special Session in Continuum Theory and for assembling the following list of questions posed at the Problem Session in Continuum Theory.

Problems in Continuum Theory

Below are questions posed at the Problem Session in Continuum Theory. Included are some remarks of the poser and others.

Question 1. Is each homogeneous, aposyndetic curve either a simple closed curve or an inverse limit of Menger curves with covering bonding maps?

Question 2. Is every homogeneous tree-like curve a pseudo-arc?

Question 3. Does every homogenous indecomposable cyclic curve decompose onto a solenoid with fibers being tree-like homogenous curves or points?

Note: *Curve* means one-dimensional continuum; X is *cyclic* provided that $H^1(X, \mathbf{Z}) \neq 0$.

Note: "Yes" to the first three questions yields a classification of homogeneous curves.

Question 4. Is every homogeneous indecomposable nondegenerate continuum one-dimensional?

Question 5. Is every hereditarily decomposable homogeneous continuum a simple closed curve?

Questions 6 and 7 by Janusz R. Prajs

Question 6. Let X be a homogeneous curve. Must X contain either an arc or a nondegenerate, proper terminal subcontinuum?

Question 7. Let X be a homogeneous continuum. Must X contain either an arc or a nondegenerate hereditarily indecomposable subcontinuum?

Question by R. H. Bing

Question 8. Let X be a planar continuum with the fixed-point property. Does $X \times [0,1]$ have the fixed point property?

Classical Question

Question 9. Let T be a simple triod (a tree). Do there exist maps $f, g : T \to T$ such that $f \circ g = g \circ f$ and $f(x) \neq g(x)$ for each $x \in T$?

Question by Eric McDowell

Question 10. A small-point hyperspace of a metric continuum, X, is given by $C_\varepsilon(X, d) = \{A \in C(X) : \text{diam}(A) \leq \varepsilon\}$, where d is a metric giving the topology on X and $\varepsilon > 0$. Is $C_\varepsilon(X, d)$ always countable closed set aposyndetic?

Note: A continuum X is said to be *countable closed set aposyndetic* if for any $p \in X$ and any countable closed subset F of X such that $p \notin F$ there exists a subcontinuum M of X such that $p \in \text{Int}M$ and $M \cap F = \emptyset$. (See [4, p. 238]. J. T. Goodykoontz, Jr. showed in 1973 that $C(X)$ is always countable closed set aposyndetic [2].

Questions 11-16 from Wayne Lewis's talk; most were originally posed by others.

Question 11. Does there exist a nondegenerate indecomposable non-metric continuum with infinitely many composants which admits a Borel transversal to its composants?

Question 12. If α is a cardinal number with $2 < \alpha < 2^\omega$, does there exist a non-metric indecomposable continuum with exactly α composants?

Question 13. Does there exist a nondegenerate hereditarily indecomposable non-metric continuum with only one composant?

Question 14. For which sets X is the Stone-Čech remainder $\beta X \setminus X$ indecomposable?

Question 15. What techniques other than Stone-Čech remainders and inverse limits with uncountably many factors produce non-metric indecomposable continua?

Question 16. How many distinct, non-metric pseudo-arcs can be formed as hereditarily indecomposable continua by inverse limits of pseudo-arcs with ω_1 factors?

Questions 17-22 concern constrained maps.

Question by Sam B. Nadler, Jr. and Lewis E. Ward, Jr.

Question 17. (the big one) Is any tree-like continuum the 2-to-1 image of a continuum?

Questions 18 and 19 by Jo Heath and Van Nall

Question 18. Is any dendroid the at-most-2-to-1 image of any indecomposable continuum?

Question 19. Is any λ-dendroid the 2-to-1 image of a continuum?

Question by Jerzy Krzempek

Question 20. Is any decomposable continuum the 2-to-1 image of any indecomposable continuum?

Question by Piotr Minc

Question 21. Is it true that a chainable continuum is hereditarily decomposable if it admits an at-most-2-to-1 map onto a dendroid?

Question by Jerzy Mioduszewski

Question 22 (A golden oldie). Can the pseudo-arc map 2-to-1 onto any continuum?

Question 23. Let W be a widely connected set. (A connected set W is *widely connected* iff every nondegenerate connected subset of W is dense in W.) (a) Is βW an indecomposable continuum? (b) If W is metrizable and separable, does W have a metric compactification which is an indecomposable continuum? (c) If W is separable and metrizable does W have a metric compactifacation γW such that for every composant C of γW, $C \cap W$ is (1) totally disconnected? (2) finite? (3) a singleton?

Question by James T. Rogers, Jr., John C. Mayer, and D. K. (Doug) Childers

Question 24. Let S be the Sierpiński plane curve, X and Y be subcontinua of X in the buried (irrational) points of S and assume that X and Y are equivalently embedded in the plane. Does this imply that X and Y are equivalently embedded in S ? ("Yes," if X and Y are arcs. The arc case is applied to rational Julia sets.)

Question 25. Let M_n^k ($k \geq 1$) be the "intermediate" Menger continuum. How many homogeneity classes does M_n^k have?

R. D. Anderson: M_3^1 has 1 class (the Menger curve is homogeneous).

J. Krasinkiewicz: M_2^1 (the Sierpiński curve) has 2 classes.

M. Bestvina: M_n^k for $n \geq 2k+1$ has 1 class.

W. Lewis: M_n^k for $k+1 < n < 2k+1$ has more than 1.

Questions 26-28 by Paul Bankston

Question 26. Are co-existential maps always confluent?

Remark: A map of $f : X \to Y$ between compacta is *co-existential* if there is an ultracopower (Y^*, p^*, Y) over Y and a continuous map $g : Y^* \to X$ such that $f \circ g = p^*$. A precise specification of Y^* and p^* is as follows: First, pick an index set I and an ultrafilter $\mathcal{D} \in \beta(I)$. Next, with $p : X \times I \to X$ and $q : X \times I \to I$ the projection maps, the $\mathcal{D}$-ultracopower of Y is defined to be an inverse image of $\mathcal{D} \in \beta(I)$ under $\beta(q)$. The (co-diagonal) map p^* is then the restriction of $\beta(p)$ to the ultracopower contained in $\beta(X \times I)$. This construction where X is an arc and $\mathcal{D}$ is a free ultrafilter on ω was first used by J. Mioduszewski in the mid 1970s to study the composant structure of $\beta([0,1))$. Later on, M. Smith and J. P. Zhu used it to construct indecomposable continua in $\beta([0,1))$. (See [3, pp. 317-352].) At about the same time as Mioduszewski, I independently started to study ultracopowers and ultracoproducts in general. My motivation was largely model-theoretic and category-theoretic, whence the terminology. (Indeed, one can alternatively obtain the $\mathcal{D}$-ultracopower of Y by first taking the lattice $F(Y)$ of closed subsets of Y, by second taking $\mathcal{D}$-ultracopower $F(Y)^I/\mathcal{D}$, and by third taking the Wallman filter of the ultracopower lattice. For more details see [1].

Co-existential maps are actually topological analogues of existential embeddings in model theory. (Imagine one field being algebraically closed in another.) Moreover, if $f : F(Y) \to F(X)$ is an existential embedding, then f induces a co-existential map from X to Y. Co-existential maps are always weakly confluent. They are monotone if the image space is locally connected. They need not be monotone in general, but all known examples of nonmonotone co-existential maps are confluent.

Question 27. Do co-existential maps preserve chainability?

Remark: Both monotone and open maps preserve chainability. A. Lelek's corresponding question is whether the same is true for confluent maps.

Question 28. Is every continuous map of a metrizable continuum onto the pseudo-arc co-existential?

Remark: A *metrizable* continuum Y is said to be in *Class(C)* iff every continuous function from any *metrizable* continuum onto Y is confluent (Lelek). This class has been characterized as being the class of hereditarily indecomposable metrizable continua (Lelek and D. R. Read). Let us refer to our corresponding class of (not necessarily metrizable) continua as Class(CE). Then every continuum in this class is hereditarily indecomposable, of covering dimension 1. Class(CE) contains continua of all weights; at least one metrizable member is nonchainable. What is particularly interesting to us about this class is that its specification is exactly dual to that of the class of existentially closed models of a first-order theory in logic. For example, the existentially closed fields, i.e., those fields that embed only existentially in other fields, are precisely the algebraically closed fields. The existentially closed linear orders are the dense orders without end-points.

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General/Set-Theoretic Topology

The General/Set-Theoretic Topology session at this conference illustrated the remarkable diversity of the field via its applications to and interaction with other areas, such as logic and set theory, topological dynamics, topological groups and topological algebra, Banach spaces, real and functional analysis, and descriptive set theory. Emerging new directions include characterizations of complete Erdös spaces and their applications in topological dynamics, and fresh attacks on some fundamental "basis problems" for general topological spaces. The outline below of results reported on at the conference is divided into sections loosely related to the areas of interaction.

I. Homeomorphism Groups, Topological Dynamics. One of the plenary lectures was given by Jan Dijkstra, who, with Jan van Mill, recently obtained a deep topological characterization of Erdös space, a space that is well-known in general topology as a counterexample in dimension theory. They used it in a striking solution to the difficult problem of describing certain homeomorphism groups of classical manifolds. Work continues on a complete version of Erdös space, which has surfaced in topological dynamics as the endpoint set of the Lelek fan and in the Julia sets of various functions. Now, through the efforts of Dijkstra and van Mill, general topology is about to produce topological characterizations of complete Erdös space and similar spaces that may be of considerable interest in topological dynamics.

II. Logic and Set Theory. Interactions between set-theoretic topology and logic/set theory were continually on display at the conference. Justin Moore announced his remarkable solution to S. Shelah's "basis problem" for ordered sets. He proved that it follows from the Proper Forcing Axiom of set theory that there are 5 uncountable linearly ordered sets such that every uncountable linear order contains a copy of one of these. This has already led Moore and others to consider applying his methods to a topological basis problem of G. Gruenhage: Is it consistent that every uncountable regular space contains a copy of either an uncountable discrete space, an uncountable subspace of the real line, or an uncountable subspace of the Sorgenfrey line? This problem is about 18 years old, but up until now, no one has had any meaningful way to approach the problem. A positive solution would solve a number of interesting problems on the structure of perfectly normal compacta and of continuous images of separable metric spaces.

In another plenary lecture, Ken Kunen used the technique of elementary submodels to obtain a solution to a problem of A. V. Arhangel'skii on the existence of locally compact linearly Lindelöf spaces; in this setting, the problem is equivalent to asking for a subtle kind of limit behavior at a point in a compact Hausdorff space.

In other talks, Alan Dow obtained an interesting relationship between the smallest measurable cardinal and maximal realcompact topologies, and Andrzej Roslanowski discussed several stong ccc type properties of partially ordered sets used in the set-theoretic technique of forcing, some natural examples of which come from interesting topologies on the reals.

III. Real Analysis, Descriptive Set Theory. With Juris Steprans, Márton Elekes showed that a certain compact null set in the real line, originally described by P. Erdös and S. Kakutani in 1955 and a fairly well-known example in geometric measure theory, has the property that, in some models of set theory, fewer than continuum many translations of it cover the line. This answered a question of Gruenhage. Elekes also answered a question of D. Mauldin by proving that there is no natural invariant Borel measure for the set of Liouville numbers. Arnold W. Miller answered a question of M. Scheepers on γ-sets, which are subsets of the real line with special properties that have relevance for (among other things) convergence properties of certain function spaces. Michael Hrušák discussed which cardinals could be the minimal cardinal of a cover of a separable metric space by nowhere dense sets, relating this cardinal to known cardinal invariants of the continuum. Maxim Burke showed that it is consistent that every non-meager set in a Polish space have relative non-meager intersection with some nowhere-dense Cantor set.

IV. Banach Spaces, Functional Analysis. Stoyu Barov (with Dijkstra) studied when sets with convex projections in Hilbert space ℓ_2 are convex. V. V. Uspenskij showed that P. S. Urysohn's universal separable metric space is homeomorphic to ℓ_2. Peter

Nyikos showed that locally-finite-in-the-norm families in a separable Banach space have weak topology extensions which are norm locally finite.

V. Topological Algebra, Topological Groups. In an invited lecture, A. V. Arhangel'skii obtained interesting results concerning when a topological group has a metrizable remainder in its Stone-Čech compactification. Jan van Mill characterized the coset spaces of separable metric topological groups. Gary Gruenhage answered several questions in the literature on cozero complemented spaces, a property equivalent to an algebraic property of the ring $C(X)$ of all continuous real-valued functions on X.

Gary Gruenhage, with contributions by Alan Dow, Justin Moore, and Jan van Mill, is to be thanked for writing the above summary and assembling the following list of problems posed at the General/Set-Theoretic Problem Session at the conference.

Problems in General and Set-Theoretic Topology

Below are problems posed at the Problem Session in General/Set-Theoretic Topology. Included are some remarks of the poser and others. Some problems sent to me (Gruenhage) by email by Justin Moore and Alan Dow are also included. Please note that the poser of a problem at the problem session is not necessarily the one who asked the question originally.

Problems 1 and 2 by Jan van Mill

Problem 1. Is there a non-trivial zero-dimensional homogeneous subspace of the real line with the fixed-point property for homeomorphisms?

Related to Problem 1 is:

Problem 2. Is there a nontrivial compact zero-dimensional first countable homogeneous space with the fixed-point property for homeomorphisms?

These questions are rather old and asked many times. I believe they are interesting since zero-dimensionality and the fixed-point property are more or less 'orthogonal' properties.

Problems 3 – 7 by Justin T. Moore

The solution to the basis problem for linear orders (i.e., PFA implies that there are five specific uncountable linear orders such that every uncountable linear order isomorphically contains one) is relevant to problems such as the conjecture (L) (no L spaces), (PFA) every perfect compactum is premetric of degree 2, (PFA) the uncountable regular spaces have a 3-element basis. Perhaps I am being a bit ambitious but I would file this in the "promising new directions" category. As for problems:

Problem 3. Does Martin's Maximum imply that every compact space either contains an uncountable discrete set or continuously maps 2-1 onto a metric space?

Problem 4. Does Martin's Maximum imply that every hereditarily Lindelöf regular space is hereditarily separable?

Problem 5. Does Martin's Maximum (MM) imply that the uncountable regular spaces have a 3-element basis?

Problem 6. Do any of the three previous conclusions have non-trivial consistency strength? In particular, do they imply the existence of $O^{\#}$?

So far, PFA has been sufficient, but I am starting to suspect that MM may also be relevant in studying these problems via the new techniques used in proving a 5-element basis for the uncountable linear orders from PFA. I have listed the first three problems in the order that I feel they are most tractable (whatever that means).

Remark by Gruenhage: Whether or not the conclusion of Problem 3 is consistent with ZFC is originally due to D. Fremlin. Whether or not the 3-element basis problem for topological spaces is consistent is originally due to me; I have also noted that if a positive answer to the 3-element basis problem is consistent with PFA, this also would give a positive answer to Fremlin's problem.

Problem 7. If X is compact and X^2 is T_5, must X be separable?

Remark: $MA(\omega_1)$ implies "yes"; this could be a nice ZFC fact though.

Problems 8 – 11 by A. V. Arhangel'skii

A "space" means a "Tychonoff space." A base $\mathcal{B}$ of a space X is said to be of *countable order*, if, for every $x \in X$ and for every strictly decreasing sequence $\eta = \{U_n : n \in \omega\}$ of elements of $\mathcal{B}$ containing x, η is a local base at x in X.

H. H. Wicke and J. Worrell have established the following remarkable property of bases of countable order: If a space X has a base of countable order locally, then it has it globally. In particular, every locally metrizable space has a base of countable order. Hence, ω_1 has a base of countable order.

Problem 8. Is every linearly Lindelöf space X with a base of countable order Lindelöf?

Note that if the answer is "yes," then every such X is separable and metrizable, since every paracompact space with a base of countable order is metrizable [2]. Note also that every locally metrizable linearly Lindelöf space is separable metrizable [4].

Problem 9. Is the product of two (of arbitrary countable family) of linearly Lindelöf p-spaces linearly Lindelöf?

The product of any countable family of Lindelöf p-spaces is Lindelöf, since every Lindelöf p-space admits a perfect mapping onto a separable metrizable space [1]. However, not every linearly Lindelöf locally compact space is Lindelöf, as shown by K. Kunen at this conference. On the other hand, A. Karpov has proved that the product of any countable family of Čech-complete linearly Lindelöf spaces is linearly Lindelöf.

A space X is said to be *discretely Lindelöf* if the closure of every discrete subspace in X is Lindelöf. Every discretely Lindelöf space is linearly Lindelöf.

Problem 10. Is every discretely Lindelöf space Lindelöf?

V. V. Tkachuk, who presented the above problem at the conference, made the following remark about it. The motivation for this question is that any discretely Lindelöf space is linearly Lindelöf; besides, if we introduce "discrete compactness" in an analogous way, then it will coincide with compactness.

Problem 11. Is every locally compact discretely Lindelöf space X Lindelöf?

This problem is especially interesting in connection with Kunen's result noted above.

Problem by Jan Dijsktra

A space is called *almost zero-dimensional* if every point has a neighborhood basis consisting of sets that can be written as intersections of clopen subsets of the space. A space is called *cohesive* if every point has a neighborhood that contains no nonempty subset that is clopen in the space. If a space X is totally disconnected then the clopen subsets of the space serve as a basis for a zero-dimensional Tychonoff topology ζ_X on the space. Remark 4.8 in [6] states that for any cohesive, almost zero-dimensional, separable metric space X the topology ζ_X has uncountable character at every point. The standard example of such a space is Erdös space $\mathfrak{E}$, which consists of the vectors in Hilbert space ℓ^2 that have only rational coordinates.

Problem 12. Is the character (or weight) of $\zeta_{\mathfrak{E}}$ equal to $2^{\aleph_0}$?

Problems 13 and 14 by Strashimir Popvassilev

Call a space X *base-base paracompact* (J. E. (Ted) Porter) if X has a base $\mathcal{B}$ such that every base $\mathcal{B}' \subseteq \mathcal{B}$ has a locally finite subcover $\mathcal{C} \subseteq \mathcal{B}'$.

Problem 13. Is every subspace X of the Sorgenfrey line S base-base paracompact? Are the irrationals with the topology induced from S base-base paracompact? What about a Bernstein subspace of S?

It is known that every F_σ subset of S is base-base paracompact, as is every Lusin subset, and, under MA, every subset of cardinality $< 2^{\aleph_0}$.

The above question is a special case of the following one asked by Porter.

Problem 14. Is every paracompact Hausdorff space base-base paracompact?

Remark by Porter: Every base-base paracompact space is a D-space.

Related problem included by Gary Gruenhage

Problem 15. Must every paracompact Hausdorff space X be base-paracompact? I.e., must there be a base $\mathcal{B}$ with $|\mathcal{B}| = w(X)$ such that every open cover of X has a locally finite refinement by members of $\mathcal{B}$?

Remark: Base-paracompactness was introduced and studied by Porter in [10].

Problem by Melvin Henriksen

The problem that follows was posed to me by J. Nagata at the conference in Matsue, Japan.

Problem 16. If X is a Tychonoff space, let $C(X)$ denote the set of all continuous real-valued functions defined on X. Characterize metrizability of X algebraically in case $C(X)$ is considered as either a ring, a lattice, or a multiplicative semigroup.

Remarks: (1) One should expect to assume that X is realcompact and hence that each of its closed discrete subspaces has non-measurable cardinality. (2) The characterization should be internal involving only algebraic invariants of $C(X)$, but not those of any larger algebras.

Problems 17 and 18 by Peter J. Nyikos

Here is the first question I posed in the problem session, the way J. Lawson told it to me last summer:

Problem 17. Begin with the compact-open topology on the set of continuous functions from N^N to N, and take the sequential modification (in which a set is closed iff it is sequentially closed, i.e., contains all limits of sequences converging from it). Is the resulting space 0-dimensional?

Related questions: Is it regular? If it is regular, is it 0-dimensional? Is its complete regularization (meaning: take the weak topology generated by continuous real-valued functions) 0-dimensional? All these questions are open; we do not have even consistency results for any of them. It is, however, known that the two topologies mentioned in Problem 17 are not the same; in fact, 5.12 of [7] gives an example of a clopen subspace in the sequential modification that is neither open nor closed in the compact-open topology.

The first few paragraphs of [9] give a very readable informal account of what this has to do with computer science. It talks about two kinds of infinite languages for specifying real numbers, real-valued functions of a real variable, and so on up a hierarchial line; a nice example of a function on the next step in the hierarchy is the definite integral of a continuous function $f : \mathbb{R} \to \mathbb{R}$. The two approaches have been shown to be equivalent this far up the hierarchy but the very next step is still open; the two kinds of languages are equivalent at this step iff the answer to the last of the related problems is "Yes"; of course, that would follow from an affirmative answer to Problem 17.

For an account of some earlier work, already showing how equivalence follows from a "Yes" answer to Problem 17, see [5].

Normann [9] shows that the equivalence continues all the way up the hierarchy as long as the higher analogues of the last related problem are true. For each step in the hierarchy there is a certain cosmic space for which the key question is whether its complete regularization is 0-dimensional.

Now for something completely different:

Problem 18. Is it consistent that compact homogeneous T_5-spaces are first countable? Could this even be true in ZFC? Is it true in ZFC that they are of cardinality $\leq \mathfrak{c}$? (It is true in the model obtained by adding $2^{\aleph_1}$ Cohen reals to a model of ZFC. See [3].)

Related problems: Is every homogeneous compact space of countable tightness of cardinality $\leq \mathfrak{c}$? first countable? Arhangel'skii conjectured "yes" answers in [3].

The Proper Forcing Axiom (PFA) implies "yes" answers to both. (See [8].)

Problem by Márton Elekes

Problem 19. Characterize the possible order types of the linearly ordered subsets of $B_1[0,1]$, where $B_1[0,1]$ is the class of real-valued Baire class 1 functions defined on $[0,1]$, partially ordered under the natural pointwise ordering. (That is, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$, so $f < g$ iff $f(x) \leq g(x)$ for all $x \in X$ and $f(x) \neq g(x)$ for at least one $x \in X$.)

Remarks: The answer to the corresponding problem concerning $C[0,1]$ is easy; exactly the real order types are possible (that is, orders that are order isomorphic to some set of reals). For Borel functions, in fact for Baire class 2 or higher, even the problem of well-ordered subsets is independent. A partial answer for Baire class 1 is that under MA the possible order types of cardinality less than 2^ω are exactly the ones not containing ω_1 or ω_1^*.

Problem by Alexander Shibakov

Problem 20. Can there be a Fréchet topological group G and a compact Fréchet space X such that $G \times X$ is not Fréchet?

Such a group G could be chosen to be countable without loss of generality (if it exists).

Remark by Gruenhage: There are, in ZFC, two Fréchet groups whose product is not Fréchet, as well as two compact Fréchet spaces whose product is not Fréchet, but there seem to be no known counterexamples, even consistent ones, for this problem. And P. Nyikos reminded me of the relevant information (noting that the group G would be non-metrizable) that it is an unsolved problem of V. I. Malykhin whether or not there is in ZFC a countable Fréchet non-metrizable group.

Problems 21 – 28 by Alan Dow

Problem 21. Is every compact ccc extremally disconnected image of ω^* separable (under PFA?)

Problem 22. If a compact X has a closed G_δ subsets mapping onto βN, must X map onto βN?

Problem 23. Suppose X is compact of countable tightness, does there exist a discrete subset D whose closure has full cardinality? In particular, can X be written as a $\mathfrak{c}$-fold union of closures of discrete sets?

Remark: This is a special case of an old problem of B. A. Efimov.

Problem 24 (Scarborough-Stone). Must the product of every family of regular sequentially compact spaces be countably compact?

Remark by Nyikos: Actually, C. T. Scarborough and A. H. Stone did not include any separation axioms in the statement of their problem. I obtained a counterexample in the class of Hausdorff spaces.

Problem 25. Is it consistent that countably compact first-countable separable regular spaces are compact?

Remark by Nyikos: I have offered \$1,000 for a solution to Problem 25, which originally is due to S. P. Franklin and M. Rajagopalan.

Problem 26. If a normal space X is the countable union of open metrizable subspaces, must X be metrizable?

Problem 27. Is it consistent that (normal) first-countable ω_1-collectionwise Hausdorff spaces are collectionwise Hausdorff?

Problem 28. Is the normal Moore space conjecture consistent with $\mathfrak{c} = \omega_2$?

Problem by Oleg Pavlov

Problem 29. Is there a regular maximal space that is a P-space?

Recall that X is maximal if X is dense-in-itself but no stronger topology is. A space answering the question, if it exists, would have cardinality that is Ulam measurable.

Problems 30 – 32 by Sheldon Davis

A symmetric on a set X is a function $d : X \times X \to [0, \infty)$ such that the following are true:

(1) $d(x, y) = 0$ iff $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

A topological space X is *symmetrizable* iff there is a symmetric d on X such that a subset $U \subset X$ is open iff for each $x \in U$ there exists $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subset U$. As usual, the ball $B(x, \varepsilon_x) = \{y \in X : d(x, y) < \varepsilon_x\}$.

There are three old questions about symmetrizable spaces which remain open in spite of quite a large amount of work.

Problem 30. Is every point of a regular symmetrizable space a G_δ -set?

This is an old question of E. A. Michael and A. V. Arhangel'skii.

There is a Hausdorff, non-regular, counterexample, obtained by taking a Tychonoff symmetrizable space with a closed set which is not a G_δ, and shrinking the closed set to a point (done by Gruenhage and Nyikos).

Problem 31. Is there a symmetrizable Dowker space?

If there is a symmetrizable Dowker space, then one can attach an additional point to it to obtain a regular symmetrizable space in which that new point is not a G_δ. Almost conversely, R. M. Stephenson, Jr. has shown that if a point x of a Hausdorff symmetrizable space X is not a G_δ, then $X\setminus\{x\}$ is not countably metacompact.

Problem 32. Is it consistent that there are no symmetrizable L-spaces?

An old result of S. Nedev shows that there are no symmetrizable S-spaces. D. Shakhmatov has constructed a model which contains a symmetrizable L-space. Z. Balogh, D. Burke, and S. Davis have constructed, in ZFC, a Hausdorff (non-regular) symmetrizable space which is hereditarily Lindelöf and not separable.

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Dynamical Systems

The Special Session in Dynamical Systems consisted of 24 lectures. Of the speakers, 10 were graduate students and 3 were recent (in the last 5 years) PhD's. In addition, 2 of the plenary speakers (J. W. Cannon and Michael Handel) and 4 of the semi-plenary speakers (Lukas Geyer, Lex Oversteegen, Kevin M. Pilgrim, and Evelyn Sander) spoke on topics in Dynamical Systems (widely defined).

Dynamical systems studies evolving systems in the most general sense. As such, it is central to the analysis of fundamental models used in all areas of sciences. Topological methods have long been a basic ingredient in Dynamical Systems and this Special Session contained talks from a variety of sub-areas including (in no particular order):

- Three dimensional flows (Krystyna Kuperberg; Akio Noguchi; Mike Sullivan)
- General topological dynamics (David Richeson; Jim Wiseman; Stewart Baldwin)
- Zero and one-dimensional dynamics (Louis Block; Ethan M. Coven)
- Tilings (Marcy Barge; Charles Holton; Brian F. Martensen; Megan Smith)
- Holomorphic dynamics (Carlos Cabrera; Doug Childers; Robert L. Devaney; Alexandra Kaffl; Daniel M. Look; Monica Moreno-Rocha; Johannes Rueckert)
- Group actions including superrigidity phenomena (Alex Clark; Michael Handel; Kamlesh Parwani)
- Applications to fluid mechanics (James Halbert; Rafal Komendarczyk)
- Applications of Dynamical Systems (Problem Session)

We isolate the last four as areas of particular interest and progress.

I. Tilings and Dynamics on Surfaces. Tilings have emerged in recent years as an exciting nexus, combining ideas and methods from Dynamics, C^*-Algebras, and Continuum Theory. Progress on classifying the natural flows on tiling spaces continues (Holton, Martensen, Smith), and connections to surface dynamics are developed further (Barge). A thirty-year-old problem of M. Hirsch asks whether pseudo-Anosov surface homeomorphisms can embed into hyperbolic toral endomorphisms. This turns out to reduce to a concrete and simple question concerning interval exchange transformations. Progress is announced by Barge as a consequence of constructions which embed tiling systems into toral endomorphisms.

II. Holomorphic Dynamics. Holomorphic dynamics continues to be one of the most active areas within dynamics. New methods from geometric group theory are being used to provide a unified approach for studying the dynamics of both rational maps and group actions on two-spheres arising from Cannon's program for the geometrization of three manifolds (Cabrera, Cannon, Geyer, Pilgrim).

Previously known connections between this field and other disciplines, such as continuum theory, continue to be developed further. Major results in this vein include the following.

> Wandering triangles exist (A. Blokh/Oversteegen). A problem concerning the dynamics of polynomials posed by Thurston and open for 20 years is resolved.
>
> Combinatorial encoding of dynamics on Sierpiński carpets and gaskets (Devaney, et al.). Abstract models for dynamics are extended from polynomials to certain families of rational maps.

Surprising applications of complex dynamics, such as to problems in gravitational lensing (Geyer), continue to be found.

III. Superrigidity Phenomena. Using new methods, such as the notion of *distortion* of group elements, progress continues to be made on the vastly ambitious Zimmer Program: *classifying ergodic measure preserving group actions on manifolds*. Loosely, a pair (G, M) is *superrigid* in a dynamical sense if any such action of G on M factors through an action which is in some sense uninteresting or trivial. Results include:

> (J. Franks/Handel.) Suppose f is a diffeomorphism of a closed, oriented surface S which preserves a Borel measure μ. If f is a distortion element in the group of such diffeomorphisms which are isotopic to the identity, then the dynamics of f is severely restricted. Applications include super-rigidity-type results for almost simple groups with distortion elements. Question: is there a surface-dynamics

proof that there are no distortion elements in the group of rigid motions of S^2?

(Parwani.) Any smooth action of $SL_n(Z)$ where $n > 2$ on an r-dimensional mod 2 homology sphere factors through a finite group action if $r < n - 1$. This result supports a recent conjecture of B. Farb and P. Shalen, which is that for such n and r the conclusion holds for actions on arbitrary compact manifolds.

IV. Applications of Dynamics. Dynamics has always been intimately connected to applications. Indeed, it has its origins in the analysis of solutions of differential equations and the study of iterative algorithms like Newton's method. The session and its problem session covered a diverse collection. Coven spoke about connections between the combinatorics of DNA sequences and the dynamics of cellular automata. Two speakers (Halbert and Komendarczyk) and one problem (by Phil Boyland) concerned the analysis of fluid flows as dynamical systems. Another problem (by Brian Raines) connected inverse limit spaces to economic models which allowed branching into alternative futures. In his semi-plenary talk, Geyer pointed to an application of complex dynamics to gravitational lensing.

Kevin Pilgrim, Phil Boyland, and Beverly Diamond are to be thanked for writing the above summary and assembling the following list of problems posed at the Dynamical Systems Problem Session at the conference.

Problems in Dynamical Systems

Below are problems posed at the Problem Session in Dynamical Systems. Included are some remarks of the poser and others.

Problem by Alexander Blokh, John Mayer, and Lex Oversteegen

Problem 1. (Invariant laminations (in the sense of Thurston). What ω-limit sets can occur for wandering N-gons in laminations of the unit disk?

Blokh and G. Levin gave bounds for the number of such polygons and for the number of sides of such polygons in terms of the degree d [4]. Later, they showed that if there exists a wandering polygon

P, there must exist at least one recurrent critical leaf whose ω-limit set coincides with that of the polygon [5]. Also, Blokh showed that if a cubic lamination has a wandering triangle then it must have exactly two critical leaves which are both recurrent and have the same limit set [3]. D. Childers (1) gives upper bounds on the number of wandering polygons and the number of sides in terms of the number of recurrent critical leaves, and (2) shows that if P is a wandering N-gon then in the topological Julia set $\omega(P) = \omega(c_i)$ for $N - 1$ distinct recurrent critical leaves c_i [7].

Problem by David Richeson and Jim Wiseman

Problem 2 (Equivalence of expanding properties). Let X be a noncompact, locally compact metric space and $f : X \to X$ be a homeomorphism. How are the following properties of the dynamical system (X, f) related? Expanding (E) (with respect to some compatible metric); Positively expansive (PE) (with respect to some compatible metric); Topologically positively expansive (TPE)?

A homeomorphism f is called *expanding* if there exist $\varepsilon > 0$ and $\lambda > 0$ such that $0 < d(x, y) < \varepsilon$ implies that $d(f(x), f(y)) > \lambda d(x, y)$; f is called *positively expansive* if there exists $\rho > 0$ such that for any distinct $x, y \in X$ there exists $n \geq 0$ with $d(f^n(x), f^n(y)) > \rho$; f is *topologically positively expansive* if, given any open neighborhood $U \subset X \times X$ of the diagonal Δ, there exists a closed neighborhood $N \subset U$ with the properties $N \subset \mathrm{Int}(F(N))$ and $\mathrm{Inv}(N) = \Delta$ (here $F = f \times f$ and Inv denotes the maximal invariant subset).

When X is a compact space all three notions are equivalent (and imply that X is finite). (See [11], [12], and [2] for proofs and references.) What if X is not compact? We know that TPE $\Longrightarrow$ PE: every TPE homeomorphism is PE with respect to some compatible metric ([12]). Also, PE $\not\Longrightarrow$ TPE: there is an example of a space X and a homeomorphism $f : X \to X$ such that f is PE but not TPE ([12]). It is easy to see that E $\Longrightarrow$ PE. We conjecture that E $\Longrightarrow$ TPE $\Longrightarrow$ PE and PE $\not\Longrightarrow$ TPE $\not\Longrightarrow$ E. In particular:

Conjecture 1: E $\Longrightarrow$ TPE: every E homeomorphism is TPE.

Conjecture 2: TPE $\not\Longrightarrow$ E: there is a space X and a homeomorphism $f : X \to X$ such that f is TPE, but f is not E with respect to any compatible metric.

Problem by Philip Boyland

Problem 3 (Time-periodic Euler fluid flows). Let $f : M^2 \to M^2$ be a C^k-diffeomorphism, $\alpha : M^2 \to \mathbb{R}$ be a C^r function such that $\alpha \circ f = \alpha$, i.e., the orbits of f are contained in the level sets of α. Find sufficient conditions for the topological entropy of f to vanish.

One such condition is the following: $r = 1, k = 1 + \epsilon$, and α has finitely many critical points. In applications, one knows only that the measure of the critical set is zero.

A variant: Let $f : M^3 \to M^3$ be a C^k diffeomorphism, Y a C^r-invariant vector field, i.e., $f_* Y = Y$. Relate dynamical properties of Y to those of f. For example, does the vanishing of the entropy of y (e.g., when $y = \nabla\phi$) imply the vanishing of the entropy of f?

Problems 4 and 5 by R. F. Williams

Problem 4 (Indecomposable continua in smooth dynamics). There exist n-dimensional continua M arising in dynamics such that (i) each point of M has a neighborhood U which is a product of an n-disk and a Cantor set, (ii) each such disk lies in a topologically smoothly embedded homeomorphic copy of $\mathbb{R}^n$ which is dense in U.

When $n = 1$ this implies that M is indecomposable. For $n = 2$, this is no longer the case. Is there a way of treating this in terms of continuum theory? I have tried, without much success; in particular, a problem with the following is that it perhaps implies, via induction on the dimension of M, that M is a bundle over a torus, with fiber a Cantor set, which would be false.

Conjecture: For $n = 2$, M is a bundle over S^1 with indecomposable fibers.

A solution to the conjecture implies that one can inductively understand such M.

(J. Rogers) Is there a Siegel disk G of a rational map such that the boundary of G is an indecomposable continuum? Does there exist a rational map with a Siegel disk G such that the prime ends of G form a nontrivial decomposition of the boundary of G?

Problem 5 (Dimension raising maps). Is there a map $f : D^2 \to X$ which raises the dimension of every subcontinuum?

Problem by Marcy Barge

Problem 6 (Interval exchange maps). Suppose that $T : I \to I$ is an interval exchange transformation. Does there exist an open interval $J \subset I$ and an integer $n > 0$ such that $T|_{T^i(J)}$ is continuous for $j = 0, 1, \ldots, n-1$ but $T^n(J) \cap J \neq \emptyset$?

An affirmative answer would imply that pseudo-Anosov surface homeomorphisms cannot embed into Anosov toral endomorphisms.

Problem by Stewart Baldwin

Problem 7 (Equivalence of regular measure spaces). The Oxtoby-Ulam theorem asserts that if λ, μ are two Borel probability measures on S^2 such that (i) points have measure zero, and (ii) nonempty open sets have positive measure, then $\mu = f^*(\lambda)$ for some homeomorphism $f : S^2 \to S^2$.

Can similar theorems be found for indecomposable continua, e.g., Knaster continua? One such require composants to have zero measure, and other conditions to account for the lack of homegeneity.

Problem by Evelyn Sander

Problem 8 (Explosion points). Let f_λ be a real one-parameter family of dynamical systems on X. A pair (λ_*, x) is called an *explosion point* if (i) $\lambda > \lambda_* \implies x$ is chain recurrent and (ii) $\lambda < \lambda_* \implies x$ has a neighborhood consisting of non-chain-recurrent points [1]. (Alternatively, this definition can be made using nonwandering points rather than chain recurrent points [10].)

J. Palis and F. Takens have conjectured that explosions in planar systems are due to tangencies and saddle-node bifurcations [10]; the statement fails in three dimensions [8]. What about noninvertible maps in one real dimension?

Problem by Kevin M. Pilgrim

Problem 9 (Finitely presented dynamics). Following D. Fried [9], we say that a dynamical system (X, f) is finitely generated provided there exists a subshift of finite type $\sigma : \Sigma \to \Sigma$ and a continuous map $\pi : \Sigma \to X$ such that $f \circ \pi = \pi \circ \sigma$; it is finitely presented if in addition the equivalence relation $\sim \subset \Sigma \times \Sigma$ given by $x \sim y \iff \pi(x) = \pi(y)$, as a dynamical system equipped with the map $\sigma \times \sigma : \sim \to \sim$, is conjugate to a subshift of finite type.

Question: Which quadratic invariant laminations are finitely presented (as dynamical systems on their corresponding quotient spaces)? In particular, are there any examples whose minor leaf is irrational?

Problem by Robert L. Devaney

Problem 10 (Sierpiński carpet Julia sets). Consider the family $f_\lambda(z) = z^n + \lambda/z^m$, $\lambda \in \mathbb{C}$, $n, m \geq 2$, $(n, m) \neq (2, 2)$. For λ near zero the Julia sets are homeomorphic to a product of a Cantor set and a quasicircle. Let Q be the connected hyperbolic component containing the origin in this family. Is ∂Q a Jordan curve?

Problem by Ethan M. Coven

Problem 11 (Dynamics of cellular automata). Consider one-sided cellular automata, i.e., sliding block codes without memory, defined on the space of all one-sided sequences from a finite alphabet. For simplicity, consider only a two-symbol alphabet and maps which are "linear in the first variable." An example is given by $(x_i)_{i=0}^\infty \mapsto (x_i + x_{i+1}x_{i+2})_{i=0}^\infty$, arithmetic in $\mathbb{Z}/2\mathbb{Z}$.

The problem is to say something intelligent about the dynamics of such maps.

All such maps have dense periodic points (see [6]) and preserve Bernoulli $(\frac{1}{2}, \frac{1}{2})$-measure. Except for maps which are also "linear in the last variable," little more is known about their dynamics. (For a counterexample to the last statement, see the papers of A. Maass.) The example above is the simplest which is linear in the first but not the last variable. Its topological entropy is unknown and it is unknown whether is is topologically transitive.

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