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MAXIMAL REALCOMPACT (AND OTHER) TOPOLOGIES

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ABSTRACT. Let **RC** be the class of realcompact Tychonoff spaces, $\mathfrak{m}$ the first uncountable measurable cardinal (which is the first Ulam measurable cardinal), and $\mathbf{P}(\mathfrak{m})$ the class of spaces in which each intersection of fewer than $\mathfrak{m}$ open sets is open. We begin with a simple theorem: If $X \in \mathbf{RC}$, there is another space μX on the same set as X whose topology is maximum for being **RC** and finer than the topology of X. (Of course, μX is discrete if and only if $|X| < \mathfrak{m}$.) This gives an operator $\mu : \mathbf{RC} \to \mathbf{M} := \{X : \mu X = X\}$ which is a coreflection. It is known that the $\mathbf{P}(\mathfrak{m})$-coreflection preserves **RC**; thus $\mathbf{M} \subseteq \mathbf{P}(\mathfrak{m}) \cap \mathbf{RC}$. The reverse inclusion represents an open question[1], but we prove it for two classes of spaces: Those for which the pseudocharacter does not exceed $\mathfrak{m}$, and those with fewer than $\mathfrak{m}$ nonisolated points. Various examples of spaces in **M** are presented, indeed: for every cardinal number $\mathfrak{n}$ there are spaces in **M** with exactly $\mathfrak{n}$ nonisolated points.

Actually, these observations about **RC** and $\mathfrak{m}$ are but a special case. In the previous paragraph we may replace **RC** by any class **R** which is productive and closed-hereditary and contains the two-point space, while replacing $\mathfrak{m}$ by

$\sigma(\mathbf{R}) := \sup\{\mathfrak{k} : \mathbf{R}$ contains the discrete space with $\mathfrak{k}$ points$\}$

(which, it is known, is a measurable cardinal if not ∞).

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[1]See the note at the end of this paper for recent relevant results established by Alan Dow.

1. Introduction and preliminaries

We refer to [KT] or [CN] for discussions of measurable cardinals, but we recall the definition. The cardinal $\mathfrak{k}$ is measurable if on a set of that cardinal there is a free ultrafilter with the property that each of its subsets of cardinal less than $\mathfrak{k}$ has nonempty intersection. Thus, ω is the first measurable cardinal, and $\mathfrak{m}$ is the second.

The development will be in the general terms mentioned at the end of the Abstract, $\mathbf{R}$ and $\sigma(\mathbf{R})$, but we shall occasionally call to attention our favorite (and motivating) special case, $\mathbf{R} = \mathbf{RC}$ and $\mathfrak{m} = \sigma(\mathbf{R})$. Thus:

$\mathbf{R}$ is a productive, closed-hereditary, homeomorphism-closed class in **Haus**, the category of Hausdorff spaces; and at several points we shall assume $\mathbf{R} \subseteq \mathbf{Tych}$ (the class of Tychonoff spaces). We also view $\mathbf{R}$ as a full subcategory of **Haus**. Then $\mathbf{R}$ is epireflective [K], and conversely. That $\mathbf{R}$ is epireflective means: For each $X \in \mathbf{Haus}$, there are $rX \in \mathbf{R}$ and continuous $\mathbf{R}$-extendible $r_X : X \to rX$ with $r_X[X]$ dense in rX. (To say that r_X is $\mathbf{R}$-extendible means, that whenever $f \in C(X,Y)$ with $Y \in \mathbf{R}$, there is $\overline{f} \in C(rX,Y)$ with $\overline{f} \circ r_X = f$.) So $r : \mathbf{Haus} \to \mathbf{R}$ is a functor, the *epireflection* of **Haus** into $\mathbf{R}$. When $\mathbf{R} = \mathbf{RC} \subseteq \mathbf{Tych}$ and $X \in \mathbf{Tych}$, rX is the familiar *Hewitt realcompactification* υX of X (see [H2], [GJ]).

For $\mathbf{A} \subseteq \mathbf{Haus}$, $\mathbf{R}(\mathbf{A})$ denotes the *epireflective hull* in $\mathbf{A}$, the class of closed subspaces of products of sets of spaces from $\mathbf{A}$ [HS].

For $\mathfrak{k}$ a cardinal, $D(\mathfrak{k})$ denotes the set, or discrete space, with $\mathfrak{k}$ points. Note that $D(2) \in \mathbf{R}$ if and only if $\mathbf{R}$ contains some space with at least two points. We shall assume that $D(2) \in \mathbf{R}$.

Definition 1.1 [HM]**.** If $\{\mathfrak{k} : D(\mathfrak{k}) \in \mathbf{R}\}$ has a cardinal upper bound, then $\sigma(\mathbf{R}) := \sup\{\mathfrak{k} : D(\mathfrak{k}) \in \mathbf{R}\}$. Otherwise, $\sigma(\mathbf{R}) := \infty$.

It is essentially a theorem of Mackey (see [GJ]) that $\sigma(\mathbf{RC}) = \mathfrak{m}$ (if $\mathfrak{m}$ exists).

Theorem 1.2 [HM]**.** (a) $\sigma(\mathbf{R}) = \infty$ *if and only if* $\mathbf{R}$ *contains every discrete space.*

(b) *If* $\sigma(\mathbf{R}) < \infty$*, then* $\sigma(\mathbf{R})$ *is measurable, and* $D(\mathfrak{k}) \in \mathbf{R}$ *if and only if* $\mathfrak{k} < \sigma(\mathbf{R})$.

(c) *If* $\mathfrak{l}$ *is measurable or* ∞*, then* $\sigma(\mathbf{R}\{D(\mathfrak{k}) : \mathfrak{k} < \mathfrak{l}\}) = \mathfrak{l}$.

Let $\mathfrak{k}$ be a regular cardinal. For $X \in \mathbf{Haus}$, $p(\mathfrak{k})(X)$ is the space on the same set as X whose topology has as base all intersections of fewer than $\mathfrak{k}$ X-open sets. We write

$\mathbf{P}(\mathfrak{k}) = \text{fix } p(\mathfrak{k}) := \{X : p(\mathfrak{k})X = X\}$.

$\mathbf{P}(\omega^+)$ is the well-known class of P-spaces [GJ].

In the $p(\mathfrak{k})$ context we allow $\mathfrak{k} = \infty$: $p(\infty)X$ is always discrete.

Theorem 1.3 [HM]**.** *Let $\mathfrak{k}$ be an infinite cardinal. Then*

(a) *the implication* $[\{X_i : i \in I\} \subseteq \mathbf{R}$ *and* $|I| < \mathfrak{k} \Rightarrow \Sigma_{i\in I} X_i \in \mathbf{R}]$ *holds if and only if* $\mathfrak{k} < \sigma(\mathbf{R})$*;*

(b) *if $\mathfrak{k}$ is regular and $\mathbf{R} \subseteq \mathbf{Tych}$, then the implication* $[X \in \mathbf{R} \Rightarrow p(\mathfrak{k})X \in \mathbf{R}]$ *holds if and only if* $\mathfrak{k} \leq \sigma(\mathbf{R})$.

For $\mathbf{R} = \mathbf{RC}$ and $\sigma(\mathbf{R}) = \mathfrak{m}$, 1.3(a) can be found in [GJ], and 1.3(b) (with generalizations) can be found in [CR]. See [HM] for further discussion.

We do not know if the hypothesis "$\mathbf{R} \subseteq \mathbf{Tych}$" is needed in 1.3(b); the issue is discussed in [HM].

1.4. Acknowledgement. We are indebted to Carol Wood for valuable comments, and to Javier Trigos for technical support, and to the referee for a careful and helpful reading of the paper.

2. Maximal R-topologies

Now and henceforth, $\mathbf{R}$ is an epireflective subcategory of $\mathbf{Haus}$ containing $D(2)$. We abbreviate $\sigma(\mathbf{R}) = \sigma$. For a space $X = (X, t)$ we write $X \in \mathbf{R}$, $(X, t) \in \mathbf{R}$, and $t \in \mathbf{R}$ interchangeably.

Theorem 2.1. *Let $(X, t) \in \mathbf{R}$ and set $S = \{s : (X, s) \in \mathbf{R}, s \supseteq t\}$. Then S has a largest element, denoted μt.*

Proof. Let $\vee S$ denote the supremum of S in the lattice of topologies on X. Then $(X, \vee S)$ is homeomorphic to the diagonal $\Delta \subseteq P := \Pi\{(X, s) : s \in S\}$ *via* the diagonal map derived from the family of continuous identity functions $(X, \vee S) \to (X, s)$ $(s \in S)$. Since $\mathbf{R}$ is closed under products, $P \in \mathbf{R}$. Since each $s \supseteq t$, the identity function $i : P = \Pi_{s\in S}(X, s) \twoheadrightarrow (X, t)^S$ is continuous. Since t is Hausdorff, the diagonal D in $(X, t)^S$ is closed there. Since $\Delta = i^{-1}(D)$ and i is continuous, Δ is closed in P. Since $P \in \mathbf{R}$ and $\mathbf{R}$ is closed-hereditary, $\Delta \in \mathbf{R}$.

Thus $\vee S \in S$, so $\vee S$ is the maximum of S. □

Comment 2.2. [$(X,t) \in \mathbf{R} \Rightarrow \mu t$ is discrete] if and only if $\sigma = \infty$. This can occur "intrinsically", as it were, e.g., for $\mathbf{R}$ the class of topologically complete spaces; or in the presence of the extra hypothesis "there is no uncountable measurable cardinal" (which is consistent with ZFC)—or for particular classes $\mathbf{R}$ in the presence of the appropriate local version of that hypothesis.

In any event, if there is no uncountable measurable cardinal then this paper is not interesting.

As with our writing $X \in \mathbf{R}$, $(X,t) \in \mathbf{R}$ and $t \in \mathbf{R}$ interchangeably, so shall we write either μX, $\mu(X,t)$, $(X,\mu t)$, or μt, as convenient.

Notation 2.3. $\mathbf{M} = \{(X,t) \in \mathbf{R} : t$ is a maximal $\mathbf{R}$-topology on $X\}$.
That is: $\mathbf{M} := \text{fix}\, \mu = \{X \in \mathbf{R} : \mu X = X\}$.
Just as σ abbreviates $\sigma(\mathbf{R})$, we write $\mu := \mu(\mathbf{R})$ and $\mathbf{M} := \mathbf{M}(\mathbf{R})$.

The following is immediate from 1.3(b).

Corollary 2.4. *Assume* $\mathbf{R} \subseteq \mathbf{Tych}$. *Then* $\mathbf{M} \subseteq \mathbf{P}(\sigma) \cap \mathbf{R}$.

Questions 2.5. (a) In 2.4, is the reverse inclusion valid (for every $\mathbf{R}$)? For particular $\mathbf{R}$? That is, is every $\mathbf{P}(\sigma) \cap \mathbf{R}$-topology $\mathbf{R}$-maximal (for every $\mathbf{R}$)? For particular $\mathbf{R}$?
(b) Is $\mathbf{M}(\mathbf{RC}) \supseteq P(\mathfrak{m}) \cap \mathbf{RC}$ valid? That is, is every $P(\mathfrak{m}) \cap \mathbf{RC}$-topology a maximal $\mathbf{RC}$-topology?

2.6. Comments. 1. We suspect that the answers are "no", but we have no real evidence.
2. Of course it is conceivable that the answer to (b) is "yes" and to (a) is "no", but we have found nothing to distinguish the special case $\mathbf{RC}$ from general $\mathbf{R}$.
3. In [C] (page 62), reference is made to a preliminary version of this paper and to special cases of the questions in this paper.
4. For some particular $\mathbf{R}$ it is (of course) true that $\mathbf{M} \supseteq \mathbf{P}(\sigma) \cap \mathbf{R}$, for example if $\sigma = \infty$ (where $\mathbf{P}(\sigma)$ is the class of discrete spaces); or if $\mathbf{R}$ is the class of compact Hausdorff spaces (where $\sigma = \omega$, $\mathbf{P}(\sigma) = \mathbf{Haus}$, and every compact space is compact-maximal.) One might also suspect for general $\mathbf{R}$ that $\sigma = \omega$ implies $\mathbf{M} \supseteq \mathbf{P}(\sigma) \cap \mathbf{R}$, but we do not know that.

We conclude this section with a brief digression, an isolated parallel to 2.1. Let $\mathbf{R}_d$ be the class of spaces $X \in \mathbf{R}$ such that X is dense-in-itself (i.e., X has no isolated points).

Theorem 2.7. *Let* $(X,t)\in\mathbf{R}_d$*, and set* $S_d := \{s : (X,s)\in\mathbf{R}_d, s\supseteq t\}$*. Then* S_d *has maximal elements.*

Proof. The set S_d is partially ordered by inclusion. According to Zorn's Lemma it suffices to show for every chain $C \subseteq S_d$ that $\vee C$, the supremum of C in the lattice of topologies on X, is in S_d. The argument in the proof of 2.1 shows $\vee C \in \mathbf{R}$. And, since no $s \in S$ contains a singleton, and C is a chain, the set $\vee C$ itself contains no singleton. □

2.8. Remarks. (a) When $\mathbf{R} = \mathbf{Tych}$ in 2.7, the indicated construction is as given by Hewitt [H1]; as he showed, the maximal topologies $s \in S_d$ are extremally disconnected spaces, and are *irresolvable* in the sense that (X,s) admits no pair of complementary dense subsets.

(b) Take $\mathbf{R} = \mathbf{RC}$ and let $(X,t) \in \mathbf{RC}$ be the usual real line; what then are the maximal elements of S_d? (The Sorgenfrey Line shows that t is not maximal.)

(c) The analogue of 2.7 holds for the class $\mathbf{R}_c$ of connected spaces in $\mathbf{R}$.

We hope to return to these issues in a later communication.

3. μ AS A COREFLECTION

There is a further aspect of the situation, which we discuss briefly in this section. If the answer to 2.5(a) or (b) is "no", we may return to this in the future.

In a category $\mathbf{B}$, a full subcategory $\mathbf{C}$ is *monocoreflective* if: For each $X \in \mathbf{B}$ there are $cX \in \mathbf{C}$ and a monic morphism $c_X : cX \to X$ (called the *coreflection* of X into $\mathbf{C}$) such that, for each $f \in \mathbf{B}(Y,X)$ with $Y \in \mathbf{C}$ there is (unique) $\overline{f} \in \mathbf{B}(Y, cX)$ with $c_X \circ \overline{f} = f$. So $c : \mathbf{B} \to \mathbf{C}$ is a functor, the *monocoreflection*.

It is easy to see that in our categories $\mathbf{R}$, monic means one-to-one, and that $p(\mathfrak{k}) : \mathbf{Haus} \to \mathbf{P}(\mathfrak{k})$ (or $p(\mathfrak{k}) : \mathbf{Tych} \to \mathbf{P}(\mathfrak{k}) \cap \mathbf{Tych}$) is a monocoreflection.

Proposition 3.1. (a) *For each* $X \in \mathbf{R}$*,* $\mu(\mu X) = \mu X$*. That is,* $\mu X \in \mathbf{M}$.

(b) *Let* $X \in \mathbf{R}$*,* $Y \in \mathbf{M}$*, and let* $f : Y \to X$ *be continuous. Then* $f : Y \to \mu X$ *is still continuous; equivalently: with* $\mu_X : \mu X \to X$ *denoting the continuous identity function on the set* X*, there is (unique)* $\overline{f} \in C(Y, \mu X)$ *with* $\mu_X \circ \overline{f} = f$*.*

(c) *For each* $X \in \mathbf{R}$*,* $\mu_X : \mu X \to X$ *is a monocoreflection of* X *into* $\mathbf{M}$*. That is,* $\mu : \mathbf{R} \to \mathbf{M}$ *is a monocoreflection.*

Proof. (a) is immediate.

(b) Given such $f : (Y, u) \to (X, t)$ (with topologies made explicit for clarity), consider the diagonal map $h := \langle \mathrm{id}_Y, f\rangle : (Y, u) \to (Y, u) \times (X, t)$ defined by $\pi_1 \circ h = \mathrm{id}_Y$ and $\pi_2 \circ h = f$. This is an embedding since id_Y is. That is, $h : (Y, \mu) \to (h[Y], (u \times t)|h[Y])$ is a homeomorphism, so $h[Y] \in \mathbf{M}$. Further, the image $h[Y]$ is closed in $Y \times X$: If $(y, x) \notin h[Y]$ then $x \neq f(y)$ so there are disjoint t-open neighborhoods U and V of x and y, respectively; then $(y, x) \in f^{-1}(V) \times U \in u \times t$, and $f^{-1}(U) \times V$ misses $h[Y]$.

So, $h[Y]$ is also closed in the finer topology $u \times \mu t$, and thus $h[Y]$ inherits an $\mathbf{R}$-topology from $u \times \mu t$; call this topology s. Since $s \supseteq (u \times t)|h[Y] \in \mathbf{M}$, we have $s = (u \times t)|h[Y]$. That means that $h : (Y, u) \to (Y, u) \times (X, \mu t)$ is continuous, so $f = \pi_2 \circ h : (Y, u) \to (X, \mu t)$ is continuous.

(c) This is (a) and (b). □

We now make some remarks of categorical nature whose intent is to shed light on $\mathbf{M}$ and μ (in the case that the answers to questions 2.5 are "no"). That intent is realized only partially, but interesting questions arise. In an attempt to be brief, we refer the reader to [HS] for most unexplained terms.

The following is easy to prove.

Proposition 3.2. *If* $\mathbf{C}$ *is a monocoreflective subcategory of* $\mathbf{R}$*, then*

(a) $\mathbf{C}$ *contains all singletons, and each coreflection map* $c_X : cX \to X$ *is one-to-one and onto; and*

(b) $\mathbf{M} \subseteq \mathbf{C}$*.*

Then we may rephrase the issue in Question 2.5.

Corollary 3.3. *Suppose* $\mathbf{R} \subseteq \mathbf{Tych}$*, so that* $p(\sigma)\mathbf{R} \subseteq \mathbf{R}$ *and* $\mathbf{M} \subseteq \mathbf{P}(\sigma) \cap \mathbf{R}$*. Then these are equivalent.*

(a) $\mathbf{M} = \mathbf{P}(\sigma) \cap \mathbf{R}$.

(b) $p(\sigma)$ *is the largest monocoreflection in* **Haus** *(or* **Tych***) which preserves* $\mathbf{R}$.

(c) $\mu = p(\sigma)|\mathbf{R}$.

(In (b), "largest" refers to this order on monocoreflections: $c \leq d$ if and only if for each X there is $f : dX \to cX$ with $c_X \circ f = d_X$. By 3.2, one may construe c_X and d_X as just the identity function on the set X; so $c \leq d$ just means that, for every X, dX carries a finer topology than cX.)

The rest of this section represents thoughts intended to extract information about $\mathbf{M}$ from the fact that $\mathbf{M}$ is the smallest monocoreflective subcategory of $\mathbf{R}$. Perhaps the reader will see how to actually do that.

From 34.1 and 37.4 of [HS], we see that in a "sufficiently complete" category $\mathbf{B}$: A full subcategory $\mathbf{C}$ is monocoreflective if and only if $\mathbf{C}$ is closed under the sums and extremal epics of $\mathbf{B}$. (The morphism e is called *extremal epic*, or just *e.e.*, or perhaps $\mathbf{B}$-*e.e.* for clarity, if (1) e is epic and (2) $e = mf$ with m monic implies that m is an isomorphism.) Each object class $\mathbf{A}$ is contained in a least monocoreflective subcategory $\mathbf{C}(\mathbf{A})$, and $C \in \mathbf{C}(\mathbf{A})$ if and only if there is an *e.e.* : $\Sigma_{i \in I} A_i \to C$ for some set $\{A_i : i \in I\} \subseteq \mathbf{A}$.

It is easy to see that (any of our) $\mathbf{R}$ is sufficiently complete, and the $\mathbf{R}$-sum is $r\Sigma_i X_i$, with $\Sigma_i X_i$ denoting the topological sum. (See §4 for more on this.) It follows now that $\mathbf{M} = \mathbf{C}(\{p\})$, and

Corollary 3.4. *For* $X \in \mathbf{R}$*, these are equivalent.*

(a) $X \in \mathbf{M}$ *(i.e., each* $\mathbf{R}$*-map* $Y \to X$ *which is monic and a surjection is a homeomorphism);*

(b) *each* $\mathbf{R}$*-map* $Y \to X$ *which is a surjection is* $\mathbf{R}$*-e.e.; and*

(c) *the natural map* $rdX \to X$ *(extending the identity function from* dX*, the set* X *with the discrete topology) to* X *is* $\mathbf{R}$*-e.e.*

Now what does that mean? A major problem is not knowing what epic, or *e.e.*, means for $\mathbf{R}$. In **Haus**, in fact for any $\mathbf{R} \subseteq \mathbf{Tych}$ for which $X \subseteq rX$ for each $X \in \mathbf{Tych}$, epic = dense image.

But, not so for certain other **R** [S]. And for **Haus**, *e.e.* = quotient (so *e.e.* implies surjective). But even for **R** = **RC** there are non-surjective *e.e.*'s (e.g., an example in [E](page 220)), and we do not know what **RC**-*e.e.* means. But one still hopes that the idea, e.g., 3.4(c), will yield something. Here is an attempt in that direction.

In a concrete category, the notion of epic may be more natural than that of surjection. So we define, in **R**: $X \in \mathbf{M}^+$ means that each **R**-map $Y \to X$ which is monic and epic is a homeomorphism onto X. Parallel to 3.4 we have

Proposition 3.5. *For $X \in \mathbf{R}$, these are equivalent.*

(a) $X \in \mathbf{M}^+$*;*

(b) *each* **R***-epic* $Y \to X$ *is* **R**-*e.e.; and*

(c) *for each* **R***-epically embedded* $Z \subseteq X$*, the natural map* $rdZ \to X$ *is* **R**-*e.e.*

Of course, $\mathbf{M}^+ \subseteq \mathbf{M}$. We do not know if $\mathbf{M}^+ = \mathbf{M}$, either for general **R** or **R** = **RC**. We can show that $\mathbf{M}^+$ is closed under **R**-sums; thus $\mathbf{M}^+ = \mathbf{M}$ if and only if $\mathbf{M}^+$ is closed under **R**-*e.e.*'s if and only if $\mathbf{M}^+$ is monocoreflective in **R**. We can also show that all the **M**-examples in §4, save 4.13, actually lie in $\mathbf{M}^+$.

Summing up: *e.e.*'s and $\mathbf{M}^+$ have accomplished nothing toward understanding **M** and resolving 2.5, but the above sketch is presented to that end—perhaps a reader can make some progress. For now, we abandon the approach.

4. Examples of spaces in M

Among other things in this section, we show that for each cardinal $\mathfrak{n}$ (including $\mathfrak{n} = 0$—see 4.13) there are spaces $X \in \mathbf{M}$ with exactly $\mathfrak{n}$ nonisolated points.

As noted in 3.4(a), $X \in \mathbf{M}$ if and only if $X \in \mathbf{R}$ and every continuous bijection $f : Y \twoheadrightarrow X$ with $Y \in \mathbf{R}$ is a homeomorphism. Sometimes this is convenient.

Several of our examples involve **R**-sums: The **R**-*sum* of $\{X_i : i \in I\} \subseteq \mathbf{R}$ is $r\Sigma_{i\in I} X_i$, where $\Sigma_{i\in I} X_i$, here for brevity usually denoted $\Sigma_i X_i$, is the usual topological sum (the "disjoint union"). We noted in §3 that it follows from somewhat involved categorical ideas that $[\{X_i\} \subseteq \mathbf{M} \Rightarrow r\Sigma_i X_i \in \mathbf{M}]$. But we give a bare-hands proof of this.

Lemma 4.1. *If $\{X_i\} \subseteq \mathbf{R}$, then the reflection map $r : \Sigma_i X_i \to r\Sigma_i X_i$ is an embedding (so we can write $\Sigma_i X_i \subseteq r\Sigma_i X_i$).*

Proof. It suffices that $X := \Sigma_i X_i$ embeds into some **R**-space, and for that it suffices that $C(X, \mathbf{R})$ separates points from closed sets (by the Embedding/Diagonal Lemma [E]). For each i, choose $p_i \notin X_i$ and define $f_i : X \to X_i + \{p_i\}$ by: $f_i(x) = x$ if $x \in X_i$, $f_i(x) = p_i$ if $x \notin X_i$. Then $\{f_i\}$ separates points from closed sets, and $X_i + \{p_i\} \in \mathbf{R}$ because: $D(2) \in \mathbf{R}$ and $X_i \in \mathbf{R}$, so $X_i \times D(2) = X_i \times \{0, 1\} \in \mathbf{R}$ and (for arbitrary $p \in X_i$) $X_i + \{p_i\}$ is homeomorphic to $(X_i \times \{0\}) \cup \{(p, 1)\}$, a closed subspace of $X_i \times D(2)$. □

Theorem 4.2. *If $\{X_i\} \subseteq \mathbf{M}$, then $r\Sigma_i X_i \in \mathbf{M}$.*

Proof. Let $X = \Sigma_i X_i$, and let $f : Y \to rX$ be a continuous bijection, with $Y \in \mathbf{R}$. First, we show each X_i is closed in rX. Each of the following inclusion maps is **R**-extendible: X_i in X (easy); X in rX (by definition); X_i in rX (by transitivity); X_i in its closure in rX (by restriction). Thus $\overline{X_i}^{rX} = rX_i$; but $rX_i = X_i$ since $X_i \in \mathbf{M} \subseteq \mathbf{R}$.

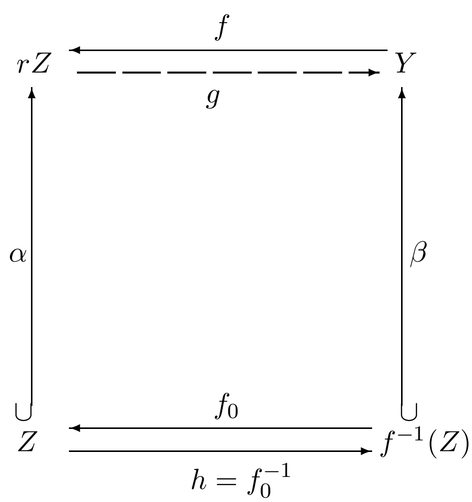
Hence $f^{-1}(X_i)$ is closed in Y, so $f^{-1}(X_i) \in \mathbf{R}$. Since the restriction $f : f^{-1}(X_i) \to X_i$ is a bijection, it is a homeomorphism (since $X_i \in \mathbf{M}$).

Further, each X_i is open in X, so $f^{-1}(X_i)$ is open in $f^{-1}(X) = \cup_j f^{-1}(X_j)$. This shows $f^{-1}(X) = \Sigma_i f^{-1}(X_i)$. Hence the restriction $f : \Sigma_i f^{-1}(X_i) \to X = \Sigma_i X_i$ is a homeomorphism on each (clopen) "ith piece", and is therefore a homeomorphism.

We are in the situation of the following curious lemma, which concludes the proof of Theorem 4.2.

Lemma 4.3. *Suppose $Z \subseteq rZ$, and $Y \in \mathbf{R}$. Let $f : Y \to rZ$ be continuous, and have the property that f is one-to-one and $f|f^{-1}(Z)$ is a homeomorphism (onto Z). Then f is a homeomorphism onto rZ.*

Proof. Consider the diagram



in which: α, β are labels for the inclusions; $f_0 := f|f^{-1}(Z)$, so that $\alpha \circ f_0 = f \circ \beta$; f_0 is assumed a homeomorphism, so f_0 has inverse h, and $f_0 \circ h = \mathrm{id}_Z$; α is the **R**-reflection map and $Y \in \mathbf{R}$, so there is g with $g \circ \alpha = \beta \circ h$. Then

$$f \circ g \circ \alpha = f \circ \beta \circ h = \alpha \circ f_0 \circ h = \alpha \circ \mathrm{id}_Z = \alpha = \mathrm{id}_{rZ} \circ \alpha.$$

Since α is epic (i.e., dense), $f \circ g = \mathrm{id}_{rZ}$. This means that f is a retraction. But f is also one-to-one, hence monic, and a monic retraction is an isomorphism, i.e., a homeomorphism onto. (See [HS].) $\square$

Corollary 4.4. (a) *If D is discrete, then $rD \in \mathbf{M}$.*
(b) *If D and E are discrete, then $r(E \times rD) \in \mathbf{M}$.*

Proof. (a) $D = \Sigma\{\{x\} : x \in D\}$ and each $\{x\} \in \mathbf{M}$, so 4.2 applies.
(b) $E \times rD = \Sigma\{\{y\} \times rD : y \in E\}$, and each $\{y\} \times rD \in \mathbf{M}$ by (a), so again 4.2 applies. $\square$

When $\mathbf{R} = \mathbf{RC}$, 4.4(a) is the statement that the Hewitt realcompactification υD of a discrete space D admits no larger realcompact topology. It is easily seen that $\upsilon(E \times \upsilon D) = \upsilon E \times \upsilon D$ fails for discrete D, E with $|D|, |E| \geq \mathfrak{m}$.

Now we consider a different kind of extension of a sum. Let $\mathfrak{k}$ be a cardinal. Given $\{X_i : i \in I\} \subseteq \mathbf{R}$, take $p \notin \Sigma_i X_i$. The space $\Sigma_i X_i \overset{\mathfrak{k}}{\cdot} p$ is $\Sigma_i X_i \cup \{p\}$, with basic neighborhoods of points in $\Sigma_i X_i$ as usual and with basic neighborhoods of p all sets of the form $\{p\} \cup \Sigma_{i \in I \setminus J} X_i$ with $|J| < \mathfrak{k}$. Note that if each X_i is a point, so $\Sigma_i X_i$ is a discrete space D, then, with αD denoting the one-point compactification, we have $\Sigma_i X_i \overset{\mathfrak{k}}{\cdot} p = p(\mathfrak{k})\alpha D$.

Theorem 4.5. (a) [HM] $[\{X_i\} \subseteq \mathbf{R} \Rightarrow \Sigma_i X_i \overset{\mathfrak{k}}{\cdot} p \in \mathbf{R}]$ *if and only if* $\mathfrak{k} \leq \sigma$.

(b) $[\{X_i\} \subseteq \mathbf{M} \Rightarrow \Sigma_i X_i \overset{\mathfrak{k}}{\cdot} p \in \mathbf{M}]$ *if and only if* $\mathfrak{k} = \sigma$.

Proof. (a) See [HM].

(b) If $\mathfrak{k} > \sigma$ the implication fails because singletons are in $\mathbf{M}$, $D(\mathfrak{k})$ is the sum of its points, and $D(\mathfrak{k}) \overset{\mathfrak{k}}{\cdot} p$ is discrete of cardinal $\mathfrak{k} > \sigma$, thus not even in $\mathbf{R}$.

If $\mathfrak{k} < \sigma$ the implication fails because $D(\sigma)$ is the sum of its points, $D(\sigma) \overset{\mathfrak{k}}{\cdot} p \in \mathbf{R}$ by (a), but $D(\sigma) \overset{\mathfrak{k}}{\cdot} p \notin \mathbf{M}$ because $D(\sigma) \overset{\sigma}{\cdot} p \in \mathbf{R}$ and its topology strictly extends that of $D(\sigma) \overset{\mathfrak{k}}{\cdot} p$.

So it remains to show $\{X_i : i \in I\} \subseteq \mathbf{M} \Rightarrow \Sigma_i X_i \overset{\sigma}{\cdot} p \in \mathbf{M}$.

If $|I| < \sigma$, then p is isolated and $\Sigma_i X_i \overset{\sigma}{\cdot} p = (\Sigma_i X_i) + \{p\} \in \mathbf{R}$ (as in the proof of 4.1) and the result follows from 4.2.

When $|I| \geq \sigma$, the topology of $\Sigma_i X_i \overset{\sigma}{\cdot} p$ has a maximality feature depending only on σ, not fully depending on $\mathbf{R}$. This will arise again later. Here we interpolate 4.6, 4.7 and 4.8 before completing the present proof.

Definition 4.6. Let $\mathfrak{k}$ be a regular cardinal. Then $\{\mathrm{cd} < \mathfrak{k}\}$ is the class of Hausdorff spaces in which every closed discrete subspace has cardinality $< \mathfrak{k}$.

So $X \in \{\mathrm{cd} < \omega\}$ means X is countably compact. Always $\mathbf{R} \subseteq \{\mathrm{cd} < \sigma\}$, and some of 4.5(a) is included in the following easily proved fact.

Proposition 4.7. $\Sigma_i X_i \overset{\mathfrak{k}}{\cdot} p \in \{\mathrm{cd} < \mathfrak{n}\}$ *($\mathfrak{n}$ regular) if and only if*

(a) $X_i \in \{\mathrm{cd} < \mathfrak{n}\}$ *for each i, and*

(b) *either* $|I| < \mathfrak{n}$ *or* $\mathfrak{k} \leq \mathfrak{n}$.

But our point is:

Proposition 4.8. *Let $\{X_i\} \subseteq$ **Haus**, let t_i be the topology of X_i and t the topology of $\Sigma_i X_i \overset{\mathfrak{k}}{\cdot} p$. Suppose $|I| \geq \mathfrak{k}$. Let s be another topology on the set of $\Sigma_i X_i \overset{\mathfrak{k}}{\cdot} p$ for which $s|X_i = t_i$ for each i, $s \supseteq t$ and $s \neq t$. Then $s \notin \{\mathrm{cd} < \mathfrak{k}\}$.*

Proof. There are $G \in s$ and $q \in G$ such that no $H \in t$ satisfies $q \in H \subseteq G$. Since $G_i := G \cap X_i \in s|X_i = t_i$ for each i, we have $q = p$. Thus $G = \{p\} \cup \cup_{i \in I} G_i$, and $J := \{i : G_i \neq X_i\}$ has $|J| \geq \mathfrak{k}$. For each $i \in J$, choose $x_i \in X_i \backslash G_i$ and let $P := \{x_i : i \in J\}$. Then P is s-discrete since $\{x_i\} = P \cap X_i$, and P is s-closed since $P \cap G = \emptyset$. □

Now we conclude the proof of 4.5(b). Suppose $\{X_i\} \subseteq \mathbf{M}$ and $|I| \geq \sigma$. By 4.5(a), $X := \Sigma_i X_i \overset{\sigma}{\cdot} p \in \mathbf{R}$. Let t be the topology of X and s another topology with $t \subseteq s \in \mathbf{R}$. Then for each i, $s|X_i \in \mathbf{R}$ (since X_i is t-closed and hence s-closed), and $s|X_i \supseteq t|X_i$. But $t|X_i \in \mathbf{M}$, so $s|X_i = t|X_i$. Now 4.8 applies: If s contains t strictly then $s \notin \{\mathrm{cd} < \sigma\} \supseteq \mathbf{R}$, a contradiction. □

Corollary 4.9. *Let $\{X_i\} \subseteq \mathbf{R}$ contained in* **Tych**. *Then $\Sigma_i \mu X_i \overset{\sigma}{\cdot} p = \mu(\Sigma_i X_i \overset{\omega}{\cdot} p) = p(\sigma)(\Sigma_i \mu X_i \overset{\omega}{\cdot} p)$.*

4.9 is inferred easily from 4.5 and the fact that each $\mu X_i \in \mathbf{P}(\sigma)$ (2.4). The next is perhaps more interesting, and should be read together with 4.13 below (the case $\mathfrak{n} = 0$).

Corollary 4.10. *Suppose $\sigma < \infty$. Then for any cardinal $\mathfrak{n} \neq 0$, there is $X \in \mathbf{M}$ with exactly $\mathfrak{n}$ nonisolated points.*

Proof. Let $|I| = \mathfrak{n}$, and for each $i \in I$ let $X_i = D(\sigma) \overset{\sigma}{\cdot} p$ (where we regard $D(\sigma)$ as the sum of its singleton subsets). Then $X_i \in \mathbf{M}$, and $X = \Sigma_i X_i \overset{\sigma}{\cdot} q \in \mathbf{M}$ by 4.5. □

What motivates the next example is: $\mathbf{P}(\sigma)$ is closed under finite products, hence so is $\mathbf{P}(\sigma) \cap \mathbf{R}$. So if $\mathbf{M}$ were not, then $\mathbf{M} \neq \mathbf{P}(\sigma) \cap \mathbf{R}$. Even in the special case $\mathbf{R} = \mathbf{RC}$, we do not know if the class $\mathbf{M}$ is closed under finite products. We can offer (only) 4.12 below (see also §5).

Lemma 4.11. *Let $X, Y \in \mathbf{M}$, let t be the topology of $X \times Y$, and let $t \subseteq s \in \mathbf{R}$. Then*

(a) *For each vertical or horizontal line L (i.e., for $L = V_x := \{x\} \times Y$ or $L = H_y := X \times \{y\}$), $s|L = t|L$.*

(b) *Let $(x, y) \in X \times Y$. If x is isolated in X (or if y is isolated in Y), then each s-neighborhood of (x, y) is a t-neighborhood.*

(c) *Suppose $s \supseteq t$, $s \neq t$. Then there are $p \in X$ and $q \in Y$, each not isolated, and $G \in s$ with $(p, q) \in G$ such that, whenever A is open in X containing p and B is open in Y containing q, then there are $x \neq p$ and $y \neq q$ with $(x, y) \in (A \times B)\backslash G$.*

Proof. (a) L is t-closed, thus s-closed, so $s|L \in \mathbf{R}$. But $(L, t|L)$ is homeomorphic to Y or to X, hence is in $\mathbf{M}$.

(b) If $(x, y) \in G \in s$, then $G \cap V_x \in s|V_x = t|V_x$ by (a), and $G \cap V_x \in t$ since V_x is t-open (since x is isolated). Since $G \supseteq G \cap V_x$, G is a t-neighborhood of (x, y).

(c) Some point (p, q) has an s-neighborhood which is not a t-neighborhood. By (b), neither p nor q is isolated. Now given A and B, let $W := (A \cap \pi_1(G \cap H_q)) \times (B \cap \pi_2(G \cap V_p))$. Then $W \not\subseteq G$, but $W \cap L = (A \times B) \cap G \cap L$ for $L = H_q$ or $L = V_p$. $\square$

Theorem 4.12. *Let D and E be discrete. Then $D \overset{\sigma}{\cdot} p \times E \overset{\sigma}{\cdot} q \in \mathbf{M}$.*

Proof. Let $X = D \overset{\sigma}{\cdot} p$ and $Y = E \overset{\sigma}{\cdot} q$. If $|D| < \sigma$, then X is discrete and $X \times Y = \Sigma\{\{x\} \times Y : x \in X\} \in \mathbf{M}$ since $Y \in \mathbf{M}$ and $|X| < \sigma$ (by 4.5 and 1.2).

So let $|D|, |E| \geq \sigma$ and suppose $t \subsetneq s \in \mathbf{R}$, and let $(p, q) \in G \in s\backslash t$ be as given by 4.11. By induction, we construct (faithfully indexed) $\{x_\alpha : \alpha < \sigma\}$ and $\{y_\alpha : \alpha < \sigma\}$ such that $P := \{(x_\alpha, y_\alpha) : \alpha < \sigma\}$ is s-discrete and s-closed. Since $|P| = \sigma$, this contradicts $s \in \mathbf{R}$.

Choose $(x_0, y_0) \in (X \times Y)\backslash G$, $x_0 \neq p$ and $y_0 \neq q$ (using 4.11(c)). Suppose $\beta < \sigma$, and for each $\alpha < \beta$ we have distinct x_α and y_α with $(x_\alpha, y_\alpha) \notin G$ and $x_\alpha \neq p$ and $y_\alpha \neq q$. Let $A = X\backslash\{x_\alpha : \alpha < \beta\}$ and $B = Y\backslash\{y_\alpha : \alpha < \beta\}$. These are open neighborhoods of p in X and q in Y, respectively. Then again by 4.11(c) there is $(x_\beta, y_\beta) \in (A \times B)\backslash G$ with $x_\beta \neq p$ and $y_\beta \neq q$.

So we have $P = \{(x_\alpha, y_\alpha) : \alpha < \sigma\}$, and $P \subseteq D \times E$, so P is t-discrete, thus s-discrete. We will show that $P = \overline{P}^s$. Surely $(p,q) \notin \overline{P}^s$ (from $P \cap G = \emptyset$), and no $(x,y) \in (D \times E)\backslash P$ is in $\overline{P}^s$. If $x \neq p$ and $(x,q) \in \overline{P}^s$, then every s-neighborhood of (x,q) meets P, and in particular $V_x \cap P \neq \emptyset$. So in particular there is α with $x = x_\alpha$. But then $V_x \cap (\{x\} \times (Y \backslash \{y_\alpha\})) \cap P \neq \emptyset$ and thus contains some point (x_γ, y_γ). So $x_\gamma = x = x_\alpha$, so $\gamma = \alpha$, while $y_\gamma \neq y_\alpha$ so $\alpha \neq \gamma$. This contradiction shows that $x \neq p \Rightarrow (x,q) \notin \overline{P}^s$; symmetrically, $y \neq q \Rightarrow (p,y) \notin \overline{P}^s$. □

The nondiscrete examples of $X \in \mathbf{M}$ displayed so far have a dense set of isolated points. However:

Proposition 4.13. (a) *For any* $\mathfrak{k} \geq \sigma$, $X = 2^{\mathfrak{k}} = (D(2))^{\mathfrak{k}}$ *has these features:* $X \in \mathbf{R}$; $|X| \geq \sigma$; X *is homogeneous.*

(b) *If* Y *is homogeneous, then* μY *is homogeneous.*

(c) *If* X *is any space with the features in (a), then* μX *is a space in* $\mathbf{M}$ *without isolated points.*

Proof. (a) $2 \in \mathbf{R}$ implies $2^{\mathfrak{k}} \in \mathbf{R}$; for cardinals, $\mathfrak{k} \geq \sigma$ implies $2^{\mathfrak{k}} \geq 2^\sigma > \sigma$; and the space $2^{\mathfrak{k}} = (D(2))^{\mathfrak{k}}$ is a topological group, hence is homogeneous.

(b) For any coreflection c in any category $\mathbf{B}$ and for any object Y, any endomorphism h of Y "lifts" to an endomorphism $\overline{h}$ of cY; that is easily proved. In our present situation, the coreflection map $\mu_Y : \mu Y \to Y$ is the identity function, so (b) follows.

(c) Since $|X| \geq \sigma$ and $\mu X \in \mathbf{R}$, μX is not discrete; and since μX is homogeneous, no point is isolated. □

5. Sometimes $\mathbf{P}(\sigma) \cap \mathbf{R}$ implies $\mathbf{M}$

We show that this is so under either of two smallness hypotheses, thus partially answering question 2.5. In this section we make the blanket assumption "all $\mathbf{P}(\sigma)\cap\mathbf{R}$-spaces are zero-dimensional"; this would follow from assuming "$\mathbf{R} \subseteq$ **Tych**", or just "every $\mathbf{P}(\omega^+)$-space is zero-dimensional", which holds in **Tych**.

The first theorem reveals the complete structure of certain spaces.

Theorem 5.1. *If $X \in \mathbf{P}(\sigma) \cap \mathbf{R}$, and if X has exactly $\mathfrak{n} < \sigma$ nonisolated points ($\mathfrak{n} > 0$) $\{p_\delta : \delta < \mathfrak{n}\}$, then there are discrete subspaces D_δ of X ($\delta < \mathfrak{n}$) such that $X\backslash\{p_\delta : \delta < \mathfrak{n}\} = \Sigma_{\delta<\mathfrak{n}} D_\delta$ and $X = \Sigma_{\delta<\mathfrak{n}} D_\delta \overset{\sigma}{\cdot} p_\sigma$. Thus $X \in \mathbf{M}$.*

Proof. [We give the proof in stages, (a), (b) and (c).]

(a) The case $\mathfrak{n} = 1$. Let p be the one nonisolated point of X, and consider

$$f : X = (X\backslash\{p\}) \cup \{p\} \to \alpha(X\backslash\{p\}) = (X\backslash\{p\}) \cup \{p\},$$

where $\alpha(X\backslash\{p\})$ is the one-point compactification of the discrete space $X\backslash\{p\}$, the point at infinity being called p, and f is the (continuous) identity function.

Since $X \in \mathbf{P}(\sigma)$, $f : X \to p(\sigma)\alpha(X\backslash\{p\})$ is also continuous. But the latter space is $(X\backslash\{p\}) \overset{\sigma}{\cdot} p$, as noted before 4.5, and this is in $\mathbf{M}$.

Since $X \in \mathbf{R}$, f is a homeomorphism, so the topology of X is that of $(X\backslash\{p\}) \overset{\sigma}{\cdot} p$.

(b) Lemma. If $X \in \mathbf{P}(\mathfrak{l})$, then for any infinite $\mathfrak{k} < \mathfrak{l}$ the space X is "$\mathfrak{k}$-Hausdorff" in the sense that for every (faithfully indexed) $\mathfrak{k}$-sequence $A = \{p_\alpha : \alpha < \mathfrak{k}\}$ in X there are disjoint clopen sets U_α with each $p_\alpha \in U_\alpha$.

Proof of Lemma. The set A is discrete, so each p_α has a clopen neighborhood V_α which misses $A\backslash\{p_\alpha\}$. The neighborhoods $U_\alpha := V_\alpha \backslash \cup_{\beta\neq\alpha} V_\beta$ are then as required.

(c) We conclude the proof of the theorem. By (b), the nonisolated points p_δ ($\delta < \mathfrak{n}$) have disjoint clopen neighborhoods U_δ. Define $V = X\backslash \cup_{\delta<\mathfrak{n}} U_\delta$; this is clopen and discrete. Redefine U_0 to be the union of V with the old set U_0. Then $X = \Sigma_{\delta<\mathfrak{n}} U_\delta$.

For each δ, $U_\delta \in \mathbf{P}(\sigma) \cap \mathbf{R}$ and U_δ has p_δ as its unique nonisolated point. By (a), $U_\delta = D_\delta \overset{\sigma}{\cdot} p_\delta$ with $D_\delta := U_\delta\backslash\{p_\delta\}$. So $X = \Sigma_{\delta<\mathfrak{n}} D_\delta \overset{\sigma}{\cdot} p_\delta$. So $X \in \mathbf{M}$ by 4.5(b) and 4.2. □

A space $X \in \mathbf{P}(\sigma)$ with exactly $\mathfrak{n}$ nonisolated points ($0 < \mathfrak{n} < \sigma$) evidently has Cantor-Bendixon degree 1 in the sense that the removal of its isolated points yields a nonempty discrete space. This suggests an ordinal-indexed sequence of theorems and conjectures which, however, we have not turned into a characterization of those $\mathbf{P}(\sigma) \cap \mathbf{R}$-spaces which lie in $\mathbf{M}$. We turn instead to another invariant, the *pseudocharacter*. Recall that

$\psi(p,(X,t)) := \min\{|u| : u \subseteq t, \cap u = \{p\}\}$, and
$\psi((X,t)) := \sup\{\psi(p,(X,t)) : p \in X\}$.
As usual we also write $\psi(p,X)$ and $\psi(X)$. Analogous to our notation $\{\mathrm{cd} < \mathfrak{k}\}$ in §4, we define $X \in \{\psi \leq \mathfrak{k}\}$ to mean that $X \in \mathbf{Haus}$ and $\psi(X) \leq \mathfrak{k}$.

Our result is that $\mathbf{R} \cap \mathbf{P}(\sigma) \cap \{\psi \leq \sigma\} \subseteq \mathbf{M}$, another partial response to Question 2.5. But this largely depends just on the fact that $\mathbf{R} \subseteq \{\mathrm{cd} < \sigma\}$:

Theorem 5.2. *Let $\mathfrak{k}$ be uncountable and regular. Then* (a) *implies* (b) *and* (b) *implies* (c).

(a) *$X \in \mathbf{P}(\mathfrak{k}) \cap \{\psi \leq \mathfrak{k}\}$;*

(b) *if $Y \subseteq X$ and Y is not closed, then Y can be partitioned into $\mathfrak{k}$ nonempty (relatively) clopen sets;*

(c) *if $Y \subseteq X$ and Y is not closed, then $Y \notin \{\mathrm{cd} < \mathfrak{k}\}$.*

We prove 5.2 shortly, but first we quickly derive two corollaries, the second containing our desired result.

Corollary 5.3. *Let $X \in \{\mathrm{cd} < \mathfrak{k}\} \cap \mathbf{P}(\mathfrak{k}) \cap \{\psi \leq \mathfrak{k}\}$. For $Y \subseteq X$, these are equivalent.*

(a) *Y is closed;*

(b) *$Y \in \{\mathrm{cd} < \mathfrak{k}\}$;*

(c) *each partition of Y into clopen sets has size $< \mathfrak{k}$.*

Proof. (a) ⇒ (b). $\{\mathrm{cd} < \mathfrak{k}\}$ is closed-hereditary.

(b) ⇒ (c). This is obvious (and is the same as (b) ⇒ (c) in 5.2).

(c) ⇒ (a). We are assuming $X \in \mathbf{P}(\mathfrak{k}) \cap \{\psi \leq \mathfrak{k}\}$. If Y is not closed, then 5.2(b) obtains, so (c) fails. □

Corollary 5.4. *Let $X \in \mathbf{P}(\sigma) \cap \{\psi \leq \sigma\}$. Then*

(a) *For $Y \subseteq X$, if $Y \in \mathbf{R}$ then Y is closed.*

(b) *X admits no properly finer $\mathbf{R}$-topology.*

(c) *If $X \in \mathbf{R}$ then $X \in \mathbf{M}$.*

Proof. (a) If Y is not closed, then $Y \notin \{\mathrm{cd} < \sigma\}$ by 5.3, so $Y \notin \mathbf{R} \subseteq \{\mathrm{cd} < \sigma\}$.

(b) Let t be the topology of X and let s be a strictly finer $\mathbf{R}$-topology on the set X. There is $Y \subseteq X$ such that Y is s-closed and not t-closed. Then $(Y, s|Y) \in \mathbf{R} \subseteq \{\text{cd} < \sigma\}$. But then $(Y, t|Y) \in \{\text{cd} < \sigma\}$ (since any class $\{\text{cd} < \mathfrak{k}\}$ is closed under continuous images), so Y is t-closed in X by 5.3, a contradiction.

(c) follows from (b). □

Corollary 5.5. *If $X \in \mathbf{R} \cap \mathbf{P}(\sigma)$ and $|X| \leq \sigma$, then $X \in \mathbf{M}$.*

Proof. $|X| \leq \sigma$ implies $\psi(X) \leq \sigma$, and 5.4 applies. □

Corollary 5.6. *Let $X \in \mathbf{R}$ and $\psi(X) \leq \sigma$. Then $p(\sigma)(X) \in \mathbf{M}$.*

Proof. We have $p(\sigma)(X) \in \mathbf{R}$ by 1.3(b), so 5.4(c) applies (with X replaced by $p(\sigma)(X)$). □

5.7. Let $\{X_i : i \in I\} \subseteq \mathbf{R}$ with $|I| \leq \sigma$ and each $\psi(X_i) \leq \sigma$, and let $X := \Pi_i X_i$. Then $p(\sigma)(X) \in \mathbf{M}$.

Proof. Again 5.4(a) applies, since $\psi(X) \leq \sigma^2 = \sigma$. □

5.5–5.7 show again the special case $D(\sigma) \overset{\sigma}{\cdot} p \in \mathbf{M}$ of 4.5(b), as well as $(D(\sigma) \overset{\sigma}{\cdot} p)^n \in \mathbf{M}$ for $n < \omega$, and $p(\sigma)(D(\sigma) \overset{\sigma}{\cdot} p)^\sigma \in \mathbf{M}$.

As indicated earlier, we do not know whether $\mathbf{P}(\sigma) \cap \mathbf{R} = \mathbf{M}$ (even when $\mathbf{R} = \mathbf{RC}$, $\sigma = \mathfrak{m}$). 5.7 suggests this question, also unsolved, which if settled affirmatively would provide positive evidence concerning that equality.

Question 5.8. If $\{X_i : i \in I\} \subseteq \mathbf{M}$ (perhaps with $|I| < \sigma$ or $|I| \leq \sigma$), is necessarily $p(\sigma)(\Pi_i X_i) \in \mathbf{M}$?

Now we complete the proof of 5.2.

(b) ⇒ (c) is obvious.

(a) ⇒ (b). Assume (a), and let $p \in \overline{Y} \backslash Y$. Since $\psi(p, X) \leq \mathfrak{k}$, there are open sets $\{W_\alpha : \alpha < \mathfrak{k}\}$ with $\cap W_\alpha = \{p\}$. Let $U_0 = X$, and suppose that for $\beta < \mathfrak{k}$ we have chosen clopen neighborhoods $\{U_\alpha : \alpha < \beta\}$ of p. Then $p \in W_\beta \cap \cap_{\alpha<\beta} U_\alpha$, and this set is open since $X \in \mathbf{P}(\mathfrak{k})$. Since $p \in \overline{Y} \backslash Y$, there is $x_\beta \in (W_\beta \cap \cap_{\alpha<\beta} U_\alpha) \cap Y$, $x_\beta \neq p$. Then, there is clopen U_β with $p \in U_\beta \subseteq W_\beta \cap \cap_{\alpha<\beta} U_\alpha$ with $x_\beta \notin U_\beta$.

Now let $V_0 = U_0 \cap Y$ and $V_\beta = [(\cap_{\alpha<\beta} U_\alpha) \backslash U_\beta] \cap Y$ for each $0 < \beta < \mathfrak{k}$. Then $\{V_\beta : \beta < \mathfrak{k}\}$ is the desired partition of Y. □

We conclude with some observations involving
weight: $wX := \min\{|\mathcal{U}| : \mathcal{U}$ is a base for $X\}$; and
local weight: $\chi(p, X) := \min\{|\mathcal{U}| : \mathcal{U}$ is a local base for p in $X\}$.
Of course, $wX \geq \chi(p, X) \geq \psi(p, X)$ for each $p \in X$.

Corollary 5.9. (a) $w(p(\sigma)2^\sigma) = \sigma$;
(b) $p(\sigma)2^\sigma \in \mathbf{M}$, *so* $\mu 2^\sigma = p(\sigma)2^\sigma$.

Proof. 5.4 shows that (a) $\Rightarrow$ (b), so it suffices to prove (a). Let $\mathcal{B}$ be a base for 2^σ with $|\mathcal{B}| = \sigma$. Then $\mathcal{B}_\sigma := \{\cap\mathcal{U} : \mathcal{U} \subseteq \mathcal{B}, |\mathcal{U}| < \sigma\}$ is a base for $p(\sigma)2^\sigma$. But [HM] shows σ is measurable, so $\mathfrak{k} < \sigma \Rightarrow 2^{\mathfrak{k}} < \sigma$, so $|\mathcal{B}_\sigma| \leq \Sigma\{2^{\mathfrak{k}} : \mathfrak{k} < \sigma\} \leq \sigma^2 = \sigma$. □

The following shows that the assumptions in 5.3 and 5.4 have considerable strength.

Theorem 5.10. *Let* $\mathfrak{k}$ *be regular and uncountable and* $X \in \{\mathrm{cd} < \mathfrak{k}\} \cap \mathbf{P}(\mathfrak{k})$. *Then*
(a) *if* $\psi(p, X) \leq \mathfrak{k}$ *then* $\psi(p, X) = \chi(p, X)$;
(b) *if* $|X| \leq \mathfrak{k}$ [*resp.,* $= \mathfrak{k}$] *then* $wX \leq \mathfrak{k}$ [*resp.,* $= \mathfrak{k}$].

Proof. (a) If $\psi(p, X) < \mathfrak{k}$, then p is isolated (since $X \in \mathbf{P}(\mathfrak{k})$ and $\psi(p, X) = \chi(p, X) = 1$), so we suppose $\psi(p, X) = \mathfrak{k}$, say $\{p\} = \cap_{\alpha<\mathfrak{k}} U_\alpha$ with each U_α clopen, $U_0 = X$. Then the collection of sets of the form $V_\beta := \cap_{\alpha<\beta} U_\alpha$ is a base at p: If not there is a clopen neighborhood W of p such that $V_\beta \subseteq W$ fails for each β, and then the family of sets of the form $A_\beta := (V_\beta \backslash V_{\beta+1}) \backslash W$ (of which $\mathfrak{k}$ are nonempty) is a clopen partition of $X \backslash W$, so $W \notin \{\mathrm{cd} < \mathfrak{k}\}$ by 5.2; thus $X \notin \{\mathrm{cd} < \mathfrak{k}\}$.

(b) $|X| \leq \mathfrak{k}$ implies each $\psi(p, X) \leq \mathfrak{k}$, so each $\chi(p, X) \leq \mathfrak{k}$ by (a) and $wX = \sup\{\chi(p, X) : p \in X\} \leq \mathfrak{k}$.

If $|X| = \mathfrak{k}$, the previous sentence shows $wX \leq \mathfrak{k}$. But $wX < \mathfrak{k}$ and $X \in \mathbf{P}(\mathfrak{k})$ imply that X is discrete of size $\mathfrak{k}$, and then $X \notin \{\mathrm{cd} < \mathfrak{k}\}$. So $wX = \mathfrak{k}$. □

Note added July 6, 2004. Alan Dow (*Maximal realcompact spaces and measurable cardinals*, Topology and Its Applications (2005), to appear) has shown that (assuming $\mathfrak{m}$ exists) $\mathbf{M} = \mathbf{P}(\mathfrak{m}) \cap \mathbf{RC}$ iff $\mathfrak{m}$ is a compact cardinal. He notes (1) if a supercompact cardinal exists, then it is consistent that $\mathfrak{m}$ is compact (Magidor), and (2) if $\mathfrak{m}$ exists, then it is consistent that $\mathfrak{m}$ is not compact. Dow calls the case of $\mathfrak{m}$ not compact "the most likely case"; and certainly this becomes the most interesting.

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