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GROUPS OF HOMEOMORPHISMS AND SPECTRAL TOPOLOGY

HAWETE HATTAB* AND EZZEDDINE SALHI*

Abstract. The aim of this paper is to study some of the relationships between groups of homeomorphisms on one side, and $A.F$ C^*-algebra, unitary commutative ring on the other side.

Let G be a countable group of homeomorphisms of a locally compact second countable topological space E. The class of an orbit O is the union of all orbits O' having the same closure as O. We denote by X the quasi-orbits space (*i.e* the space of orbits classes). If every decreasing sequence of saturated closed subsets of E is finite, then X is homeomorphic to the prime spectrum of a unitary commutative ring equipped with the Zariski topology and E is the closure of the union of a finitely many orbits.

Let E be the line $\mathbb{R}$ such that every element of G is an increasing homeomorphism and let X_0 be the union of all open subsets of X homeomorphic to $\mathbb{R}$ or S^1. The space $X - X_0$ is always homeomorphic to the primitive spectrum of an $A.F$ C^*-algebra equipped with the Jacobson topology and if G has a minimal set, then it is homeomorphic to the prime spectrum of a unitary commutative ring equipped with the Zariski topology if and only if every totally ordered family of orbits has a greatest lower bound.

We give an example of a diffeomorphism of the unit 2-sphere S^2 such that the above result fails.

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0. Introduction

A topological space is called a spectral space ([11, Definition 4.9]) if it is a T_0-space and if each irreducible closed subset has a generic point. By [11, 4.17], such space is homeomorphic to the prime spectrum of the lattice of its open subsets.

Recall that a topological space is a T_0-space if for every pair of distinct points x and y, there exists a neighborhood containing one of them but not the other. A closed subset C is irreducible if it is not the union of two proper closed subsets or if the intersection of two nonempty open subsets is nonempty. An element x of C is called a generic point if the closure of the singleton $\{x\}$ is equal to C: $\overline{\{x\}} = C$.

M.Hochster proved in [10] that a topological space X is homeomorphic to the prime spectrum of a unitary commutative ring equipped with the Zariski topology if and only if it satisfies the following properties:

i) X is a quasi-compact space.

ii) X is a T_0-space.

iii) Each irreducible closed subset has a generic point.

iv) X has a basis of quasi-compact open subsets.

v) The intersection of two quasi-compact open subsets is quasi-compact.

The space X is said to be quasi-compact if it satisfies the property of Borel-Lebesgue but it is not necessarily a Hausdorff space.

In [1] it was shown that an ordered set $(Y, \leq)$ is order-isomorphic to the prime spectrum of a unitary commutative ring equipped with the inclusion if and only if there exists a topology defined on the set Y compatible with the order and satisfying the above properties i), ii), iii), iv) and v). By [12], such set satisfies the following conditions:

(K_1) Each totally ordered family of elements in $(Y, \leq)$ has a supremum and an infimum.

(K_2) For every elements $a < b$ in Y, there exist two consecutive elements $a_1 < b_1$ with $a \leq a_1 < b_1 \leq b$.

O.Bratteli and G.A.Elliott showed in [5] that a topological space X is homeomorphic to the primitive spectrum of an approximately finite-dimensional separable C^*-algebra (called $A.F$ C^*-algebra) equipped with the Jacobson topology if and only if it has a countable basis, and it satisfies the above properties ii), iii) and iv) but not necessarily the properties i) and v). An approximately finite-dimensional separable C^*-algebra is an inductive limit of a sequence of finite-dimensional C^*-algebras.

In the first paragraph, we will give some topological notions concerning quasi-homeomorphisms and some dynamical notions corresponding to an open equivalence relation: minimal set, local minimal set, proper and locally dense trajectories, class of a trajectory, ... These notions will be used in the following paragraphs.

In the second paragraph, we will study some spectral properties of a countable subgroup G of the group of homeomorphisms of a locally compact and second countable topological space E.

In the third paragraph, we will study some relationships between groups of homeomorphisms of $\mathbb{R}$ on one side, and $A.F$ C^*-algebra, unitary commutative ring on the other side.

1. Topological notions

1.1 Quasi-homeomorphisms.

Following A.Grothendieck and J.Dieudonné [8] a continuous map $f : X \to Y$ between two topological spaces is called a quasi-homeomorphism if the map which assigns to each open set $V \subset Y$ the open set $U = f^{-1}(V) \subset X$ is a bijective map. Equivalently, the map which assigns to each closed set $G \subset Y$ the closed set $F = f^{-1}(G) \subset X$ is a bijective map. In the same manner the initial topology of X coincides with the inverse topology of X under f and the space $f(X)$ is strongly dense in Y (i.e $f(X)$ meets any locally closed subset of Y).

An onto quasi-homeomorphism $f : X \to Y$ satisfies the following statements [3]:

1) The map f is open, closed and for every locally closed subset $A \subset X$, we have $A = f^{-1}(f(A))$.

2) For every $x, y \in X$, we have the following implication:

$$f(x) = f(y) \;\; \Rightarrow \;\; \overline{\{x\}} \; = \; \overline{\{y\}}$$

We deduce that if moreover X is a T_0-space, then f is a homeomorphism.

3) The notions of quasi-compact open subset, irreducible closed subset are stable by direct image and by inverse image under f.

From these properties we obtain the following statements:

4) For each $A \subset X$, $B \subset Y$, we have $f(\overline{A}) = \overline{f(A)}$ and $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$ (f is a continuous open closed map).

5) For every open sets $U \subset X$, $V \subset Y$ and for every $A \subset X$, $B \subset Y$, we have $f(U \cap A) = f(U) \cap f(A)$. Thus $f(U \cap \overline{A}) = f(U) \cap \overline{f(A)}$ and $f^{-1}(V \cap \overline{B}) = f^{-1}(V) \cap \overline{f^{-1}(B)}$.

It follows immediately from these properties the following lemma:

Lemma 1.1.1. *Let $f : X \to Y$ be an onto quasi-homeomorphism. Then:*

a) If A is a locally closed (resp. locally dense) subset of X, then $f(A)$ is a locally closed (resp. locally dense) subset of Y.

b) If B is a locally closed (resp. locally dense) subset of Y, then $f^{-1}(B)$ is a locally closed (resp. locally dense) subset of X.

By using the previous lemma we obtain the following proposition:

Proposition 1.1.2. *Let $f : X \to Y$ be an onto quasi-homeomorphism. Then the following are equivalent:*

a) X is a Baire space.

b) Y is a Baire space.

Proof. Suppose that X is a Baire space, and let (V_n) be a sequence of open dense subsets of Y. By lemma 1.1.1, it follows that $(f^{-1}(V_n))$ is a sequence of open dense subsets of X. Therefore, $\overline{\bigcap f^{-1}(V_n)} = X$ and hence

$$X = \overline{f^{-1}(\bigcap V_n)} = f^{-1}(\overline{\bigcap V_n}).$$

Thus $Y = f(X) = \overline{\bigcap V_n}$.

On the other hand, suppose that Y is a Baire space, and let (U_n) be a sequence of open dense subsets of X. From lemma 1.1.1 and property 1), it follows that $(f(U_n))$ is a sequence of open dense subsets of Y. Then $\overline{\bigcap f(U_n)} = Y$ and so

$$X = f^{-1}(\overline{\bigcap f(U_n)}) = \overline{f^{-1}(\bigcap f(U_n))} = \overline{\bigcap f^{-1}(f(U_n))}.$$

Thus $X = \overline{\bigcap U_n}$. □

1.2 Dynamical properties of an open equivalence relation.

1.2.1 Dynamical notions.

Let $\mathcal{R}$ be an open equivalence relation on a topological space X. In this paragraph, we will give some dynamical notions which will be used in the following paragraphs. These notions are borrowed from notions of codimension-one foliation on a closed manifold.

Throughout this paper, we will call trajectory at a point x instead of equivalence class at x and we write it $\mathcal{R}(x)$.

Recall that if $A \subset X$, the saturation $Sat_{\mathcal{R}}(A)$ of A is the union of all trajectories meeting A. The subset A is called saturated (or invariant) if we have $A = Sat_{\mathcal{R}}(A)$. Since the relation $\mathcal{R}$ is open the interior, the closure, the boundary of each saturated subset is also saturated.

Let us now give these dynamical notions:

1) The *class* $Cl(T)$ of a trajectory T is the union of all trajectories T' having the same closure as T.

2) A *minimal* set (M.S) is a minimal element of the family of nonempty saturated closed subsets (ordered by inclusion). Equivalently, a minimal set (M.S) is a nonempty saturated subset $S \subset X$ such that the closure of every trajectory $T \subset S$ is equal to S.

3) Let U be a nonempty saturated open subset. A *minimal set in* U is a minimal set of the restriction of the relation $\mathcal{R}$ to U. Then a minimal set in U is a nonempty saturated subset $S \subset U$ such that for each trajectory $T \subset S$ we have $\overline{T} \cap U = S$.

4) A *local minimal set* (L.M.S) is a minimal set in some nonempty saturated open subset. A trajectory T is contained in a local minimal set if and only if $\overline{T} - Cl(T)$ is closed. In this case the class $Cl(T)$ of T is the local minimal set containing T. Otherwise, we have $\overline{\overline{T} - Cl(T)} = \overline{T}$. In general, we have $\overline{T} = \overline{Cl(T)}$.

5) A trajectory T is called *locally closed at a point* $x \in T$ if there exists a nonempty open subset U containing x such that $\overline{T} \cap U = T \cap U$. We say that a trajectory T is *locally closed* if it is locally closed at each point $x \in T$. We have the following equivalences:

The trajectory T is locally closed $\Leftrightarrow$ the subset $\overline{T} - T$ is closed $\Leftrightarrow$ there exists an open subset U such that $\overline{T} \cap U = T$.

6) A trajectory T is called *proper* if it is locally closed with $int(\overline{T}) = \emptyset$.

7) A trajectory T is called *locally dense at a point* x if there exists a nonempty open subset U containing x such that $\overline{T} \cap U = U$. We say that a trajectory T is *locally dense* if it is locally dense at each point $x \in T$. We have the following equivalences:

The trajectory T is locally dense $\Leftrightarrow$ the subset $int(\overline{T})$ is nonempty $\Leftrightarrow$ there exists an open subset U such that $T \subset U \subset \overline{T}$.

Remark 1.2.1.1. Let S be a (L.M.S), precisely a (M.S) in an open nonempty connected saturated subset U. Then:

. If S contains a locally closed trajectory T, then $T = S$, and T is closed in U; indeed we have $S = Cl(T)$ (T is contained in the (L.M.S) S) and $T = Cl(T)$ (T is a locally closed trajectory).

. If S contains a locally dense trajectory T, then $S = U = Cl(T)$.

Remark 1.2.1.2. The trajectories classes determine an equivalence relation $\widetilde{\mathcal{R}}$ on X defined by: $\widetilde{\mathcal{R}}(x) = Cl(\mathcal{R}(x))$. Since For each open subset $U \subset X$ we have $Sat_{\mathcal{R}}(U) = Sat_{\widetilde{\mathcal{R}}}(U)$, it is obvious that the new equivalence relation $\widetilde{\mathcal{R}}$ is open.

The quotient space $X/\widetilde{\mathcal{R}}$ (called the quasi-trajectories space) is always a T_0-space but the space of trajectories $X/\mathcal{R}$ is not in general a T_0-space. The space $X/\widetilde{\mathcal{R}}$ is the universal T_0-space associated to the space $X/\mathcal{R}$ as in Bourbaki [4, Exercise 27 p: I-104].

The map $\varphi : X/\mathcal{R} \to X/\widetilde{\mathcal{R}}$ which assigns to each trajectory T its class $Cl(T)$ is an onto quasi-homeomorphism. Let $q : X \to X/\mathcal{R}$ and $p : X \to X/\widetilde{\mathcal{R}}$ be the canonical projections.

We denote by T_s the saturated topology on X formed by the saturated open subsets of X. A saturated open subset $U \subset X$ is called *compact by saturation* if it is quasi-compact for the saturated topology T_s. That is, every covering (U_i) of U by saturated open subsets U_i contains a finite sub-covering.

Lemma 1.2.1.3. *Let $\mathcal{R}$ be an open equivalence relation on a topological space X. An open subset V of the quotient space $X/\mathcal{R}$ is quasi-compact if and only if the open subset $U = q^{-1}(V)$ is compact by saturation.*

Proof. Suppose that V is quasi-compact, and let $(U_i, i \in I)$ be a covering of $U = q^{-1}(V)$ by saturated open subsets. Thus the open subsets $(q(U_i))$ cover V, and some finite number of these, $q(U_{i_1}), ..., q(U_{i_n})$, cover V. Because every U_i is saturated, $q^{-1}(q(U_i)) = U_i$ and hence $U = U_{i_1} \cup ... \cup U_{i_n}$.

Conversely, let $(V_i, i \in I)$ be a family of open subsets of $X/\mathcal{R}$ such that $V = \bigcup_i V_i$. Since $q^{-1}(V_i)$ is a saturated open subset and U is compact by saturation, it follows that $U = q^{-1}(V_{i_1}) \cup ... \cup q^{-1}(V_{i_n})$ which implies that $V = V_{i_1} \cup ... \cup V_{i_n}$. □

Lemma 1.2.1.4. *Let $\mathcal{R}$ be an open equivalence relation on a locally compact second countable topological space Y. Let U be a saturated open subset. Then the following are equivalent:*

a) U is compact by saturation.

b) There exists a compact subset $K \subset U$ with $U = Sat(\overset{\circ}{K})$.

Proof. $a) \Rightarrow b)$ There exists an increasing sequence $\{K_n\}$ of compact subsets with $U = \bigcup_{n\in\mathbb{N}} \overset{\circ}{K}_n$. Thus $U = \bigcup_{n\in\mathbb{N}} Sat(\overset{\circ}{K}_n)$. Since U is compact by saturation and the relation $\mathcal{R}$ is open, there exists an integer p such that $U = Sat(\overset{\circ}{K}_p)$ (the sequence $\{K_n\}$ is increasing).

$b) \Rightarrow a)$ We suppose that there exists a compact $K \subset U$ such that $U = Sat(\overset{\circ}{K})$. Let $\{U_i : i \in I\}$ be a family of saturated open subsets such that $U = \bigcup_i U_i$. The family $\{K_i = K \cap U_i : i \in I\}$ is an open covering of the compact K, then we have $K = K_{i_1} \cup ... \cup K_{i_p}$. Because $U_i = Sat(K_i)$ for each $i \in I$, we have $U = U_{i_1} \cup ... \cup U_{i_p}$; this ends the proof of lemma. □

1.2.2 The identity relation.

Let $\mathcal{R}$ be the identity relation on a topological space X. We say that a point $x \in X$ satisfies a property $\mathcal{P}$ if the trajectory $\mathcal{R}(x) = \{x\}$ satisfies this property. For example :

• The class of a point x is the subset

$$Cl(x) = \{y \in X/\overline{\{y\}} = \overline{\{x\}}\}.$$

• An element x is said to be locally closed if there exists an open set U such that $\overline{\{x\}} \cap U = \{x\}$. Equivalently, $\overline{\{x\}} - \{x\}$ is closed. In this case we have $Cl(x) = \{x\}$.

• An element x is said to be locally dense if there exists an open set U such that $x \in U \subset \overline{\{x\}}$, it is equivalent to the fact that $int(\overline{\{x\}}) \neq \emptyset$.

• A subset $A \subset X$ is called a (M.S) (minimal set) if for any $x \in A$ we have $\overline{\{x\}} = A$. Equivalently, A is a minimal element of the family of nonempty closed subsets ordered by inclusion.

• A subset $A \subset X$ is called a (L.M.S) (local minimal set) if there exists an open set U such that for each $x \in A$ we have $\overline{\{x\}} \cap U = A$. In this case for each $x \in A$ we have $A = Cl(x)$ and $\overline{A} = \overline{\{x\}}$.

• A point x is contained in some (L.M.S) if and only if $\overline{\{x\}} - Cl(x)$ is closed. Otherwise we have $\overline{\overline{\{x\}} - Cl(x)} = \overline{\{x\}}$.

Sometimes we write, wrongly, x to indicate the singleton $\{x\}$.

2. Groups of homeomorphisms of a topological space and spectral topology

Let E be a topological space and Homeo(E) its group of homeomorphisms. Let G be a countable subgroup of Homeo(E). The family of orbits $G(x) = \{g(x) : g \in G\}$ by G determines an open equivalence relation on E. Let $Z = E/G$ be the space of orbits and $X = E/\widetilde{G}$ be the quasi-orbits space (an element of $E/\widetilde{G}$ is an orbit class). The canonical projections $q : E \to E/G$ and $p : E \to E/\widetilde{G}$ are open and the map $\varphi : E/G \to E/\widetilde{G}$, which associates to each orbit its class, is an onto quasi-homeomorphism.

The fact that the quasi-orbits space X is a T_0-space allows us to define an order on X by

$$a = p(x) \leq p(y) = b \ \ if \ \ \overline{G(x)} \subset \overline{G(y)}$$

It is easy to see that if O and O' are two orbits, then $O \subset \overline{O'}$ if and only if every nonempty saturated open subset U containing O contains O'. Hence the topology of X is compatible with the inverse order:. That is, every element $a \in X$ satisfies

$$\overline{\{a\}} = \{b \in X : b \leq a\}$$

Theorem 2.1. *Let E be a locally compact second countable topological space and let G be a countable subgroup of Homeo(E). If every decreasing sequence of saturated closed subsets of E is finite, then we have the following properties:*

a) X is homeomorphic to the primitive spectrum of an A.F C^*-algebra and to the prime spectrum of a unitary commutative ring.

b) The space E is the closure of the union of a finitely many orbits O_1,...,O_p. That is, $E = \overline{O_1 \cup ... \cup O_p}$.

c) Every orbit contains in its closure a minimal set.

To prove this theorem, we need the lemma 2.2.

From exercise 7, page 171 of Bourbaki (algèbre commutative, chapitre 1 à 4, Masson 1985) we know that if E is a compact metrisable topological space such that its quasi-orbits space X is irreducible, then X has a generic point.

The referee noticed as that an analogous result with the following lemma can be found in the text-book of W.Gottschalk and G.Hedlund, Topological dynamics, Amer. Math. Soc. Colloq. Publ. Vol 36, Providence, R.I. 1955 (theorem 9.20).

Lemma 2.2. *Let E be a locally compact second countable topological space and let G be a countable subgroup of Homeo(E). Then every irreducible closed subset of $X = E/\widetilde{G}$ has a generic point.*

Proof. Let A be an irreducible closed subset of $X = E/\widetilde{G}$. The subspace $B = p^{-1}(A)$ is an irreducible closed subset of E equipped with the saturated topology T_s. Let $\widehat{E} = E \cup \{\omega\}$ be the one-point compactification of E. We can suppose that G is a group of homeomorphisms of $\widehat{E}$ by putting $G(\omega) = \{\omega\}$. We denote by $C = B$ if B is a compact of E, otherwise $C = B \cup \{\omega\}$. The subset C is invariant compact contained in $\widehat{E}$.

Let $\mathcal{U} = \{U_1, ..., U_n\}$ be a finite covering of C by open nonempty invariant subsets. For each $1 \leq i \leq n$, U_i is everywhere dense in C; indeed C is an irreducible closed subset. So the open subset $\Omega_{\mathcal{U}} = \bigcap_{i=1}^{n} U_i$ is everywhere dense in C.

Let $\mathcal{S}$ be a countable basis of $\widehat{E}$ equipped with the saturated topology T_s (the space (E, T_s) is second countable). The family Σ of finite covering of C by elements of $\mathcal{S}$ is countable; we write $\Sigma = \{\mathcal{U}_n : n \in \mathbb{N}\}$. Put

$$\mathcal{V}_n = \{\bigcap_{k=1}^{n} U_{i_k} : U_{i_k} \in \mathcal{U}_k \ \ for 1 \leq k \leq n\}$$

It is obvious that $\mathcal{V}_n$ is a finite covering of C finer than $\mathcal{U}_n$ and for each $n < m$ the covering $\mathcal{V}_m$ is finer than $\mathcal{V}_n$.

The fact that C is compact implies the property:

(*) for each nonempty open subset W of C, there exists an integer n such that W contains an element of $\mathcal{U}_n$, therefore W contains an element of $\mathcal{V}_n$.

For each integer n the open invariant subset $\Omega_{\mathcal{V}_n}$ is everywhere dense in C. By Baire's theorem, it follows that the intersection $F = \bigcap_n \Omega_{\mathcal{V}_n}$ is everywhere dense in C.

Every orbit $O \subset F$ is contained in each nonempty open subset W of C (property (*)). Thus $\overline{O} = C$ and that O is a generic orbit of (B, T_s). Then $p(O)$ is a generic point of A. □

Proof of theorem.

a) First, we remark that every decreasing sequence (X_n) of closed subsets of X is finite; indeed in this case $(p^{-1}(X_n))$ is a decreasing sequence of saturated closed subsets of E. Therefore the space X satisfies one of the following equivalences:

α) Every decreasing sequence of closed subsets is finite.

β) Every increasing sequence of open subsets is finite.

γ) Every nonempty open subset is quasi-compact.

We conclude that the space X satisfies the properties i), iv) and v) given in the introduction. From lemma 2.2, it follows that X satisfies the property iii). Since X is always a T_0-space and it has a countable basis, it follows that it is homeomorphic to the primitive spectrum of an $A.F$ C^*-algebra and to the prime spectrum of a unitary commutative ring.

b) Because X satisfies one of the previous properties α), β) or γ), it is a noetherian space. Thus there exists a finitely many irreducible components $X_1, ..., X_p$ with $X = X_1 \cup ... \cup X_p$ (Bourbaki, algèbre commutative, chapitre 1 à 4). Each X_i is necessarily an irreducible closed subset, thus every X_i $(1 \leq i \leq p)$ contains a generic point α_i; that is, $X_i = \overline{\{\alpha_i\}}$ (lemma 2.2).

For any $1 \leq i \leq p$, we denote by O_i an orbit contained in $p^{-1}(\{\alpha_i\})$. It suffices to show that we have $E = \overline{O_1 \cup ... \cup O_p}$. We have $p^{-1}(\{\alpha_i\}) = Cl(O_i)$. The fact that $\overline{O_i} = \overline{Cl(O_i)}$ together with $p^{-1}(\overline{\{\alpha_i\}}) = \overline{p^{-1}(\{\alpha_i\})}$ imply the equality $p(\overline{O_i}) = \overline{\{\alpha_i\}}$ for each integer $1 \leq i \leq p$. Then we obtain the equalities:

$$p(\overline{O_1 \cup ... \cup O_p}) = \overline{\{\alpha_1\}} \cup ... \cup \overline{\{\alpha_p\}} = X = p(E).$$

Thus
$$E = p^{-1}(p(\overline{O_1 \cup ... \cup O_p})) = \overline{O_1 \cup ... \cup O_p}\ .$$

c) We consider an arbitrary orbit O. If $\overline{O}$ is not a minimal set, then there exists an orbit O_1 with $\overline{O_1} \subset \overline{O}$ and $\overline{O_1} \neq \overline{O}$. From the hypothesis and by a finite recurrence, it follows that there exits an orbit $O_p \subset \overline{O}$ such that every orbit $O' \subset \overline{O_p}$ satisfies $\overline{O'} = \overline{O_p}$. We conclude that $\overline{O_p}$ is a minimal set contained in $\overline{O}$. □

Examples 2.3.

a) Let us consider the space $X = \mathbb{N} \cup \{\omega\}$ equipped with the topology of subbasis the family $\{U_n\}_{n \in \mathbb{N}}$, where for each integer n, $U_n = \{n, n+1, ...\} \cup \{\omega\}$. The space X corresponds to the quasi-orbits space of the subgroup G of $\mathrm{Homeo}_+(\mathbb{R})$ given in [7, Chapter 4.4, Example 4.14]. This space satisfies the hypotheses of the above theorem and so it satisfies its three conclusions; in particular it is homeomorphic to the primitive spectrum of an $A.F$ C^*-algebra and to the prime spectrum of a unitary commutative ring.

b) Let us consider the space $X = \{\frac{1}{n} : n \geq 1\} \cup \{0\}$ equipped with the topology of subbasis $\{U_n\}_{n \in \mathbb{N}}$ such that for each integer n we have $U_n = \{\frac{1}{n}, \frac{1}{n-1}, ..., 1\}$, $U_0 = X$. Since the increasing sequence $\{U_n\}_{n \geq 1}$ is not finite, X does not satisfy the hypotheses of the previous theorem, but it is homeomorphic to the primitive spectrum of an $A.F$ C^*-algebra and to the prime spectrum of a unitary commutative ring.

We remark that this space corresponds to the quasi-orbits space of the subgroup G of $\mathrm{Homeo}_+(\mathbb{R})$ given in [7, Chapter 4.4, Remarque 4-17-iii)].

Example 2.4. Let $X = A \cup B$ be a subset of $\mathbb{R}^2$ with $A = \{a_0 = (0,0)\}$ and $B = \{b_n = (n,1) : n \in \mathbb{Z}\}$. We consider a topology T on X such that every nonempty closed subset F different from X is given by $F = A \cup B_1$ where B_1 is a finite subset of B. This topology satisfies $\overline{\{a_0\}} = \{a_0\}$ and $\overline{\{b_n\}} = \{a_0, b_n\}$, $n \in \mathbb{Z}$.

It is easy to see that this space (X, T) is a noetherian space such that every nonempty irreducible closed subset $Y \neq X$ has a generic point, but the whole space X is an irreducible closed subset without generic point. We conclude, by lemma 2.2, that it cannot be homeomorphic to any quasi-orbits space. Also, this space cannot be homeomorphic neither to a primitive spectrum nor to a prime spectrum.

Applying [11, 4.17] and lemma 2.2, it follows the following proposition:

Proposition 2.5. *If E is a locally compact second countable topological space and if G is a countable subgroup of $Homeo(E)$, then $X = E/\widetilde{G}$ is homeomorphic to the prime spectrum of the lattice of its open subsets.*

An orbit O is called a *maximal* (resp. *minimal*) orbit if every orbit O' such that $O \subset \overline{O'}$ (resp. $O' \subset \overline{O}$) has the same closure as O: $\overline{O} = \overline{O'}$. An orbit O is a minimal orbit if and only if it is contained in a minimal set.

If there exists a compact subset $K \subset E$ meeting every orbit, then by lemma 1.2.1.4, the space E is compact by saturation. Thus by lemma 1.2.1.3, the space $E/\widetilde{G}$ is quasi-compact. Also, by using Zorn's lemma, every orbit contains in its closure a minimal orbit.

In the next proposition, we will show that every orbit is contained in the closure of a maximal orbit. This result generalizes, with a different proof, a result of E. Salhi concerning groups of increasing homeomorphisms of the line $\mathbb{R}$, and transversally oriented codimension-one foliation on a closed manifold.

Proposition 2.6. *Let E be a locally compact second countable topological space, and let G be a subgroup of the group $Homeo(E)$. Then every orbit O is contained in the closure of a maximal orbit.*

The proof of this proposition needs the following lemma:

Lemma 2.7. *Let $\{a_i\}_{i\in I}$ be a totally ordered family of the quasi-orbits space X. Then there exists $a \in X$ satisfying $\overline{\{a\}} = \overline{\{a_i : i \in I\}}$, and $a = \sup\{ai, i \in I\}$.*

Proof. Firstly, we prove that the subset $A = \overline{\{a_i : i \in I\}}$ is an irreducible closed subset. For this, it suffices to show that if U and V are two nonempty open subsets of X such that each of them meets A, then the intersection meets also A. There exist two indexes i and j such that $a_i \in U$ and $a_j \in V$. Since the family $\{a_i : i \in I\}$ is totally ordered, we can suppose that $a_i \leq a_j$ which means that $\overline{\{a_i\}} \subset \overline{\{a_j\}}$. Thus the intersection $U \cap V$ contains a_j and so it meets A.

Now, by applying lemma 2.2, the subset $A = \overline{\{a_i : i \in I\}}$ contains a generic point a, that is, $\overline{\{a\}} = A$. The element a is an upper bound of the family $\{a_i : i \in I\}$; let $b \in X$ be an other upper bound of this family. We have, for each i, the inclusion $\{a_i\} \subset \overline{\{b\}}$ and hence $\overline{\{a\}} = \overline{\{a_i : i \in I\}} \subset \overline{\{b\}}$ which implies that $a \leq b$, thus $a = \sup\{a_i, i \in I\}$. □

Proof of proposition. Consider an orbit O_0 and put $a_0 = p(O_0)$. We denote by $\mathcal{C}_{a_0}$ a maximal totally ordered family formed by elements $b \in X$ such that $a_0 \leq b$. From lemma 2.7, there exists $a_m \in X$ satisfying $\overline{\{a_m\}} = \overline{\{a : a \in \mathcal{C}_{a_0}\}}$ and $a_m = \sup\{a : a \in \mathcal{C}_{a_0}\}$. Thus a_m is a maximal element of X; indeed $\mathcal{C}_{a_0}$ is a maximal totally ordered family. Let O_m be an orbit contained in the saturated subset $p^{-1}(\{a_m\})$, this orbit is a maximal orbit containing O_0 in its closure. For this, first we remark that the inequality $a_0 \leq a_m$ gives the inclusion $\overline{O_0} \subset \overline{O_m}$. Now, if O' is an orbit which $\overline{O_m} \subset \overline{O'}$ and $a' = p(O')$, then $\overline{\{a_m\}} = p(\overline{O_m}) \subset p(\overline{O'}) = \overline{\{a'\}}$. Since a_m is a maximal element, $a_m = a'$ and so $\overline{O_m} = \overline{O'}$. This ends the proof. □

Proposition 2.8. *Let E be a locally compact topological space, and let G be a subgroup of Homeo(E). Then the quasi-orbits space X and the space of orbits Z are Baire spaces.*

Proof. We start by showing that X is a Baire space. Let (V_n) be a sequence of open subsets such that each V_n being everywhere dense in X. Therefore every $p^{-1}(V_n)$ is a saturated open subset everywhere dense in E; indeed, the fact that the canonical projection p is open implies that $\overline{p^{-1}(V_n)} = p^{-1}(\overline{V_n})$. This result combined with the fact that E is a Baire space implies that $E = \overline{\bigcap_n p^{-1}(V_n)}$.

Moreover, we have the following equalities:

$$p^{-1}(\overline{\bigcap_n V_n}) = \overline{p^{-1}(\bigcap_n V_n)} = \overline{\bigcap_n p^{-1}(V_n)} = E.$$

Thus

$$X = p(E) = p(p^{-1}(\overline{\bigcap_n V_n})) = \overline{\bigcap_n V_n}.$$

We conclude that X is a Baire space. From the fact that the map $\varphi : Z \to X$ is an onto quasi-homeomorphism and from the proposition 1.1.2, it follows that the space Z is also a Baire space. □

A point $a \in X$ is a *separated point* if for every $b \notin \overline{\{a\}}$ there exists an open neighborhood U_a (resp. U_b) of a (resp. of b) such that $U_a \cap U_b = \emptyset$.

An orbit $G(x)$ is a *separated orbit* if $q(x)$ is a separated point of E/G which is equivalent to the fact that for every orbit $G(y)$ not contained in $\overline{G(x)}$, there exist two saturated open subsets $G(x) \subset U$ and $G(y) \subset V$ such that $U \cap V = \emptyset$.

We denote by $Sep(X)$ the union of all separated points of X, and by X_0 its interior.

Remark 2.9. We have the following properties:
(i) Each locally dense orbit is a separated orbit.
(ii) Each separated orbit O is a maximal orbit.
(iii) The space $Sep(X)$ is a Hausdorff space.

The next example gives a group of homeomorphisms generated by a homeomorphism on the unit 2-sphere such that $X - X_0$ does not have any basis of quasi-compact open subsets.

Example 2.10. Let S^2 be the two unit sphere

$$S^2 = \{(z, u) \in \mathbb{C} \times \mathbb{R} / |z|^2 + u^2 = 1\}.$$

We consider a diffeomorphism f on S^2 defined by $f(z, u) = (ze^{2i\pi u}, u)$. For each element $A(z_0, u_0) \in S^2$ we denote by O_A the orbit at A by f and by P_A, M_A respectively the parallel and the meridian circle containing A.

It is easy to show that the orbit O_A is contained in P_A and that it is finite if $u_0 \in \mathbb{Q} \cap [-1, 1]$ or dense in P_A if $u_0 \in [-1, 1] - \mathbb{Q}$. We will show that the space $X = S^2/\widetilde{G}$ of orbits classes, where G is the group generated by f, cannot have any basis of quasi-compact open subsets. We remark that the space X_0, which corresponds to the interior of the union of separated points of X, is empty.

According to lemma 1.2.1.3, it suffices to show the following property:

(*) The sphere S^2 cannot have any basis of open subsets compact by saturation.

Let $J =]a, b[$ be an arc contained in a meridian circle such that the parallel circles P_a and P_b, containing respectively the points a and b, are distinct. Consider the saturated open subset U formed by the points x situated between the parallel circles P_a, P_b. We claim that the open subset U does not contain any compact by saturated open subset V. If not, there exist an open saturated subset $V \subset U$ and a compact subset $K \subset V$ such that $V = Sat(\overset{\circ}{K})$ (lemma 1.2.1.4). The distance $r_0 = d(K, V^c)$ between the two compact subsets K and V^c is nonzero (V^c is the complementary of V in S^2).

Let x_0 be the supremum of points $x \in J$ where the parallel circle P_x intersects the open subset V. Put $\epsilon = \frac{r_0}{2}$, there exists $y \in]x_0 - \epsilon, x_0[\subset J$ such that the parallel circle P_y intersects V, then P_y meets also K because we have $V = Sat(\overset{\circ}{K})$, thus we have $d(K, V^c) \leq \frac{r_0}{2}$ which is absurd. So the statement (*) is true.

3. Groups of homeomorphisms on the line $E = \mathbb{R}$

We begin to give some useful results about codimension-one foliation. Roughly speaking, a codimension-one foliation on a smooth m-manifold M is an open equivalence relation $\mathcal{F}$ on M such that each trajectory (called a leaf) is weakly embedded submanifold of dimension $p = m - 1$ and such that the canonical projection of M on the space of leaves $M/\mathcal{F}$ is locally a submersion. In this case for each $x \in M$, there exists a chart (U, φ) such that $\varphi(U) = \mathbb{R}^m$ and the trajectories of the restriction of the relation $\mathcal{F}$ to U are homeomorphic to $\mathbb{R}^p \times \{y\}$, $y \in \mathbb{R}$. This chart (U, φ) is called a distinguished chart.

For notions: proper leaf, locally dense leaf, minimal set, local minimal set, class of a leaf see paragraph 1.2.1.

Let $\mathcal{F}$ be a transversally oriented codimension-one foliation of class C^r, $r \geq 0$, on a closed manifold M of dimension m. The boundary δU of a nonempty saturated connected open subset distinct of M, defined by A.R Dippolito [7, chapter 4.4], coincides with the set of points $x \in M - U$ such that there exists a path $c : [0, 1] \to M$ where $c(0) = x$ and $c(]0, 1]) \subset U$. The boundary δU of U is a union of a finitely many leaves and we have $\overline{\delta U} = \overline{U} - U$. Attracting proper leaf from one side is also defined in [7, chpt. 4.4].

Lemma 3.1 [2]**.** *Let U be a connected nonempty saturated open subset of M. Then the following are equivalent:*

a) U is compact by saturation (i.e is quasi-compact for the saturated topology T_s).

b) The two following properties are satisfied:

i) Each leaf $L \subset \delta^{\epsilon}U$, $\epsilon = \pm$, is attracting from the side ϵ (i.e L is attracting from the side of U).

ii) For each leaf $F \subset U$, the intersection $\overline{F} \cap U$ contains a minimal set in U.

Lemma 3.2 [15]**.** *For each nonempty saturated open subset $U \subset M$ and for each leaf $F \subset U$, the intersection $\overline{F} \cap U$ contains at most finitely many minimal sets in the open set U.*

Consider the space of leaves $Z = M/\mathcal{F}$ and the space of quasi-leaves $X = M/\widetilde{\mathcal{F}}$ (the space of leaves classes). Let X_0 (resp. Z_0) be the union of all open subsets of X (resp. Z) homeomorphic to $\mathbb{R}$ or S^1.

In [2], the authors proved that if $\mathcal{F}$ has a well defined height, then the space $X - X_0$ is homeomorphic to the prime spectrum of a unitary commutative ring equipped with the Zariski topology.

Recall that the statement that the height of the foliation is well defined [16] is equivalent to the fact that every totally ordered family of leaves, ordered by inclusion of their closures, is well-ordered (*i.e.* it has a minimal element).

Precisely, the authors of [2] proved that $X - X_0$ satisfies the properties:

i) $X - X_0$ is a quasi-compact space.

ii) $X - X_0$ is a T_0-space.

iii) Each irreducible closed subset has a generic point.

iv) $X - X_0$ has a basis of quasi-compact open subsets.

v') If $\mathcal{F}$ has a well defined height, then the intersection of two quasi-compact open subsets is quasi-compact.

In this section, we consider a countable subgroup G of the group $\mathrm{Diff}_+{}^r(\mathbb{R})$ of increasing C^r-diffeomorphisms ($r \geq 0$). We remark that a C^0-diffeomorphism means a homeomorphism. We can suppose that for each $x \in \mathbb{R}$, we have $G(x) \neq \{x\}$ which is equivalent to the fact that every orbit is unbounded in both sides.

Let $Z = \mathbb{R}/G$ be the space of orbits, and $X = \mathbb{R}/\widetilde{G}$ be the quasi-orbits space. We denote by Z_0 (resp. X_0) the union of all open subsets of Z (resp. X) homeomorphic to $\mathbb{R}$ or S^1. We have $Z_0 = X_0$.

Lemma 3.3. *With the above notations we have the following properties:*

i) If G has a minimal set, then $X - X_0$ is quasi-compact.

ii) $X - X_0$ is a T_0-space.

iii) Each irreducible closed subset of $X - X_0$ has a generic point.

Proof. Immediately, we have the property ii) and the lemma 2.2 gives us the property iii). For showing the property i) we consider a minimal set S of G. If S is a closed orbit, then there exists an element $f \in G$ without fixed point. If not, S is the only minimal set and we have $S \subset \overline{O}$, for each orbit O [15]. In both cases, there exists a compact set $K \subset \mathbb{R}$ with $Sat(K) = \mathbb{R}$. Thus we deduce by lemma 1.2.1.3 and by lemma 1.2.1.4 that X is quasi-compact and so $X - X_0$ is also quasi-compact; indeed $X - X_0$ is a closed subset of X. □

For each $x \in \mathbb{R}$ and $\varepsilon > 0$, we denote by $I^{\eta}_{x,\varepsilon}$ the open interval $]x, x+\varepsilon[$ if $\eta = +$, and $]x-\varepsilon, x[$ if $\eta = -$. Recall that G_x is the isotropic subgroup of G at x.

We say that a proper orbit $O = G(x)$ is *attracting* from the side $\eta = \pm$ if there exists $\varepsilon > 0$ such that for every $y \in I^{\eta}_{x,\varepsilon}$ we have $G_x(y) \neq \{y\}$; this means that for every $y \in I^{\eta}_{x,\varepsilon}$ we have $x \in \overline{G(y) \cap I^{\eta}_{x,\varepsilon}}$.

We say also that this orbit $O = G(x)$ is *stable* from the side $\eta = \pm$ if there exists $\varepsilon > 0$ such that for every $y \in I^{\eta}_{x,\varepsilon}$ we have $G_x(y) = \{y\}$. In this case we have the inclusion $p(Sat(I^{\eta}_{x,\varepsilon})) \subset X_0$. Where $p : \mathbb{R} \to X$ is the canonical projection.

It is clear that the notions attracting and stable are independent of the choice of a point x in O.

We say that O^{η} ($\eta = \pm$) is attracting, stable instead of O attracting, stable from the side η.

The proof of the following lemma is near to the proof of the proposition 3.5 of [2].

Lemma 3.4. *With the above notations the space $X - X_0$ has a countable basis of quasi-compact open subsets.*

Proof. It suffices to show that for every nonempty open subset V of $X - X_0$ and for every $\alpha \in V$, there exists a nonempty quasi-compact open subset W of $X - X_0$ satisfying the condition (*): $\alpha \in W \subset V$.

Consider an orbit O such that $p(O) = \alpha$ and an element $x \in O$. Let V' be a nonempty open subset of X such that $V = V' - X_0$. We put $U' = p^{-1}(V')$, and let $]a, b[$ be the component connected of U' containing x.

Since $G(a) \cap]a, b[= G(b) \cap]a, b[= \emptyset$, for every $y \in]a, b[$ we have $G(y) \cap]a, b[= G_a(y) = G_b(y)$.

First, we suppose that there exists $g \in G_a = G_b$ such that $g(x) \neq x$ (we can suppose that $g(x) > x$). In this case the subset $Sat([x, g(x)])$ is a nonempty compact by saturation open subset of $\mathbb{R}$; we deduce that $W = p(Sat([x, g(x)])) - X_0$ satisfies the condition (*). Otherwise, we have $G_a(x) = \{x\}$, then $G(x) \cap]a, b[= \{x\}$ and so $O = G(x)$ is a proper orbit. We complete the proof in three steps.

Step 1. If $G(x)$ is proper attracting from both sides, then there exists $\varepsilon > 0$ such that for every $y \in]x - \varepsilon, x + \varepsilon[$ we have $x \in \overline{G(y)}$. Thus $Sat([x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]) = Sat(]x - \varepsilon, x + \varepsilon[)$. We conclude that the open subset $W = p(Sat(]x - \varepsilon, x + \varepsilon[)) - X_0$ is a quasi-compact open subset of $X - X_0$ and so satisfies (*).

Step 2. The orbit $O = G(x)$ is proper attracting from one side and not attracting from the other side (we can suppose that O^- is attracting and O^+ is not attracting).

- If O^+ is not stable, then there exists an infinite sequence (y_n) in $]x, b[$ converging to x and such that for every integer n, $G_x(y_n) \neq \{y_n\}$. We put $FixG_x = \{y \in]a, b[: G_x(y) = \{y\}\}$. For every integer n we denote by $]a_n, b_n[$ the component connected of $]x, b[- FixG_x$ containing y_n. Since O^+ is not attracting the sequences (a_n), (b_n) are infinite and converge to x. Thus for some integer (large enough) we have $Sat(]x - \varepsilon, b_n[) = Sat([x - \frac{\epsilon}{2}, a_n + \frac{b_n - a_n}{2}])$ and hence the subset $W = p(sat(]x - \varepsilon, b_n[)) - X_0$ satisfies (*).

- If O^+ is stable, then

$$p(Sat(]x - \varepsilon, x + \varepsilon[)) - X_0 = p(Sat([x - \frac{\epsilon}{2}, x])) - X_0$$

and so the quasi-compact open subset $W = p(Sat(]x - \varepsilon, x + \varepsilon[)) - X_0$ satisfies the condition (*).

Step 3. The orbit $O = G(x)$ is proper not attracting from both sides. Since $\alpha = p(G(x))$ is not contained in X_0 the orbit $G(x)$ is not stable from one side at least (we suppose that O^- is not stable). As in the step 2, we can construct the quasi-compact open subset W satisfying (*) according to the fact that O^+ is stable or not stable. This ends the proof. □

Remark 3.5. We can associate to the group G a group H of $\mathrm{Homeo}_+(\mathbb{R})$ generated by two elements T and h such that the orbits of G and those of H have the same nature (proper, dense, exceptional, ...).

By taking $S^1 = \overline{\mathbb{R}}/-\infty \sim +\infty$ where $\overline{\mathbb{R}} = [-\infty, +\infty]$ and by suspension of the new group H, we obtain a transversally oriented codimension-one foliation of class C^0 on the closed 3-manifold $M = V_2 \times S^1$, where V_2 is a closed surface of genus 2.

Proof of remark 3.5. By taking $\overline{\mathbb{R}} = [-\infty, +\infty]$ we can suppose that $E = [0, 1]$. Since G is countable, we can write $G = \{g_n / n \in \mathbb{N}\}$. We put $T(x) = x + 1$ for every $x \in \mathbb{R}$, $h(x) = x$ if $x \leq 0$ and $h(x) = g_n(x - n) + n$ if $x \in [n, n + 1]$, $n \in \mathbb{N}$. The orbit $H(0)$ is closed and equal to $\mathbb{Z}$, for every $x \in \mathbb{R} - \mathbb{Z}$ and for the integer n with $n < x < n + 1$, we have $H(x) \cap]0, 1[= G(x - n)$; indeed, we have $H(x) \cap]n, n + 1[= H_n(x)$ where H_n is the isotropic subgroup of H at n, the subgroup H_n is equal to the subgroup containing elements of H having a fixed point in $\mathbb{R}$ [14]. We conclude that the orbits $G(x)$ and $H(x - n)$ have the same nature. □

Lemma 3.6. *If G has a minimal set, then the intersection of two quasi-compact open subsets of $X - X_0$ is quasi-compact if and only if every totally ordered family of orbits (ordered by inclusion of their closures) has an infimum.*

Proof. By lemma 3.3 and lemma 3.4, the space $X - X_0$ satisfies the properties i), ii), iii), iv) of the introduction. If moreover, the intersection of two quasi-compact open subsets of $X - X_0$ is quasi-compact then, as in the introduction, $X - X_0$ satisfies the condition (K_1) of Kaplansky and so every totally ordered family of orbits has an infimum.

Conversely, by the above construction in remark 3.5, we can study the intersection of two nonempty open saturated subsets U, V of $M = V_2 \times S^1$ which are compact by saturation.

We must show, by using lemma 3.1, that the intersection $W = U \cap V$ is also compact by saturation. Since $\delta^\epsilon W \subset \delta^\epsilon U \cup \delta^\epsilon V$, $\epsilon = \pm$, it suffices to show that the open subset W satisfies the property $b - ii$) of lemma 3.1. Consider a leaf $F \subset W$, we can suppose that W is connected, otherwise we can take the connected component of W containing the leaf F.

Let $\{F_i\}$ be a maximal totally ordered family of leaves contained in $\overline{F} \cap W$ and let L be the infimum of this family (from the hypothesis, this leaf L exists). Using lemma 3.2 and lemma 3.1-b-ii), there exists a minimal set E_1 (resp. E_2) of $\mathcal{F}$ restricted to U (resp. V) contained in the closure $\overline{F_i}$ for each index i. If L_1 ,L_2 are two leaves with $L_1 \subset E_1$, $L_2 \subset E_2$, then $\overline{L_1} = \overline{E_1} \subset \overline{L}$ and $\overline{L_2} = \overline{E_2} \subset \overline{L}$ and so $L \subset U$ and $L \subset V$, thus $L \subset W$ and $\overline{L} \cap W$ is a minimal set of $\mathcal{F}$ restricted to W. Otherwise, there exists a leaf $L' \subset \overline{L} \cap W$ with $\overline{L'} \neq \overline{L}$ which contradicts the fact that the family $\{F_i\}$ is maximal in $\overline{F} \cap W$. We conclude that W satisfies the two facts of the property b) of lemma 3.1, thus it is compact by saturation. □

Remark 3.7. As in the proof of the lemma 3.4 we can state and give a proof of lemmas 3.1 and 3.2, where we replace in the statement of these lemmas the hypothesis “codimension-one foliation” by the hypothesis “groups of increasing homeomorphisms of the line $E = \mathbb{R}$” without using foliation. Hence we can give a direct proof of the lemma 3.6 without returning to foliation. But to state the theorem 3.9 we need the proof of lemma 3.6 in terms of foliation.

By lemmas 3.3, 3.4 and 3.6, the characterization of primitive and prime spectrum spaces, given in the introduction, permit us to state:

Theorem 3.8. *Let G be a countable subgroup of $Homeo_+(\mathbb{R})$ having a minimal set. Let X be the quasi-orbits space and X_0 be the union of all open subsets of X homeomorphic to $\mathbb{R}$ or S^1. Then:*

a) The space $X - X_0$ satisfies always the properties i), ii), iii) and iv) as in the introduction.

b) The space $X - X_0$ satisfies also the property v) if and only if every totally ordered family of orbits has a greatest lower bound.

In the same manner we obtain also the following theorem which generalizes and completes the main result of [2]:

Theorem 3.9. *Let $\mathcal{F}$ be a transversally oriented codimension-one foliation of class C^r, $r \geq 0$, on a closed manifold M. Let X be the quasi-leaves space and X_0 be the union of all open subsets of X homeomorphic to $\mathbb{R}$ or S^1. Then:*

a) The space $X - X_0$ satisfies always the properties i), ii), iii) and iv) as in the introduction.

b) The space $X - X_0$ satisfies also the property v) if and only if every totally ordered family of leaves has a greatest lower bound.

Example 3.10. The example 2.10 shows that the two conclusions of these theorems fail if E is a closed surface or if $\mathcal{F}$ is of codimension greater than one; indeed, in these cases the space $X - X_0$ does not have any basis of quasi-compact open subsets.

The two examples of 2.3 satisfy the conclusions a) and b) of the theorem 3.8. The next example gives a space which satisfies a) but not b).

Example 3.11. Let us consider the space $X = \{\frac{1}{n} : n \geq 1\} \cup \{a, b\}$ equipped with the topology of subbasis the family $\{U_n : n \geq 1\} \cup \{V_a, V_b\}$, where $V_a = X - \{b\}$, $V_b = X - \{a\}$ and $U_n = \{\frac{1}{n}, \frac{1}{n-1}, ..., 1\}$ for each integer $n \geq 1$. This space corresponds to the quasi-orbits space of the subgroup G of [2, Example 2.9], it satisfies the property a) but not the property b) of the theorem 3.8. Indeed the intersection $V_a \cap V_b = X - \{a, b\}$ of the two quasi-compact open subsets V_a and V_b is not quasi-compact.

Next, we will give some conditions on a countable subgroup G of $\mathrm{Homeo}_+(\mathbb{R})$ such that $X - X_0$ satisfies the two properties of theorem 3.8.

Proposition 3.12 [14]**.** *A countable subgroup G of $Homeo_+(\mathbb{R})$ has a minimal set and each totally ordered family of orbits has an infimum under one of the following conditions:*

a) G is polycyclic (i.e every subgroup of G is of finite type). In particular if G is abelian of finite type.

b) G is an abelian subgroup of $Diff_+{}^r(\mathbb{R})$ ($r \geq 2$) (of finite type or not).

c) G is a subgroup of $Diff_+{}^\omega(\mathbb{R})$ (abelian or not, of finite type or not).

d) The action of G on $\mathbb{R}$ is free.

e) G is a finite type subgroup of $Diff_+{}^r([0,1])$, ($r \geq 2$).

Each totally ordered family of orbits has a finite length under the conditions a), b), c), and d) of the previous proposition. In the case of the property e) every totally ordered family is well-ordered of order type less than or equal to the first limit ordinal ω.

Using examples of [14] and of [9], and the method used in [2, Example 2-9] we can construct some countable groups of $Homeo_+(\mathbb{R})$ which do not satisfy the property b) of theorem 3.8 in one of these conditions:

a) G is abelian (necessarily of non finite type).

b) G is a subgroup of $\mathrm{Diff}_+{}^\infty(\mathbb{R})$ generated by two elements (necessarily non abelian).

c) G is a subgroup of $\mathrm{Diff}_+{}^\infty([0,1])$ (necessarily of non finite type).

Example 3.13. Consider the set $Y = \{a, b_1, b_2, b_3, c\}$ formed by elements distinct two-two. We consider a topology on Y generated by the family $\{Y, \{b_i, c\} : i = 1, 2, 3\}$. We have $\overline{\{a\}} = \{a\}$ and $\overline{\{b_i\}} = \{a, b_i\}$, $i = 1, 2, 3$.

Since the closure of $\{c\}$ contains three minimal sets $\{b_1\}$, $\{b_2\}$, $\{b_3\}$ in the open subset $V = Y - \{a\}$, we deduce that Y cannot be homeomorphic to any $X - X_0$ as in theorem 3.8. Indeed, under the hypotheses of this theorem every orbit contains in its closure at most two minimal sets in a given nonempty saturated open subset [15].

Nevertheless, it is easy to see that every T_0-space having only a finitely many elements satisfies the five properties given in the introduction and so it is homeomorphic to a primitive spectrum and to a prime spectrum. However, we don't know when such spaces are homeomorphic to a space $X - X_0$ as in theorem 3.9.

In infinite case, we may build some countable topological spaces having an infinitely many elements which cannot be homeomorphic to any $X - X_0$ as in theorem 3.9. However, these spaces satisfy the five properties cited above [2, example 3.20].

Proposition 3.14. *We denote by G, X, X_0 the same objects as in theorem 3.8. If every orbit is contained in a local minimal set (L.M.S), then the space $X - X_0$ is homeomorphic to the primitive spectrum of a postliminary A.F C^*-algebra.*

Proof. The fact that each orbit is contained in a (L.M.S) implies that each point of $X - X_0$ is isolated in its closure. Indeed, on one side for every $x \in X$ and for every orbit O contained in $p^{-1}(\{x\})$, we have $p^{-1}(\overline{\{x\}} - \{x\}) = \overline{O} - Cl(O)$, and on the other side an orbit O is contained in a (L.M.S) if and only if $\overline{O} - Cl(O)$ is a closed subset (property 4 of paragraph 1.2.1). Applying [5, Theorem p. 79] and lemmas 3.3, 3.4, it follows that the space $X - X_0$ is homeomorphic to the primitive spectrum of a postliminary $A.F$ C^*-algebra. $\square$

Proposition 3.15. *Under the hypotheses of theorem 3.8, the union Y of all locally closed points of X is a strongly dense subset of X. In particular, Y is everywhere dense in X.*

A subset is said to be strongly dense if it meets every nonempty locally closed subset[8].

For notions related to closed point, locally closed point, minimal set, local minimal set in a topological space we refer to the paragraph 1.2.2.

Proof. In the beginning, we show the following statement:

(*) If a nonempty topological space Z is a quasi-compact T_0-space, then it contains a closed point $z \in Z$.

Since Z is a quasi-compact space, by Zorn's lemma, there exists a minimal set S of Z. Thus for every $z \in S$ we have $\overline{\{z\}} = S$. The fact that Z is a T_0-space implies that S is a singleton $\{z\}$ (indeed $\overline{\{a\}} = \overline{\{b\}} \Rightarrow a = b$). Therefore, we have (*).

For showing the proposition, it suffices to prove that every nonempty locally closed subset of X contains a locally closed point. Let A be a locally closed subset of X. There exists a(n) open (resp. closed) subset U (resp. C) of X such that $A = U \cap C$. Since the open subset X_0 is Hausdorff, it follows that its points are locally closed. First, we suppose that A meets X_0, that is, A contains a locally closed point. Otherwise, we suppose that A is contained in $X - X_0$. From lemma 3.4, it follows that the space $X - X_0$ has a basis of quasi-compact open subsets. Then for each $x \in A$, there exists a quasi-compact open subset V such that $x \in V \subset U$. Since the subset $V \cap C$ is closed in the quasi-compact V, then $V \cap C$ is quasi-compact. By the statement (*), there exists a closed point y in $V \cap C \subset U \cap C = A$. We deduce that y is a locally closed point containing in A. $\square$

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Institut Supérieur d'Informatique et du Multimedia, Route de Mharza, Km 1 B.P:1030 Sfax 3018, Tunisie

E-mail address: `thwtwtt@yahoo.fr`

Faculté des Sciences de Sfax, Département de mathématiques, 802 Sfax 3018 Tunisie

E-mail address: `Ezzeddine.Salhi@fss.rnu.tn`