Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 28, No.2, 2004
Pages 527-540

EXTENSION OF CONTINUOUS FUNCTIONS ON PRODUCT SPACES, BOHR COMPACTIFICATION AND ALMOST PERIODIC FUNCTIONS

SALVADOR HERNÁNDEZ*

Dedicated to Professor Wis Comfort

Abstract. The Bohr compactification is a well known construction for (topological) groups and semigroups. Recently, this notion has been investigated for arbitrary structures in [2] where the Bohr compactification is defined, using a set-theoretical approach, as the maximal compactification which is compatible with the structure involved. Here, we give a characterization of the continuous functions defined on a product space that can be extended continuously to certain compactifications of the product space. As a consequence, the Bohr compactification of an arbitrary topological structure is obtained as the Gelfand space of the commutative Banach algebra of all almost periodic functions. Previously, almost periodic functions f are defined in terms of translates of f with no reference to any compactification of the underlying structure. An application is given to the representation of isometries defined between spaces of almost periodic functions.

2000 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.

Key words and phrases. Bohr compactification, almost periodic functions, product spaces, extension of continuous functions.

*Research partially by the Spanish Ministry of Science (including FEDER funds), grant MTM2004-07665-C02-01; and the Generalitat Valenciana, grant GV04B-019.

1. Introduction

Hart and Kunen in [2] have defined and investigated the Bohr compactification and topology of an arbitrary discrete structure (see also [4]). Even though their approach to the question is set-theoretical, they also comment that the name of Bohr is attached to this construction stems from the fact that, for topological groups, this compact structure can be defined via almost periodic functions that were introduced by Harald Bohr. In fact, in [2] is also implicit the question of defining the Bohr compactification of arbitrary structures using "appropriately" defined almost periodic functions (see [2, 2.3.12]). The motivation for this paper is to study this question. We define almost periodic functions f directly in terms of the translates of f without any previous reference to any compactification of the underlying structure. We also characterize the continuous functions defined on a product space that can be extended to certain compactifications of the product space. As a consequence, it is proved that the Bohr compactification introduced in [2] is canonically equivalent to the Gelfand or structure space associated to the commutative Banach algebra of all almost periodic functions. In [6, 7, 8, 9] I. Prodanov established a theory of almost periodic function for certain general topological structures that he called *continous algebraic structures*. Prodanov's results are related to the ones in this paper and, in fact, our definition of almost periodic function is similar to the one considered by him. Thus, part of our results can also be obtained using Prodanov's approach.

The basic definitions and terminology are taken from [2, 1, 10, 11, 12]. Firstly, we recall some basic facts that will be used along the paper.

Let X be a set and $\Delta = \Delta(X) \subset X \times X$ the diagonal on X. For $B, C \subset X \times X$, $S \subset X$ we define $B \circ C = \{(x,z) \in X \times X : (x,y) \in B \text{ and } (y,z) \in C \text{ for some } y \in X\}$, $B^{-1} = \{(x,y) : (y,x) \in B\}$, $B[S] = \{y \in X : (x,y) \in B \text{ for some } x \in X\}$. A *uniformity* on X is a set μ of subsets of $X \times X$ satisfying the following conditions: (i) $B \supset \Delta$ for all $B \in \mu$; (ii) if $B \in \mu$, then $B^{-1} \in \mu$; (iii) if $B \in \mu$, there exists $C \in \mu$ such that $C \circ C \subset B$; (iv) the intersection of two members of μ also belongs to μ; (v) any subset of $X \times X$, which contains a member of μ, itself belongs to μ.

The members of μ are called *vicinities* (of Δ). By a *base* for a uniformity μ is meant a subset $\mathcal{B}$ of μ such that a subset of $X \times X$ belongs to μ if and only if it contains a set belonging to $\mathcal{B}$. A *uniform space* μX is a pair comprising a set X and a uniformity μ on X. If μX is a uniform space, one may define a topology τ on X by assigning to each point x of X the neighborhood base comprised of the sets $B[x]$, B ranging over the uniformity (see [10]).

The most basic properties of Banach algebras can be found in [11]. It suffices to say here that every commutative Banach algebra with unity $\mathcal{A}$ has associated a compact space $K = K(\mathcal{A})$ (*the Gelfand space*), which consists of all non null complex homomorphisms of $\mathcal{A}$. The algebra $\mathcal{A}$ is isomorphic to a subalgebra of $C(K)$ by means of a map $f \longrightarrow \hat{f}$, given by $\hat{f}(\chi) = \chi(f)$ for all $\chi \in K$. We call $\hat{f}$ the *Gelfand transform* of f. If $\| \cdot \|_\infty$ denotes the norm of uniform convergence on K, it holds that $\| \hat{f} \|_\infty \leq \| f \|$ for all $f \in \mathcal{A}$.

In what follows $\mathcal{L}$ is a set (possibly empty) of symbols of constants and symbols of functions; every function symbol has arity ≥ 1. Using the symbols of $\mathcal{L}$ and the predicate "=" one may construct logical formulae in the usual way. Only the predicate "=" will be used here. A *structure* $\mathcal{U}$ for $\mathcal{L}$ is a non empty set A (the domain) together with elements (of) and functions (defined on) A corresponding to the symbols in $\mathcal{L}$. E.g., when we talk about groups, it is understood that $\mathcal{L} = \{\cdot, i, 1\}$ (symbols of the product, inverse element and identity). Thus, groups are displayed as $\mathcal{U} = (A; \cdot, i, 1)$.

Let $\mathcal{U}$ be a structure for $\mathcal{L}$ and $f : A \to X$. If $\Phi \in \mathcal{L}$ is an n-ary function symbol, then $f(\Phi_{\mathcal{U}})$ denotes $\{(f(a_1, ..., f(a_n), f(b)) : (a_1, ..., a_n, b) \in \Phi_{\mathcal{U}}\}$. Here $\Phi_{\mathcal{U}}$ is identified to the graph of Φ. We have that $f(\Phi_{\mathcal{U}}) \subset X^{n+1}$ but is not necessary the graph of an n-ary function.

A *topological structure* for $\mathcal{L}$ is a pair $(\mathcal{U}, \tau)$ where $\mathcal{U}$ is a structure for $\mathcal{L}$, and τ is a topology on A making all functions in $\mathcal{U}$ continuous. We write $\mathcal{U}$ for $(\mathcal{U}, \tau)$ if the topology is understood.

Let $\mathcal{U}$ and $\mathcal{V}$ be two topological structures of $\mathcal{L}$, and $f : A \to B$. The map f is a *homomorphism* from $\mathcal{U}$ to $\mathcal{V}$ iff f is continuous, $f(\Phi_{\mathcal{U}}) \subset \Phi_{\mathcal{V}}$ for each function symbol Φ of $\mathcal{L}$, and $f(c_{\mathcal{U}}) = c_{\mathcal{V}}$ for each constant symbol c of $\mathcal{L}$.

A *compact structure* for $\mathcal{L}$ is a topological structure $(\mathcal{U}, \tau)$ in which τ is a compact Hausdorff topology.

Let A be any non-empty set. A *compactification* of A is a pair (X, φ), where X is a compact space, $\varphi : A \longrightarrow X$, and $\varphi(A)$ is dense in X. If (X, φ) and (Y, ψ) are two compactifications of A, then $(X, \varphi) \leq_\Gamma (Y, \psi)$ means that $\Gamma : Y \longrightarrow X$ is a continuous function and $\Gamma \circ \psi = \varphi$. $(X, \varphi) \leq (Y, \psi)$ means that $(X, \varphi) \leq_\Gamma (Y, \psi)$ for some Γ.

If $(\mathcal{U}, \tau)$ is a topological structure and (X, φ) is a compactification of the set A, then (X, φ) is *compatible* with $(\mathcal{U}, \tau)$ iff φ is continuous and there is a topological structure $\mathcal{X}$ built on the set X such that φ is a homomorphism.

The *Bohr compactification*, $(b(\mathcal{U}, \tau), \Phi_{(\mathcal{U},\tau)})$, of a given topological structure $(\mathcal{U}, \tau)$, is the maximal compatible compactification. The τ is omitted when it is clear from context.

Here on, we consider a topological structure $(\mathcal{U}, \tau)$ for an arbitrary but fixed set $\mathcal{L}$ of constants and functions. If A is the set underlying the structure $\mathcal{U}$, we have that (A, τ) is a topological space such that for each $\Phi \in \mathcal{L}$, the map $\Phi : A^n \to A$ is continuous on A^n, here "n" is the arity of Φ. In order to simplify the notation in the above situation later on, we consider the following symbolism: for $\Phi \in \mathcal{L}$ of arity "n", let $A_i = A$ for $i = 1, ..., n$ and consider $\Phi : A_1 \times A_2 \times ... \times A_n \to A$ canonically defined. Thus, with some notational abuse, each $(a_{i_1}, a_{i_2}, ..., a_{i_m}) \in A_{i_1} \times A_{i_2} \times ... \times A_{i_m}$, $1 \leq m \leq n$, defines the map $\Phi_{(a_{i_1}, a_{i_2}, ..., a_{i_m})} : A_{j_1} \times A_{j_2} \times ... \times A_{j_{(n-m)}} \to A$ by $\Phi_{(a_{i_1}, a_{i_2}, ..., a_{i_m})}(a_{j_1}, ..., a_{j_{(n-m)}}) = \Phi(a_1, a_2, ..., a_n)$. Using this symbolism, if we take any j with $1 \leq j \leq n$ and an arbitrary but fixed $\overrightarrow{x} \in \prod\{A_i : 1 \leq i \leq n, \ i \neq j\}$, we define a *translation* $t^\Phi_{\overrightarrow{x}}$ on A ($t_{\overrightarrow{x}}$ for short if there is no possible confusion) by the rule $t_{\overrightarrow{x}}(a) = \Phi_{\overrightarrow{x}}(a) = \Phi(\overrightarrow{x}; a)$, here on the symbol ";" is used to mean that the *variable* a is placed at the coordinate "j". We say that $t_{\overrightarrow{x}}$ is a *simple translation* on A. In case Φ is 1-ary, we define the translation $t^\Phi_\emptyset$ to be Φ. Simple translations can be multiplied using the ordinary composition of mappings. Thus, the set of all simple translations generates the semigroup of (general) translations $S(\mathcal{U})$ with the composition law defined by ordinary function composition. To emphasize the fact that simple translations are defined on A, we shall use the symbol $t^\Phi_{\overrightarrow{x}}$ when $\overrightarrow{x} \in \prod\{A_i : 1 \leq i \leq n, \ i \neq j\}$, $1 \leq j \leq n$. Otherwise, we shall use the most standard symbol $\Phi_{\overrightarrow{x}}$. In the sequel, if X is a topological space, we denote by $C_\infty(X)$ the set of all complex valued continuous functions on X equipped with the supremum norm.

Definition 1. The map $f : A \to \mathbb{C}$ is said to be almost periodic when is bounded, continuous and it holds that the set $\{(f \circ \tau \circ \Phi_{a_j}) : a_j \in A\}$ is relatively compact in $C_\infty(\prod\{A_i : 1 \leq i \leq n,\, i \neq j\})$ for all $\Phi \in \mathcal{L}$, all $\tau \in S(\mathcal{U})$, and all j with $1 \leq j \leq n$, with n being the arity of Φ.

As a consequence of the definition above, for all f almost periodic and $\tau \in S(\mathcal{U})$, it holds that $f \circ \tau$ is almost periodic. The set of all almost periodic functions on a topological structure $\mathcal{U}$ is denoted by $AP(\mathcal{U})$. Obviously, when the composition of any two simple translations yields a simple translation, we have that the set of all simple translations is itself a semigroup. This happens, for example, for groups and semigroups.

In principle, our approach for defining the Bohr compactification of an arbitrary structure generalize the one given by Loomis [5] and Semadeni [12] for topological groups. Nevertheless, the obvious complication that arise when we want to extend the operations of a given algebraic structure to its Bohr compactification stems from the fact that there are *many* arbitrary operations of different arity in general. This means that the set of simple translations is far from being a semigroup and, as a consequence, the usual proofs given for groups and semigroups do not work here. Thus, our approach is also topological since it is based on Proposition 3 below, which is a result about extension of continuous functions defined on a compact space.

2. Main results

The goal now is to study the properties of $AP(\mathcal{U})$. We want $AP(\mathcal{U})$ to be a commutative Banach algebra so that its structure or Gelfand space be isomorphic to $b\mathcal{U}$ as it was defined in [2]. The proof of the proposition below is more or less standard, we include part of it for the reader's sake (see [12, §14.7]).

Proposition 1. *The set $AP(\mathcal{U})$ is a closed subalgebra of $C_\infty(A)$ containing the constants.*

Proof. We only check that $AP(\mathcal{U})$ is closed in $C_\infty(A)$. Let $\{f^{(n)}\}$ be a sequence in $AP(\mathcal{U})$ converging to f for the uniform convergence topology, $\Phi \in \mathcal{L}$, $\tau \in S(\mathcal{U})$, and $\{a_n\}$ a sequence in A. In order to prove that $f \in AP(\mathcal{U})$, we must show that $\{(f \circ \tau \circ \Phi_{a_n})\}$ has a convergent subsequence. Let us denote $g^{(n)} = (f^{(n)} \circ \tau \circ \Phi)$, $g = (f \circ \tau \circ \Phi)$ and $g_a^{(n)} = (f^{(n)} \circ \tau \circ \Phi_a)$, to simplify the notation. Thus, $g_{a_n}^{(1)} = (f^{(1)} \circ \tau \circ \Phi_{a_n})$

Given $\epsilon > 0$, there is $n_1 < \omega$ such that $\| g - g^{(n)} \| < \frac{\epsilon}{3}$ for $n > n_1$, where $\| \, . \, \|$ denotes the supremum norm. The sequence $\{g_{a_n}^{(1)}\}$ will contain a convergent subsequence , say $\{g_{a_{(n,1)}}^{(1)}\}$. There is no loss of generality in assuming that $\| g_{a_{(n,1)}}^{(1)} - g_{a_{(m,1)}}^{(1)} \| < \frac{1}{2}$ for all $n, m < \omega$. We now define inductively a collection of convergent sequences $\{g_{a_{(n,j)}}^{(j)}\}$, $j < \omega$, satisfying:

- $\{g_{a_{(n,j+1)}}^{(j+1)}\}$ is a subsequence of $\{g_{a_{(n,j)}}^{(j)}\}$;
- $\| g_{a_{(n,j)}}^{(j)} - g_{a_{(m,j)}}^{(j)} \| < \frac{1}{2^j}$ for all $n, m < \omega$.

Take the diagonal subsequence $\{g_{a_{(n,n)}}\}$ of $\{g_{a_n}\}$, and let $n_0 < \omega$ such that $n_0 \geq n_1$ and $\frac{1}{2^{n_0}} < \frac{\epsilon}{3}$. For any $n, m \geq n_0$, we have

$$\| g_{a_{(n,n)}} - g_{a_{(m,m)}} \| \leq$$
$$\| g_{a_{(n,n)}} - g_{a_{(n,n)}}^{n_0} \| + \| g_{a_{(n,n)}}^{n_0} - g_{a_{(m,m)}}^{n_0} \| + \| g_{a_{(m,m)}}^{n_0} - g_{a_{(m,m)}} \| \leq 3 \cdot \frac{\epsilon}{3}.$$

Hence, $\{g_{a_{(n,n)}}\}$ is a Cauchy sequence in $C_\infty(A)$ what completes the proof. □

We have just shown that $AP(\mathcal{U})$ is a commutative Banach algebra of continuous functions on A (the base space of $\mathcal{U}$) with the supremum norm. Therefore, we can state the following.

Definition 2. The compact Gelfand space associated to the commutative Banach algebra $AP(\mathcal{U})$ is denoted by $b\mathcal{U}$.

Next result is our characterization of the Bohr compactification. The proof is split in several lemmas and propositions.

Theorem 1. *The space $b\mathcal{U}$ is a realization of the Bohr compactification of the topological structure $(\mathcal{U}, \tau)$.*

According to the definition of the Bohr compactification of a general topological structure, in order to prove the result above, the following facts need be established:

- B1 there is a map δ from A into $b\mathcal{U}$ such that $\delta(A)$ is dense in $b\mathcal{U}$.
- B2 there is an algebraic structure on $b\mathcal{U}$ compatible with $\mathcal{U}$.
- B3 $b\mathcal{U}$ is the maximal compactification of $\mathcal{U}$.

The verification of these properties requires some previous work. Firstly, we state the following proposition on compact structures.

Proposition 2. *For each compact structure $\mathcal{U} = (X, \mathcal{L})$ it holds that $C(X)$ is contained in $AP(\mathcal{U})$.*

Proof. For any $f \in C(X)$, let Φ be an arbitrary but fixed element of $\mathcal{L}$, $\tau \in S(\mathcal{U})$, and let n be the arity of Φ. For any index j such that $1 \leq j \leq n$, define the map $\gamma : X_j \to C_\infty(\prod\{X_i : 1 \leq i \leq n,\ i \neq j\})$ by $\gamma(a) = f_a$, $a \in X_j$, where $X_i = X$ for $1 \leq j \leq n$ and $f_a = (f \circ \tau \circ \Phi_a)$. Taking into account that the topology of uniform convergence on compact sets is proper (that is to say, the continuity of any map $\varphi : X \times X \longrightarrow C$ yields automatically the continuity of the map defined by $x \mapsto \varphi(x, .)$), it follows that γ is continuous on X_j (see [1]).

Since X is compact, we obtain the compactness of $\gamma(X_j)$ in $C_\infty(\prod\{X_i : 1 \leq i \leq n,\ i \neq j\})$. This completes the proof. $\square$

Next lemma verifies item B1 above in the definition of the Bohr compactification.

Lemma 1. *There is a continuous map δ sending A into a dense subset of $b\mathcal{U}$.*

Proof. Since $b\mathcal{U}$ is the structure space of $AP(\mathcal{U})$, it follows that $b\mathcal{U}$ is the set of all multiplicative complex functionals on $AP(\mathcal{U})$ equipped with topology of pointwise convergence on $AP(\mathcal{U})$ (cf. [11, Appendix D]). Therefore, the evaluation mapping $\delta : A \longrightarrow b\mathcal{U}$ defined by $\delta(a)[f] = f(a)$ for all $f \in AP(\mathcal{U})$ sends A into $b\mathcal{U}$. Moreover, given that $AP(\mathcal{U}) \subset C(A)$, the map δ is clearly continuous.

The algebra $AP(\mathcal{U})$ may be identified to a subalgebra of $C(b\mathcal{U})$ by means of the Gelfand transform in the following way: for each $f \in AP(\mathcal{U})$, its Gelfand transform is a map $\widehat{f} \in C(b\mathcal{U})$ defined by $\widehat{f}(p) = p(f)$ for all $p \in b\mathcal{U}$. Moreover, for each $f \in AP(b\mathcal{U})$, it holds that $\|\widehat{f}\|_\infty \leq \|f\|_\infty$ (here, $\| \ \|_\infty$ denotes the suppremum norm on either space, $b\mathcal{U}$ and A). On the other hand, $AP(\mathcal{U})$ is a Banach algebra of continuous functions on A equipped with the supremum norm. Since the evaluation mapping δ sends A into $b\mathcal{U}$, it follows that $\|f\|_\infty \leq \|\widehat{f}\|_\infty$ for all $f \in AP(\mathcal{U})$. That is to say, the Gelfand transform, in this case, is an isometry that sends $AP(\mathcal{U})$ isometrically into $C(b\mathcal{U})$. In fact, the Stone-Weierstrass theorem shows that the Gelfand transform is an isometry onto $C(b\mathcal{U})$. Now, observe that $\delta(A)$ separates this algebra since, for each $\widehat{f}$,$\widehat{g} \in C(b\mathcal{U})$ with $\widehat{f} \neq \widehat{g}$, there is $a \in A$ such that $\widehat{f}(\delta(a)) \neq \widehat{g}(\delta(a))$. This property yields the density of the subspace $\delta(A)$ in $b\mathcal{U}$. □

We have just verified that $b\mathcal{U}$ is a compactification of A. In order to verify that $b\mathcal{U}$ coincides with the Bohr compactification of $\mathcal{U}$, the next step is to equip $\delta(A)$ with an algebraic structure $\delta(\mathcal{U})$ for $\mathcal{L}$ such that δ is a continuous homomorphism.

Lemma 2. *The space $\delta(\mathcal{A})$ may be provided with an algebraic structure, $\delta(\mathcal{U})$, for $\mathcal{L}$ such that the map $\delta : A \longrightarrow \delta(A)$ is a continuous homomorphism.*

Proof. Let Φ be any function in $\mathcal{L}$ with arity n. We may assume WLOG that $n > 1$ since the case $n = 1$ is easier to deal with. For $(a_1, ..., a_n) \in A^n$, we set $\Phi(\delta(a_1), ..., \delta(a_n)) = \delta(\Phi(a_1, ..., a_n))$. We must check that Φ is properly defined on $\delta(A)$. Take $(b_1, ...b_n) \in A^n$ with $\delta(b_i) = \delta(a_i)$ for $1 \leq i \leq n$, and let us see that $\delta(\Phi(a_1, ..., a_n)) = \delta(\Phi(b_1, ..., b_n))$. Observe that it suffices to verify that $\delta(\Phi(a_1, a_2..., a_n)) = \delta(\Phi(b_1, a_2..., a_n))$. Otherwise, we split the procedure in several iterates affecting one single coordinate each one of them. Now, since δ is the evaluation mapping, we only need to show that $f(\Phi(a_1, a_2..., a_n)) = f(\Phi(b_1, a_2..., a_n))$ for all $f \in AP(\mathcal{U})$. Let f be any fixed element in $AP(\mathcal{U})$ and consider the simple translation $t = \Phi_{(a_2,...a_n)}$. We have that $f \circ t$ is in $AP(\mathcal{U})$ and $\delta(a_1) = \delta(b_1)$. Hence, $(f \circ t)(a_1) = (f \circ t)(b_1)$. It follows that Φ is well defined on $\delta(A)^n$. Moreover, the definition yields automatically that δ is a continuous (Φ)-homomorphism. This completes the proof. □

We now verify that item B2 stated above holds. It will suffice to prove that every operation $\Phi \in \mathcal{L}(\mathcal{U})$ extends to a continuous operation $\Phi^b \in \mathcal{L}(b\mathcal{U})$. That is, if Φ has arity n, we must extend Φ, defined on A^n, to a continuous map Φ^b defined on $(b\mathcal{U})^n$. At this point, there is no loss of generality in assuming that A is a subspace of $b\mathcal{U}$ since, otherwise, we would replace A by $\delta(A)$ and $\mathcal{U}$ by $\delta(\mathcal{U})$. The first step in this direction is a lemma concerning the extension of mappings defined on products. We refer to [1] for this result. Since we will need (and extend) it in the following we include a sketch of the proof here. In the sequel, if X is a topological space and μZ is a uniform space, we denote by $C_\infty(X, \mu Z)$ the space of all continuous functions from X to μZ equipped with the topology of uniform convergence. That is to say, for $g \in C_\infty(X, \mu Z)$ a basic neighbourhood base consists of sets of the form $N(g, B) = \{h \in C_\infty(X, \mu Z) : (g(x), h(x)) \in B,\ x \in X,\ B \in \mu\}$. Observe that this topology is generated by the uniformity $\mu(\infty)$ which has a base consisting of the sets $B^+ = \{(f, g) \in (C_\infty(X, \mu Z))^2 : (f(x), g(x)) \in B, x \in X\}$, with $B \in \mu$.

Lemma 3. *Let X, Y and K be topological spaces with K being a compactification of X , and let μZ be a uniform space. Let $f : X \times Y \to \mu Z$ be a continuous map such that, for all $y \in Y$, the canonical map $f(., y) : X \to \mu Z$ admits a continuous extension $\overline{f}(., y) : K \to \mu Z$. Consider the following properties:*

(a) *The family $\{f(x, .) : x \in X\}$ is relatively compact in $C_\infty(Y, \mu Z)$;*
(b) *The family $\{f(x, .) : x \in X\}$ is equicontinuous;*
(c) *The map $\gamma : Y \to C_\infty(X, \mu Z)$ defined by $\gamma(y)(x) = f(x, y)$ $(x \in X)$ is continuous;*
(d) *f extends continuously to $K \times Y$.*

Then we have: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

Proof. ($a \implies b$) Let B be an arbitrary vicinity in μ. The family $\{N(g, B) : g \in C(Y, \mu Z)\}$ is an open cover of $C_\infty(Y, \mu Z)$. Since $C = \{f(x, .) : x \in X\}$ is relatively compact, it follows there is a finite subfamily, say $\{N(g_l, B) : 1 \leq l \leq m\}$, that covers C. Now, for each $y \in Y$ there is a neighbourhood $V_{(y,l)}$ such that $(g_l(y), g_l(y')) \in B$ for all $y' \in V_{(y,l)}$. Take $V = \cap_{l=1}^m V_{(y,l)}$. It is easily verified that $(f(x, y), f(x, y')) \in B^2$ for all $y' \in V$ and $x \in X$. This proves the equicontinuity of C.

($b \implies c$) Assuming that $\{f(x, .) : x \in X\}$ is equicontinuous. Given $B \in \mu$ and $y_0 \in Y$, there is a neighbourhood V of y_0 such that $(f(x, y_0), f(x, y)) \in B$ for all $y \in V$. That is, $\gamma(V) \subset N(\gamma(y_0), B)$ and this yields the continuity of γ.

($c \implies d$) Let $\overline{f}(., y) : K \to \mu Z$ be the continuous extension of $f(., y)$ for each $y \in Y$. In order to prove the continuity of $\overline{f} : K \times Y \to \mu Z$, it suffices to show that, for every $p \in K$, the mapping $\overline{f} : (X \cup \{p\}) \times Y \to \mu Z$ is continuous. Now, given $B \in \mu$, take and arbitrary point $y \in Y$. Since $\overline{f}(., y)$ is continuous, there is a neighbourhood $U_{(p,y)}$ of p such that for all $x \in U_{(p,y)}$ it holds $(\overline{f}(p, y), f(x, y)) \in B$. Since $\gamma : Y \to C_\infty(X, \mu Z)$ is continuous, there is a neighbourhood V_y such that $(f(x, y), f(x, y')) \in B$ for all $y' \in V_y$ and $x \in X$. Hence, for each $(x, y') \in U_{(p,y)} \times V_y$ we have that $(f(p, y), f(x, y')) \in B^2$. This proves the continuity of $\overline{f}$. □

Next proposition is one of the main results of this note. It extends the lemma above to finite products.

Proposition 3. *Let $\{X_i : 1 \leq i \leq n\}$ be a finite family of topological spaces and let K_i be a fixed compactification of X_i for $1 \leq i \leq n$. If μZ is a complete uniform space and $g : \prod\{X_i : 1 \leq i \leq n\} \to \mu Z$ is a continuous mapping satisfying:*

(a) *$g_{\overrightarrow{x}}$ can be extended to a continuous map $\overline{g}_{\overrightarrow{x}} : K_j \to \mu Z$, for all $\overrightarrow{x} \in \prod\{X_i : 1 \leq i \leq n,\ i \neq j\}$, $1 \leq j \leq n$; and*

(b) *$\{g_{x_j} : x_j \in X_j\}$ is relatively compact in $C_\infty(\prod\{X_i : 1 \leq i \leq n,\ i \neq j\}, \mu Z)$, $1 \leq j \leq n$.*

Then g can be extended to a continuous map $\overline{g} : \prod\{K_i : 1 \leq i \leq n\} \to \mu Z$.

Proof. In order to simplify the notation, we treat the case $n = 3$ only, as this is representative for the general case. The proof for $n > 3$ is obtained proceeding by induction.

It is clear from Lemma 3 that g extends with continuity to $\overline{g}_1 : K_1 \times X_2 \times X_3 \longrightarrow \mu Z$. Now, we prove that g also extends with continuity to $\overline{g}_2 : K_1 \times K_2 \times X_3 \longrightarrow \mu Z$. This will suffice since the same arguments apply to extend g to the whole product of compact spaces.

Denote by $\overline{g}_{x_2}$ to the continuous extension to $K_1 \times X_3$ of g_{x_2} for each $x_2 \in X_2$.

(i) Firstly, we prove that $\{\overline{g}_{x_2} : x_2 \in X_2\}$ is relatively compact in $C_\infty(K_1 \times X_3, \mu Z)$.

Indeed, the restriction mapping $r : C_\infty(K_1 \times X_3, \mu Z) \longrightarrow C_\infty(X_1 \times X_3, \mu Z)$ is one-to-one and uniformly continuous. By Lemma 3, each map f in $C_\infty(X_1 \times X_3, \mu Z)$ may be extended to a continuous map $\overline{f}$ in $C_\infty(K_1 \times X_3, \mu Z)$. It is easily verified that for all $B \in \mu$, if $(f, g) \in B^+$ then $(\overline{f}, \overline{g}) \in \overline{B}^{\,+} \subset (B \circ B)^+$ (here $\overline{B}^{\,+}$ denotes the closure of B in $\mu Z \times \mu Z$). Thus, r is an onto uniform isomorphism. This yields (i).

(ii) For each $p_0 \in K_1$ and $\{x_\delta : \delta \in D\}$ a net in X_1 converging to p_0, it holds that $\{g_{(x_\delta, a)} : \delta \in D\}$ converges to $\overline{g}_{(p_0,a)}$ in $C_\infty(X_2, \mu Z)$ for all $a \in X_3$.

Indeed, since $\{\overline{g}_{x_2} : x_2 \in X_2\}$ is relatively compact in $C_\infty(K_1 \times X_3, \mu Z)$, given a vicinity $B \in \mu$, there is a finite subset $\{b_j : 1 \leq j \leq l\} \subset X_2$ such that for each fixed $b \in X_2$ there is an b_j with

$$(\overline{g}(p, b, a), \overline{g}(p, b_j, a)) \in B \tag{I}$$

for all $p \in K_1$ and $a \in X_3$.

On the other hand, the map $g_{(b_j,a)}$ can be extended with continuity to K_1 for all (b_j, a), $1 \leq j \leq l$. Therefore for each fixed $a \in X_3$ and j with $1 \leq j \leq l$, there is $\delta_j \in D$ such that, for $\delta \geq \delta_j$, it holds

$$(\overline{g}(p_0, b_j, a), \overline{g}(x_\delta, b_j, a)) \in B. \tag{II}$$

Let $\delta_0 \in D$ be such that $\delta_0 \geq \delta_j$ for $1 \leq j \leq l$. Applying (I), for every $x_2 \in X_2$ there is an element b_j such that

$$(\overline{g}(p_0, x_2, a), \overline{g}(p_0, b_j, a)) \in B$$

and

$$(\overline{g}(x_\delta, b_j, a), \overline{g}(x_\delta, x_2, a)) \in B$$

for all $\delta \in D$. Appying also (II), we obtain

$$(\overline{g}(p_0, x_2, a), \overline{g}(x_\delta, x_2, a)) \in B^3$$

for all $x_2 \in X_2$ and $\delta \geq \delta_0$. This proves (ii).

Since $g_{(x_\delta,a)}$ can be extended with continuity to $\overline{g}_{(x_\delta,a)} : K_2 \to \mu Z$ for all $\delta \in D$ and $a \in X_3$, it follows from (ii) that $\overline{g}_{(p,a)}$ can also be extended with continuity to K_2 for all $p \in K_2$ and $a \in X_3$. The latter property with (i) and Lemma 3 implies that g can be extended with continuity to $K_1 \times K_2 \times X_3$. This completes the proof. □

We are now ready to establish item B2.

Proposition 4. *Every $\Phi \in \mathcal{L}(\mathcal{U}\ )$, of arity n, can be extended to a continuous map Φ^b from $(b\mathcal{U})^n$ into $b\mathcal{U}$.*

Proof. Let f be an arbitrary element of $AP(\mathcal{U})$ and let $g = f \circ \Phi$. By the definition of $AP(\mathcal{U})$, we know that for all $\overrightarrow{a} \in \prod\{A_i : 1 \leq i \leq n,\ i \neq j\}$, the map $g_{\overrightarrow{a}} = f \circ t^{\Phi}_{\overrightarrow{a}}$ belongs to $AP(\mathcal{U})$ (here $A_i = A$, $1 \leq i \leq n$). As a consequence, $g_{\overrightarrow{a}}$ can be extended with continuity to $b\mathcal{U}$. We also have that the set $\{g_{a_j} : a_j \in A_j\}$ is relatively compact in $C_\infty(\prod\{A_i : 1 \leq i \leq n,\ i \neq j\})$. Applying Lemma 3, we obtain that g can be extended to a continuous function $\overline{g} : (b\mathcal{U})^n \to \mathbb{C}$. Now, for each $\overrightarrow{p} \in (b\mathcal{U})^n$ we define $\Phi^b(\overrightarrow{p})$ to hold the equality $(\overline{f} \circ \Phi^b)(\overrightarrow{p}) = \overline{g}(\overrightarrow{p})$, for all $f \in AP(\mathcal{U})$, where $\overline{f}$ denotes the continuous extension of f to $b\mathcal{U}$. Clearly, the map Φ^b is well defined, continuous and extends Φ, since $AP(\mathcal{U})$ can be identified to the space of all continuous complex-valued functions on $b\mathcal{U}$. This completes the proof. □

Finally, we deal with item B3 in next proposition.

Proposition 5. *The compact structure $b\mathcal{U}$ is the maximal compatible compactification of $\mathcal{U}$.*

Proof. Let (K, ρ) be a compatible compactification of $\mathcal{U}$ where $\rho : A \longrightarrow K$ is the canonical injection. Observe that, since ρ is a homomorphism, we have that $f \circ \rho \in AP(\mathcal{U})$ for all $f \in C(K)$. We define the map $\Gamma : b\mathcal{U} \longrightarrow K$ to hold the equality $f(\Gamma(p)) = p(f \circ \rho)$ for all $f \in C(K)$. The map Γ is clearly continuous by definition. Moreover, if a is an arbitrary element of A, we have that $f(\Gamma \circ \delta)(a)) = f(\Gamma(\delta(a))) = \delta(a)(f \circ \rho) = (f \circ \rho)(a) = f(\rho(a))$, for all $f \in C(K)$. As a consequence, $(\Gamma \circ \delta)(a) = \rho(a)$ for all $a \in A$. This proves that $(K, \rho) \leq_\Gamma (b\mathcal{U}, \delta)$. Hence, the maximallity of $b\mathcal{U}$ has been verified. □

To finish this section we set the proof of Theorem 1.

Proof of Theorem 1: It suffices to combine Lemma 1, Proposition 4 and Proposition 5. □

3. Isometries

In this section we apply the results obtained in the previous part in order to represent linear isometries defined between spaces of almost periodic functions. Here on, $\mathcal{L}$ denotes a set of symbols of constants and functions and $\mathcal{U}$ and $\mathcal{V}$ are two topological structures for $\mathcal{L}$ whose domains are the spaces A and B, respectively.

There is no loss of generality in assuming that the sets of almost periodic functions $AP(\mathcal{U})$ and $AP(\mathcal{V})$ separate the points of A and B, respectively (otherwise, we should take the canonical quotient space, obtained by identifying the points which may not be separated by almost periodic functions). The sets A and B inherite a topology as subspaces of their respective Bohr compactifications, which is called the *Bohr topology.* These topological spaces are denoted by $A^\sharp$ and $B^\sharp$. The topological structures $\mathcal{U}$ and $\mathcal{V}$ equipped with the Bohr topologies are denoted by $\mathcal{U}^\sharp$ and $\mathcal{U}^\sharp$, respectively.

If Φ is a 2-ary function in $\mathcal{L}$, for every $y \in B$, we denote by Φ_y (resp. ${}_y\Phi$) the map defined as $\Phi_y(z) = \Phi(z, y)$ (resp. ${}_y\Phi(z) = \Phi(y, z)$) for all $z \in B$. An isometry T of $AP(\mathcal{U})$ onto $AP(\mathcal{V})$ *commutes with* (Φ) *translations* when there is a map $\lambda : B \longrightarrow A$ such that $(Tf) \circ \Phi_y = T(f \circ \Phi_{\lambda(y)})$ for all $f \in AP(\mathcal{V})$. We say that T is *non-vanishing* when $(Tf)(y) \neq 0$ for all $y \in B$ if and only if $f(x) \neq 0$ for all $x \in A$.

Theorem 2. *Let T be a non-vanishing linear isometry of $AP(\mathcal{U})$ onto $AP(\mathcal{V})$.*

(i) *There exists a continuous map h of $B^\sharp$ onto $A^\sharp$ and a continuous mapping $w : B^\sharp \longrightarrow \mathbb{C}$, $|w| \equiv 1$, such that*

$(Tf)(y) = w(y) \cdot f(h(y))$ *for all $y \in B$ and all $f \in AP(\mathcal{U})$.*

(ii) *If $\Phi \in \mathcal{L}$ is 2-ary, associative, has a neutral element 1 and T commutes with translations, then there is a singleton $b \in B$ such that $h \circ {}_b\Phi$ is a Φ-isomorphism of $\mathcal{V}$ onto $\mathcal{U}$.*

Proof. The statement (i) is consequence of the Banach-Stone theorem and the fact that T preserves non-vanishing functions (see [3] to find an analogous result for topological groups). Thus, only (ii) need to be checked. There is no loss of generality in assuming that $w = 1$ (otherwise, we should replace T by the isometry $\tilde{T}(f) = w^{-1} \cdot T(f)$). Denote by $1_\mathcal{U}$ the neutral element in $\mathcal{U}$ and let b be equal to $h^{-1}(1_\mathcal{U})$. We define the isometry R of $AP(\mathcal{U})$ onto $AP(\mathcal{V})$ by $Rf(y) = Tf(\Phi(b, y))$ for all $y \in B$. It is easy to check that R also commute with the Φ-translations. Indeed,

$$\begin{aligned}((Rf) \circ \Phi_z)(y) &= (Rf)(\Phi(y, z)) = (Tf)(\Phi(b, \Phi(y, z)) = \\ (Tf)(\Phi(\Phi(b, y), z)) &= ((Tf) \circ \Phi_z)(\Phi(b, y)) = \\ T(f \circ \Phi_{\lambda(z)})(\Phi(b, y)) &= R(f \circ \Phi_{\lambda(z)})(y).\end{aligned}$$

Applying (i) to R we obtain a map $k : \mathcal{V} \longrightarrow \mathcal{U}$ such that $Rf = f \circ k$ for all $f \in AP(\mathcal{U})$. Since $AP(\mathcal{V})$ separates the points of B, it is readily seen that $k(y) = h(\Phi(b, y))$ for all $y \in B$. Thus, $k(1_{\mathcal{V}}) = h(\Phi(b, 1_{\mathcal{V}})) = h(b) = 1_{\mathcal{U}}$. On the other hand
$f(k(y)) = Rf(y) = Rf(\Phi(1_{\mathcal{V}})) = ((Rf) \circ \Phi_y)(1_{\mathcal{V}}) = R(f \circ \Phi_{\lambda(y)})(1_{\mathcal{V}}) = (f \circ \Phi_{\lambda(y)})(k(1_{\mathcal{V}})) = (f \circ \Phi_{\lambda(y)})(1_{\mathcal{U}}) = f(\Phi(1_{\mathcal{U}}, \lambda(y)) = f(\lambda(y))$.

Again, this means that $k(y) = \lambda(y)$ for all $y \in B$. Finally
$f(k(\Phi(z, y))) = Rf(\Phi(z, y)) = ((Rf) \circ \Phi_y)(z) = R(f \circ \Phi_{k(y)})(z) = (f \circ \Phi_{k(y)})(k(z)) = f(\Phi(k(z), k(y))$.

That is to say, $k(\Phi(z, y)) = \Phi(k(z), k(y))$ for all $z, y \in B$. This proves that k is a Φ-isomorphism. □

Acknowledgement: We would like to thank the referee of an original draft of this paper for a number of very helpful comments.

References

[1] Ryszard Engelking, *General Topology*. Heldermann Verlag, Berlin. 1989.
[2] J. E. Hart and K. Kunen, *Bohr compactifications of discrete structures,* Fund. Math. **160** (1999), 101-151.
[3] S. Hernández, *Group homomorphisms induced by isometries of spaces of almost periodic functions*, Topology Proc., **27** (2) (2003), 461-477.
[4] P. Holm, *On the Bohr compactification*, Math. Ann., **156** (1964), 34-46.
[5] L.H. Loomis, *An Introduction to Abstract Harmonic Analysis*. Van Nostrand, New York, 1953.
[6] I. Prodanov, *Compact representations of continuous algebraic structures*, Annuaire Univ. Sofia Fac. Math. **60** 1965/66, 139-148 (1967) (in Bulgarian).
[7] I. Prodanov, *Compact representations of continuous algebraic structures*, General Topology and its Relation to Modern Analysis and Algebra II. Proc. of the Second Prague Topology Symposium 1966, Prague, Academia 1967, 290-294 (in Russian).
[8] I. Prodanov, *Minimal compact representations of algebras*, Annuaire Univ. Sofia Fac. Math. Méc **67** (1972/73) 507-542.
[9] I. Prodanov, *Compact representations of continuous algebraic structures*, Dr. of Sciences Dissertation, Sofia University, 1980.
[10] W. Roelcke and S. Dierolf, *Uniform structures in topological groups and their quotients*. McGraw-Hill, New York, 1981.
[11] W. Rudin *Fourier Analysis on Groups*, Wiley, New York. 1990.
[12] Z. Semadeni *Banach Spaces of Continuous Functions*, PWN, Warsaw, 1971.

Universitat Jaume I, Departamento de Matemáticas, Campus de Riu Sec, 12071-Castellón, Spain
E-mail address: hernande@mat.uji.es