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FUNCTIONAL BALANCE, DISCRETE BALANCE, AND BALANCE IN TOPOLOGICAL GROUPS

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Abstract. We consider the question of determining parameters for when topological group balance (the left and right uniformities on the group are equivalent) and functional balance (the classes of left and right uniformly continuous bounded real valued functions coincide) in topological groups are equivalent. Our main result is that a topological group G is balanced iff it is functionally balanced and discretely balanced. A topological group is discretely balanced if every left uniformly discrete subset is right uniformly discrete. This partially answers a question of T.S. Wu. The proof makes use of a theorem derived from the well known theorem of Katetov on extending real valued bounded uniformly continuous functions from a subspace of a uniform space to the whole space and a characterization of uniform separation pointed out by the Author in a previous paper. It is still unknown if the conditions that G is balanced and G is functionally balanced are equivalent.

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1. Introduction

A natural question that arises in topological groups and in uniform spaces is the following: Does the class of uniformly continuous functions on the space completely determine the uniformity on the space? In topological groups the question of interest is whether it is true that if the classes of left and right uniformly continuous real valued bounded functions coincide (the group is functionally balanced), does the left and right uniform structures on the group coincide (the group is balanced)? This question has been suggested by work of Graev [2] and Itzkowitz [4]. The earlier paper of Graev characterized balanced groups as a subgroup of a product of balanced metric groups. (See [3] and [7] for details). During the period 1988 - 92, positive answers to this question were obtained for locally compact groups, and almost metrizable groups by various researchers each using different approaches ([8], [12], [13], [15]). Since then, positive answers have been found for groups that are quasi-k spaces [16], locally connected [11], and most recently k-spaces [14]. The answer to this question in the general case is still unknown. In this paper we show the following:

Theorem. *A topological group G is balanced if and only if every real valued left uniformly continuous function on G is right uniformly continuous and every left uniformly discrete set is right uniformly discrete.*

It is well known that there are two natural uniformities on a T_0 topological group that we now describe. These are the left uniformity $\mathcal{U}_l$ and the right uniformity $\mathcal{U}_r$. Let $\mathcal{U}$ be the neighborhood system of the identity in G.

Definition. (a) Let $L_U = \{(x,y) \in G \times G | x^{-1}y \in U \in \mathcal{U}\}$. Then the *left uniformity* $\mathcal{U}_l$ is generated by the collection of sets $\{L_U | U \in \mathcal{U}\}$ (that is, $\{L_U | U \in \mathcal{U}\}$ is a neighborhood base at $\triangle$ the diagonal of $G \times G$).

(b) Let $R_U = \{(x,y) \in G \times G | yx^{-1} \in U \in \mathcal{U}\}$. Then the *right uniformity* $\mathcal{U}_r$ is generated by the collection of sets $\{R_U | U \in \mathcal{U}\}$.

Definition. A topological group G is *balanced* if given any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $Vx \subset xU$ for all $x \in G$.

Notes. (1) If G is balanced consider the entourage $L_U = \{(x,y) \in G \times G | x^{-1}y \in U \in \mathcal{U}\}$ of $\mathcal{U}_l$. From the definition there is $V \in \mathcal{U}$ such that $Vx \subset xU$ for all $x \in G$. Then $R_V = \{(x,y) \in G \times G | y \in Vx\} \subset \{(x,y) \in G \times G | y \in xU\} = L_U$. This means that $L_U \in \mathcal{U}_r$ since uniformities are closed under supersets, so that $\mathcal{U}_l \subset \mathcal{U}_r$. Since G is balanced we can also write that for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $Vx^{-1} \subset x^{-1}U$, for all $x \in G$. Thus $xV \subset Ux$ for all $x \in G$ so that $L_V \subset R_U$ and $\mathcal{U}_r \subset \mathcal{U}_l$. Therefore $\mathcal{U}_l = \mathcal{U}_r$, so the two uniformities are identical.

(2) The condition $Vx \subset xU$, for all $x \in G$, implies $V \subset xUx^{-1}$, for all $x \in G$, so that $V \subset W = \bigcap_{x \in G} xUx^{-1}$. This means that W is a neighborhood of e. Furthermore, since $yWy^{-1} = y(\bigcap_{x \in G} xUx^{-1})y^{-1} = \bigcap_{x \in G} yxUx^{-1}y^{-1} = \bigcap_{x \in G} yxU(yx)^{-1} = \bigcap_{z \in G} zUz^{-1} = W$, it follows that the collection of invariant neighborhoods $\{W | W = xWx^{-1}$, for all $x \in G\}$ is a base for the neighborhood system at the identity in a balanced group G. This is why classically, balanced groups were called SIN groups (SIN = small invariant neighborhoods).

Definition. Let f be a real valued function on a topological group G.

(1) f is *left uniformly continuous* if for each $\epsilon > 0$ there is a neighborhood W of e such that $|f(x) - f(y)| < \epsilon$ whenever $x^{-1}y \in W$ (or $y \in xW$).

(2) f is *right uniformly continuous* if for each $\epsilon > 0$ there is a neighborhood W of e such that $|f(x) - f(y)| < \epsilon$ whenever $yx^{-1} \in W$ (or $y \in Wx$).

Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be uniform spaces.

Definition. The function $f : X \longrightarrow Y$ is *uniformly continuous* if for each $V \in \mathcal{V}$ the set $U = \{(x,y) | (f(x), f(y)) \in V\}$ is a member of $\mathcal{U}$.

Definition. A topological group G is *functionally balanced* if the class of bounded left uniformly continuous real valued functions on G coincides with the class of bounded right uniformly continuous functions.

Note that it is not difficult to see that a group G is functionally balanced if and only if every real valued left uniformly continuous function is right uniformly continuous.

For the sequel we establish the following conventions:

(1) In a uniform space the entourages of the diagonal $\triangle$ that will be used are symmetric (that is, if $U \in \mathcal{U}$ then $U = U^{-1}$, where $U^{-1} = \{(y,x)|(x,y) \in U\}$).

(2) If G is a topological group, all the neighborhoods U of the identity e that we use will be assumed symmetric (that is, $U = U^{-1}$, where $U^{-1} = \{x^{-1}|x \in U\}$).

(3) All topological groups are T_0 so they are Hausdorff and completely regular.

2. Functional separation in uniform spaces

Let $(X,\mathcal{U})$ be a uniform space, let $A \subset X$, and let f be a real valued function defined on the topological space induced on X by the uniformity $\mathcal{U}$.

Definition. The subsets A and B of X are *uniformly separated* if there is $U \in \mathcal{U}$ such that $U[A] \cap U[B] = \emptyset$. (Here we are using the standard notation introduced in Kelley [10]: $U[A] = \{y|y \in U[x], x \in A\}$ and $U[x] = \{y|(x,y) \in U \in \mathcal{U}\}$.)

It is easy to see that sets A and B in X are uniformly separated if and only if there is a real valued uniformly continuous function $f : X \longrightarrow [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. It is also easy to see that two sets A and B are uniformly separated in a uniform subspace Y of X if and only if they are uniformly separated in X (see [5]). Furthermore, if $V \in \mathcal{U}$ and $A \subset X$ then A and $(V[A])^c$ are uniformly separated.

Notation. (1) If $A \subset X$ then A^c is the complement of A in X.

(2) If $(X,\mathcal{U})$ is a uniform space, and $A \subset X$ then $\mathcal{U}_A$ is the uniformity induced on A by $\mathcal{U}$.

We now make use of the well-known theorem of Katetov [9]. Proofs of this theorem can be found in Gantner [1] and Itzkowitz [5].

Theorem 2.1. *Let $(X,\mathcal{U})$ be a uniform space, and let $(A,\mathcal{U}_A)$ be a uniform subspace. Then every bounded uniformly continuous real valued function f on A extends to a bounded uniformly continuous real valued function on X without increase in norm.*

As a corollary of Katetov's theorem we have:

Corollary 2.2. *Let $(X,\mathcal{U})$ be a uniform space, and let $W \in \mathcal{U}$. Let $A \subset X$, $W[A] \neq X$, and let f be a bounded uniformly continuous real valued function defined on A. Then f has a uniformly continuous extension $\bar{f}$ on all of X such that $\bar{f}(x) = 0$ for $x \in (W[A])^c$ and $\bar{f}$ is of the same sup norm as f on A.*

Proof. Define a function f^* on $A \cup (W[A])^c$ by the condition that f^* on A coincides with f and on $W[A]^c$ it is identically zero. Then f^* is clearly uniformly continuous and by Katetov's theorem it admits a uniformly continuous extension $\bar{f}$ to all of X with the same sup norm. □

3. Extension of uniformly continuous real valued functions and balance on a topological group

Let $(X,\mathcal{U})$ be a uniform space.

Definition. $A \subset X$ is *uniformly discrete* if there is an entourage U of $\triangle$ such that $U[a] \cap U[b] = \emptyset$ if $a, b \in A$ and $a \neq b$.

In the case of a topological group G which has both left and right uniformities this concept admits the following two forms:

Definition. A set $A \subset G$ is *left uniformly discrete* if there is a neighborhood U of e satisfying $aU \cap bU = \emptyset$, if $a \neq b$ and $a, b \in A$. In this case we say that A is left uniformly discrete with respect to U and U is a left separating neighborhood of A. A is right uniformly discrete if there is a neighborhood W of e satisfying $Wa \cap Wb = \emptyset$, if $a \neq b$ and $a, b \in A$, and we say A is right uniformly discrete with respect to W (also W is a right separating neighborhood of A).

Definition. G is *discretely balanced* if every left uniformly discrete set is right uniformly discrete.

Note. If G is discretely balanced then every right uniformly discrete set is left uniformly discrete. To see this, let A be right uniformly discrete. Then there is a neighborhood W of the identity e such that $Wa \cap Wb = \emptyset$, if $a \neq b$ and $a, b \in A$. Since inversion in a topological group is a homeomorphism and W is symmetric, $a^{-1}W \cap b^{-1}W = \emptyset$, if $a \neq b$ and $a, b \in A$. This means A^{-1} is left uniformly discrete and therefore right uniformly discrete. This in turn means that A is left uniformly discrete.

3.1. **General questions.** The following are natural questions that are suggested by previous research.

(1) If G is discretely balanced, is G balanced? (Proposed by T.S. Wu.)
(2) If G is functionally balanced, is it balanced?
(3) Is there a specific class P of topological groups where one can face this problem in order to resolve it for all topological groups?

Seemingly, Graev's theorem mentioned in the introduction (See [3, 8.17] and [7] for details) may be helpful in this. (Suggested by D. Dikranjan.)

Remark 3.2. A proof appears in [6] that if G is a T_0 group in which every left uniformly discrete set is strongly neutral then G is balanced. The set A is *strongly neutral* if it is left uniformly discrete with respect to a neighborhood V of e and if there is a neighborhood U of e such that $Ua \subset aV$ for all $a \in A$. The condition of strong neutrality is clearly a stronger condition than that of discrete balance. The proof in [11, Theorem 2] suggested the result mentioned in the beginning of this remark. Since the converse of each of the statements (1) and (2) in 3.1 and the opening statement of this note are easy to show, it follows that the statements "every left uniformly discrete set in G is strongly neutral" and "G is balanced" are equivalent. This condition involving strong neutrality motivated the question of Wu.

The following result appeared in [11] and will be of use here.

Lemma 3.3. *Let A be a left uniformly discrete set with respect to the symmetric neighborhood U of e in G and suppose V is a symmetric neighborhood of e satisfying $V^2 \subset U$.*

(1) Then V is a separating neighborhood of A.
(2) For each x in G, xV can intersect at most one aV, where $a \in A$.

It is easy to see that if f and g are bounded uniformly continuous real valued functions on a uniform space $(X, \mathcal{U})$ then $f \wedge g$ is uniformly continuous on $(X, \mathcal{U})$. Thus in the case of a topological group this observation implies that if f and g are left uniformly continuous then $f \wedge g$ is left uniformly continuous (and similarly for right uniform continuity).

Theorem 3.4. *Let G be functionally balanced and discretely balanced. Then G is balanced.*

Proof. Let A be left uniformly discrete so that A is also right uniformly discrete. Then there is a symmetric neighborhood U of e that is both left and right separating on A. Let V and W be symmetric neighborhoods of e such that $W^3 \subset V$ and $V^2 \subset U$. Without loss of generality we may suppose $e \in A$. Note that both V and W are left and right separating neighborhoods of A, and for x in G, xV can meet at most one aV, for $a \in A$ and Vx can meet at most one Va for $a \in A$.

Since G is completely regular there is a left uniformly continuous real valued function f_e satisfying $f_e(e) = 1$, $f_e(W^c) = 0$, and $0 \leq f_e \leq 1$ on G. For each $a \neq e$ in A define $f_a(x) = f_e(a^{-1}x)$ so that $f_a(a) = 1$, $f_a((aW)^c) = 0$, and $0 \leq f_a \leq 1$ on G. Note that each f_a is left uniformly continuous and so right uniformly continuous on G.

Now define $g_a(x) = f_e(xa^{-1})$. Then $g_a(a) = 1$, $g_a((Wa)^c) = 0$, and $0 \leq g_a \leq 1$ on G. Since f_e is right uniformly continuous it follows that each g_a is right uniformly continuous on G and therefore left uniformly continous on G.

As noted previously, $f_a \wedge g_a$ is both left and right uniformly continuous on G, and $\text{suppt}(f_a \wedge g_a) \subset aW \cap Wa$. Furthermore if $a \neq b$ then $aW \cap Wa$ is left and right uniformly separated from $bW \cap Wb$ by W. To see this, note that $(bW \cap Wb)W \subset bW^2 \subset bW^3 \subset bV$ and $(aW \cap Wa)W \subset aW^2 \subset aW^3 \subset aV$. Since $bV \cap aV = \emptyset$, left uniform separation by W follows. Right separation by W follows similarly by multiplying by W on the left.

Now define $k(x) = \Sigma_{a \in A} f_a \wedge g_a(x)$. Note that $k|_{aV} = f_a \wedge g_a$ for $a \in A$. This is because $\text{suppt}(f_a \wedge g_a) \subset aW \cap Wa$ for $a \in A$, and $f_b(x) = 0$ if $x \in aV$ and $b \neq a$. Therefore k is left uniform continuous on aV. Furthermore, $k(x) = 0$, if $x \in aV \setminus (aW \cap Wa)$. It is easy to check that $\text{suppt}(k) \subset \bigcup_{a \in A}(aW \cap Wa)$, $k(x) = 0$ if $x \in \bigcup_{a \in A} aV \setminus (\bigcup_{a \in A}(aW \cap Wa) = \bigcup_{a \in A}(aV \;\; aW \cap Wa)$, and k is left uniformly continous on $\bigcup_{a \in A} aV$. Next we note that $(\bigcup_{a \in A}(aW \cap Wa))W \subset (\bigcup_{a \in A} aW)W = \bigcup_{a \in A} aW^2 \subset \bigcup_{a \in A} aV$, so that $\bigcup_{a \in A}(aW \cap Wa)$ is left uniformly separated from $(\bigcup_{a \in A} aV)^c = \bigcap_{a \in A} aV^c$. Therefore by Theorem 2.1, $k|_{\cup aV}$ has a left (hence right) uniformly continuous extension $\bar{k}$ to all of G such that $\bar{k} = 0$ on $\bigcup_{a\in A}(aW \cap Wa)^c$. But k already has this property so $k = \bar{k}$, Thus k is left uniformly continuous on G and therefore right uniformly continuous on G.

Finally, since k is right uniformly continuous, there is a symmetric neighborhood W_1 of e such that $xy^{-1} \in W$ implies that $|k(x) - k(y)| < 1$. Without loss of generality we may assume $W_1 \subset W$. Then if $x \in W_1a$, $a \in A$, it follows that $k(x) > 0$. Since $x \in W_1a \subset Wa$, we cannot have $x \in bW$ where $b \neq a$ (since then $k(x) = 0$). Therefore $x \in aW$. This is true for each $a \in A$ (that is, $W_1a \subset aW$). Therefore A is strongly neutral and by the theorem mentioned in 3.2, G is balanced. □

Definition. The set A in G is *symmetrically discrete* if A is both left and right uniformly discrete and there is a neighborhood U of e such that $Ua \cap Ub = aU \cap bU = \emptyset$ for $a, b \in A$ and $a \neq b$.

As a corollary to the method of proof we have:

Corollary 3.5. *If G is functionally balanced and A is symmetrically discrete then A is strongly neutral.*

Theorem 3.4 implies the following characterization of balance, which can be written in one of the following two forms:

Theorem 3.6(A). *If G is a T_0 topological group then the following are equivalent:*

(a) G is balanced.
(b) G is functionally balanced and discretely balanced.

Theorem 3.6(B). *If G is a discretely balanced T_0 group then G is balanced iff G is functionally balanced.*

Note 3.7. These results still do not answer the question posed by Ta Sun Wu on whether balance and discretely balanced are equivalent. Wu indicated to the author a proof that the following is true. Every strongly discretely balanced T_0 group G is balanced. Here a group G is strongly discretely balanced if each left uniformly discrete set A is strongly uniformly discrete. The set A is strongly uniformly discrete if there is a neighborhood U of e such that for $a \neq b$ and $a, b \in A$, $UaU \cap UbU = \emptyset$. In view of the main results, Wu's question 3.1 (1) becomes: Does discrete balance imply functional balance?

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References

[1] Gantner, T.E., *Some corollaries to the metrization lemma*, MAA Math Monthly, **76** (1969), 45-47.

[2] Graev, M.I., *Theory of topological groups I. Norms and metrics on groups. Complete groups. Free topological groups*, Uspehi Mat. Nauk, N.S. 5, vyp.2 (36) (1950), 3-56.

[3] Hewitt, E. and Ross, K.A., Abstract Harmonic Analysis I, Springer-Verlag Berlin, (1963).

[4] Itzkowitz, G.L., *Continuous measures, Baire category, and uniform continuity in topological groups*, Pac. J. of Math. **54** (1974), 115-125.

[5] Itzkowitz, G., *Uniform separation and a theorem of Katetov*, Annals N.Y. Acad.of Sci. **767** (1995), 92-96.

[6] Itzkowitz, G., *On balanced topological groups*, Topology Proceedings **23** (1998), 219-233.

[7] Itzkowitz, G., *Projective limits and balanced topological groups*, Top. and its Appl. **110** (2001), 163-183.

[8] Itzkowitz, G.L., Rothman, S., Strassberg H., and Wu, T.S., *Characterization of equivalent uniformities in topological groups*, Top. and Its Appl. **47** (1992), 9-34.

[9] Katetov, M., *On real valued functions in topological spaces*, Fund. Math. **38** (1951), 85-91.

[10] Kelley, J.L., General Topology, D.Van Nostrand (1957).

[11] Megrelishvili, (Levy) M., Nickolas, P., and Pestov, V., *Uniformities and uniformly continuous functions on locally connected groups*, Bull. Austral. Math. Soc. **56** (1997), 279-283.

[12] Milnes, P., *Uniformity and uniformly continuous functions for locally compact groups*, Proc. AMS **109** (1990), 567-570.

[13] Pestov, V.G., *A test of balance of a locally compact group*, Ukrainian Math. J. **40** (1988), 281-284.

[14] Previts,W.H. and Wu,T.S., *Notes on balanced groups*, to appear.

[15] Protasov, I.V., *Functionally balanced groups*, Math. Notes **49** (1991), 614-616.

[16] Troallic, J.P., *Sequential criteria for equicontinuity and uniformities on topological groups*, Top. and its Appl. **68** (1996), 83-95.

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