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THE URYSOHN IDENTITY FOR CLOSED SUBSETS OF SOME NONMETRIZABLE MANIFOLDS

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ABSTRACT. Let Y be a closed subspace of $M \times L$ where M is a compact manifold and L is the long line. Assuming Martin's axiom and the negation of continuum hypothesis, we prove that $\operatorname{ind} Y = \dim Y = \operatorname{Ind} Y$.

1. INTRODUCTION

There exist examples of nonmetrizable manifolds with noncoinciding dimensions [4, 5, 7]. All such examples were constructed under additional set-theoretical assumptions. The following question was posed by Fedorchuk [6, Question 3.17].

Question 1.1. *Is the Urysohn identity valid for a closed subset Y of $M \times L$, where M is a compact manifold and L is the long line?*

Here by the Urysohn identity we mean the identity $\operatorname{ind} Y = \dim Y = \operatorname{Ind} Y$. In this note we answer the above question in positive under Martin's axiom and the negation of continuum hypothesis. Earlier partial results in this direction have been obtained in [8]. Note also that under the continuum hypothesis there exists an example of closed subspace $Y \subseteq X = I \times L$, where $I = [0, 1]$, such that $1 = \operatorname{ind} Y < \operatorname{Ind} Y = 2$ (Smirnov's example, [9]). Thus a positive answer to the Question 1.1 cannot be obtained without additional set-theoretical assumptions.

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2. Preliminaries

Let X be a metrizable compactum. By $C_n(X)$ we will denote the space of all continuous maps from X to the n-dimensional cube I^n, $n \geq 1$, endowed with the topology of uniform convergence. The following statement is evident.

Proposition 2.1. *Let F_0 and F_1 be disjoint closed subspaces of X. Then $O = \{f \in C_n(X) \mid f(F_0) \cap f(F_1) = \varnothing\}$ is a non-empty open subspace of $C_n(X)$, $n \geq 1$.*

Let F be a closed subspace of X. For any subspace A of $C_n(F)$ by $\widetilde{A}$ we denote the set of all extensions of mappings from A over X. The following fact is also evident.

Proposition 2.2. *If A is open in $C_n(F)$, then $\widetilde{A}$ is open in $C_n(X)$. If A is dense in $C_n(F)$, then $\widetilde{A}$ is dense in $C_n(X)$.*

Using Proposition 2.2 it is easy to verify the following statement.

Proposition 2.3. *If A is a dense G_δ subspace of $C_n(F)$, then $\widetilde{A}$ is a dense G_δ subspace of $C_n(X)$.*

Proposition 2.4. *The set $F = \{f \in C_n(X) \mid \dim f(X) = n\}$ is F_σ subspace of $C_n(X)$ for any metrizable compactum X.*

Proof. Construct in I^n a countable collection $Q = \{q_k \mid k \in \omega\}$ of n-dimen- sional cubes with diameters converging to zero and such that the centers of these cubes form a dense set in I^n. Let $F_k = \{f \in C_n(X) \mid q_k \subseteq f(X)\}$. Then F_k is closed in $C_n(X)$. It is easy to see that $F = \cup\{F_k \mid k \in \omega\}$. □

It follows from [1, p. 247, Theorem 2] that for a compactum X with $\dim X = n \leq m$ maps from X into I^m with images contained in n-dimensional polyhedra form a dense subspace of $C_m(X)$. This fact and Proposition 2.4 imply the following observation.

Proposition 2.5. *Let X be a metrizable compactum and $\dim X \leq n-1$. Then $G = \{f \in C_n(X) \mid \dim f(X) \leq n-1\}$ is a dense G_δ subspace of $C_n(X)$.*

As usual ¬CH will denote the negation of continuum-hypotheses and $\mathfrak{c}$ will denote the cardinality of continuum. By MA we denote

Martin's axiom. We will use the following topological version of MA: no compact Hausdorff space with the countable chain condition (in particular, metrizable) can be the union of less than $\mathfrak{c}$ nowhere dense subsets. Equivalently, any family of less than continuum of dense open subsets of X has a nonempty intersection. See [11] for comprehensive discussion of Martin's axiom.

Proposition 2.6 (MA)**.** *Let M be a completely metrizable separable nonempty space. Then any collection $\{O_\alpha \mid \alpha \in \mathcal{A}\}$, $|\mathcal{A}| < \mathfrak{c}$, of open dense subsets of M has a nonempty intersection.*

Proof. Embed M into the Hilbert cube Q. Let $Z = \overline{M}$ be the closure of M in Q. Note that M, being completely metrizable, is a G_δ subspace in Z. Therefore $M = \cap\{U_n \mid n \in \omega\}$, where U_n is open and dense in Z for each n. For each $\alpha \in \mathcal{A}$, let O'_α be an open subspace of Z such that $O'_\alpha \cap M = O_\alpha$. Then O'_α is dense in Z. Let $P = (\cap\{O'_\alpha \mid \alpha \in \mathcal{A}\}) \cap (\cap\{U_n \mid n \in \omega\})$. Since Z is a metrizable compactum, MA implies that P is nonempty. The choice of O'_α implies $P = \cap\{O_\alpha \mid \alpha \in \mathcal{A}\}$. □

The following statement is an easy corollary of Proposition 2.6.

Proposition 2.7 (MA)**.** *Any family of less than continuum of dense G_δ subsets of a completely metrizable separable nonempty space has a nonempty intersection.*

3. Main Lemma

For a mapping $f\colon X \to Y$ we let $\dim f = \sup\{\dim f^{-1}(y) \mid y \in Y\}$.

Lemma 3.1 (MA)**.** *Let X be a metrizable compactum with $\dim X = n \geq 1$ and $\mathcal{F} = \{F_\alpha \mid \alpha \in \mathcal{A}\}$, $|\mathcal{A}| < \mathfrak{c}$, be a family of its closed subspaces such that $\dim F_\alpha \leq n-1$ for each $\alpha \in \mathcal{A}$. Then for any two closed disjoint subspaces A and B of X there exists a partition C between A and B such that $\dim C \leq n-1$ and $\dim(C \cap F_\alpha) \leq n-2$ for each $\alpha \in \mathcal{A}$.*

Proof. Let $O = \{f \in C_n(X) \mid f(A) \cap f(B) = \varnothing\}$, $G_\alpha = \{f \in C_n(F_\alpha) \mid \dim f(F_\alpha) \leq n-1\}$, $G = \{f \in C_n(X) \mid \dim f = 0\}$. Proposition 2.1 and completeness of $C_n(X)$ imply that O is nonempty, separable and completely metrizable. Propositions 2.3 and 2.5 imply that $\widetilde{G}_\alpha$, for each $\alpha \in \mathcal{A}$, is a dense G_δ subset in $C_n(X)$.

It follows from [3, p.114, Exercise 1.12.H] that G is also a dense G_δ subspace of $C_n(X)$. Let $G'_\alpha = O \cap \widetilde{G}_\alpha$ and $G' = O \cap G$. Then G' and G'_α, for each α, are dense G_δ subspaces of O. Apply Proposition 2.7 to find $f \in G' \cap (\cap\{G'_\alpha \mid \alpha \in \mathcal{A}\})$. Note that f possesses the following properties:

(1) $f(A) \cap f(B) = \varnothing$
(2) $\dim f = 0$
(3) $\dim f(F_\alpha) \leq n-1$ for every $\alpha \in \mathcal{A}$

Let $F'_\alpha = f(F_\alpha), \alpha \in \mathcal{A}$, and $\mathcal{F}' = \{F'_\alpha \mid \alpha \in \mathcal{A}\}$. Then $\mathcal{F}'$ is a collection of closed at most $(n-1)$-dimensional and therefore nowhere dense subspaces of I^n. Let $D = \cup \mathcal{F}'$. It follows from MA that $\dim D \leq n-1$. Indeed, suppose the opposite, i.e. $\dim D = n$. Then D contains n-dimensional closed ball B. Note that $B = \cup\{(F'_\alpha \cap B) \mid \alpha \in \mathcal{A}\}$. Applying MA we find α such that $F'_\alpha \cap B$ is not nowhere dense in B. Contradiction.

Since $\dim D \leq n-1$ and $f(A) \cap f(B) = \varnothing$ by property (1) of f, there exists a partition P between $f(A)$ and $f(B)$ in I^n such that $\dim(P \cap D) \leq n-2$, and in particular $\dim(P \cap F'_\alpha) \leq n-2$ for each α. Let $P' \subseteq P$ be a partition between $f(A)$ and $f(B)$ which does not contain interior points (relative to I^n). Then $\dim P' = n-1$. Let $C = f^{-1}(P')$. Hurewicz theorem [3, p. 200, Theorem 3.3.10] and property (2) of f imply that C is a partition between A and B in X which possesses all desired properties. □

4. Inductive Dimension

By L we denote the long line, i.e. the set of points of the form $\alpha + t$, where $\alpha < \omega_1$ is a countable ordinal and $t \in [0,1)$. This set is naturally ordered and is endowed with order topology. Note that L is a non-metrizable countably compact 1-manifold.

Let K be a metrizable compactum and $X = K \times L$. Let $p\colon X \to K$ be the projection. For each $\alpha < \omega_1$ we let $L_\alpha = [\alpha, \omega_1)$, and for any subspace S of X we let $S_\alpha = S \cap (K \times L_\alpha)$.

Theorem 4.1. *Let Y be a closed subspace of $X = K \times L$, where K is a metrizable compactum and L is the long line. Then* $\operatorname{ind} Y = 0$ *implies* $\operatorname{Ind} Y = 0$.

Proof. Let A and B be disjoint closed subspaces of Y. Since L is countably compact and K is metrizable, projection p is closed [2, p. 203, Theorem 3.10.7]. Therefore since K is hereditarily Lindelöf we can find $\beta < \omega_1$ such that $p(Y_\alpha) = p(Y_\beta)$, $p(A_\alpha) = p(A_\beta)$, and $p(B_\alpha) = p(B_\beta)$ for each $\alpha \geq \beta$. In particular $p(A_\beta) \cap p(B_\beta) = \varnothing$. Indeed, suppose the opposite and let $x \in p(A_\beta) \cap p(B_\beta)$. Then there exist two cofinal sequences $\{x_\alpha \in A \mid \alpha < \omega_1\}$ and $\{y_\alpha \in B \mid \alpha < \omega_1\}$ such that x_α and y_α are in $\{x\} \times L$ for each α. Since A and B are closed in Y we can assume that the sets $\{x_\alpha \mid \alpha < \omega_1\}$ and $\{y_\alpha \mid \alpha < \omega_1\}$ are closed in $\{x\} \times L$. Since any two closed cofinal subsets of L have a non-empty intersection we obtain $A \cap B \neq \varnothing$, contradiction.

Let $Y' = p(Y_\beta)$, $A' = p(A_\beta)$, and $B' = p(B_\beta)$. Then A' and B' are disjoint, as shown above. Since projection p is closed Y', A' and B' are closed subspaces of K. Note also that $Y_\beta \subseteq Y' \times L$.

We will show that $\operatorname{ind} Y' = 0$. Suppose the opposite, i.e. $\operatorname{ind} Y' > 0$. Let $Q = \{q_i \mid i \in \omega\}$ be a countable dense subset of Y'. By the definition of Y' for each $i \in \omega$ the set $\{q_i\} \times L_\beta$ contains a closed cofinal set of points from Y. The second coordinates of these points form a countably many closed cofinal subsets of L, which have a non-empty intersection. Therefore there exists $\gamma < \omega_1$ such that $\gamma > \beta$ and $\{(q_i, \gamma) \mid i \in \omega\} \subseteq Y \cap (K \times \{\gamma\})$. Since Y is closed the last inclusion implies $Y' \times \{\gamma\} = Y \cap (K \times \{\gamma\})$ and we obtain a contradiction with $\operatorname{ind} Y = 0$.

Thus $\operatorname{ind} Y' = 0$. Since Y' is a metrizable compactum this implies $\operatorname{Ind} Y' = 0$ and therefore there exist closed disjoint sets Z_A and Z_B such that $A' \subset Z_A$, $B' \subset Z_B$, and $Y' = Z_A \cup Z_B$. Put $T = Y \cap (K \times [0, \beta])$. Since T is metrizable compactum and $\operatorname{ind} T = 0$ we have $\operatorname{Ind} T = 0$. Hence we can find closed disjoint sets S_A and S_B such that $A \cap T \subseteq S_A$, $B \cap T \subseteq S_B$ and $T = S_A \cup S_B$. We can also assume that $S_A \cap (Z_B \times \{\beta\}) = \varnothing$ and $S_B \cap (Z_A \times \{\beta\}) = \varnothing$. Let $\Phi_A = (S_A \cup (Z_A \times L_\beta)) \cap Y$ and $\Phi_B = (S_B \cup (Z_B \times L_\beta)) \cap Y$. One can easily check that the equality $Y = \Phi_A \cup \Phi_B$ provides an empty partition between A and B in Y. □

Theorem 4.2 (MA+¬CH)**.** *Let Y be a closed subspace of $X = K \times L$, where K is a metrizable compactum and L is the long line, such that* $\operatorname{ind} Y < \infty$. *Then* $\operatorname{ind} Y = \dim Y = \operatorname{Ind} Y$.

Proof. Since Y is a normal space $\dim Y \leq \operatorname{Ind} Y$. Since Y is locally second-countable $\operatorname{ind} Y \leq \dim Y$. Therefore it is sufficient to show that $\operatorname{ind} Y = \operatorname{Ind} Y$. We will prove this last equality by induction on $n = \operatorname{ind} Y$. Theorem 4.1 shows that the equality is satisfied for $n = 0$.

Let A and B be two disjoint closed subsets of Y. As in the proof of Theorem 4.1 we find $\beta < \omega_1$ such that $p(Y_\alpha) = p(Y_\beta)$, $p(A_\alpha) = p(A_\beta)$, and $p(B_\alpha) = p(B_\beta)$ for each $\alpha \geq \beta$. Let $Y' = p(Y_\beta)$, $A' = p(A_\beta)$, and $B' = p(B_\beta)$. Again, as in the proof of Theorem 4.1 we conclude that Y', A' and B' are closed subsets of K, $A' \cap B' = \varnothing$, and that $\operatorname{ind} Y' \leq n$. Put $T = Y \cap (K \times [0, \beta])$ and consider two cases.

Case 1: $\operatorname{ind} Y' < n$.
Since Y' is a metrizable compactum this implies $\operatorname{Ind} Y < n$. Therefore there exists a partition C between A' and B' in Y' such that $\operatorname{ind} C \leq n-2$. Let C' be a partition between $A \cap T$ and $B \cap T$ in $Y \cap T$ extending $C \times \{\beta\}$, such that $\operatorname{ind} C' \leq n-1$. Put $P = (C' \cup (C \times L_\beta)) \cap Y$. Then P is a partition between A and B in Y such that $\operatorname{ind} P \leq n-1$. By inductive assumption we conclude that $\operatorname{Ind} P \leq n-1$

Case 2: $\operatorname{ind} Y' = n$.
For any two positive integers m and $k \leq m$, and for each $\alpha < \omega_1$ let $\Delta(\alpha, k, m) = [\alpha + (k-1)/m, \alpha + k/m]$. Using some enumeration we can assume that $\{\Delta(\alpha, k, m) \mid 0 < k \leq m, m \in \omega, \alpha < \omega_1\} = \{\Delta_\gamma \mid \gamma < \omega_1\}$. For each $\alpha < \omega_1$ we let

$$F_\alpha = \{x \in Y' \mid \{x\} \times \Delta_\alpha \subseteq Y_\beta\}.$$

It is easy to verify that each set F_α is closed in Y' and $\dim F_\alpha \leq n-1$. By $\neg CH$ we have $\omega_1 < \mathfrak{c}$ and therefore we can apply Lemma 3.1 to the spaces Y', A' and B', and to the family $\mathcal{F} = \{F_\alpha \mid \alpha < \omega_1\}$. Let C be a partition between A' and B' in Y' which satisfies conditions of Lemma 3.1. Note that dimension ind is defined locally. Therefore we can apply Theorem 1 from [10] to sets of the form $C \times \Delta$, where $\Delta \subset L_\beta$ is homeomorphic to the usual closed interval, to obtain $\operatorname{ind}(C \times L_\beta) \cap Y \leq n-1$. Let C' be a partition between $A \cap T$ and $B \cap T$ in $Y \cap T$ extending $C \times \{\beta\}$, such that $\operatorname{ind} C' \leq n-1$. Put $P = (C' \cup (C \times L_\beta)) \cap Y$. Then P is a partition between A and B in Y such that $\operatorname{ind} P \leq n-1$. By inductive assumption we conclude that $\operatorname{Ind} P \leq n-1$. □

The following theorem is a direct consequence of Theorem 4.2

Theorem 4.3 (MA+¬CH). *Let M be a compact manifold. Then for any closed subspace Y of $M \times L$, where L is the long line, the Urysohn identity* $\operatorname{ind} Y = \dim Y = \operatorname{Ind} Y$ *holds.*

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References

[1] P.S. Alexandrov, B.A. Pasynkov, *Introduction to Dimension Theory*, Nauka, Moscow, 1973.
[2] R. Engelking, *General Topology*, Heldermann, 1989.
[3] ———, *Theory of dimensions: Finite and infinite*, Heldermann, 1995.
[4] V.V. Fedorchuk, *A differentiable manifold with noncoinciding dimensions*, Topology Appl. **54** (1993), 221–239.
[5] ———, *A differentiable manifold with noncoinciding dimensions under the continuum-hypothesis*, Mat. Sb. **186** (1995), no. 1, 149–160 (Russian); English translation in Sb. Math. **186** (1995), no. 1, 151–162.
[6] ———, *Urysohn identity and the dimension of manifolds*, Uspekhi Mat. Nauk **53** (1998), no. 5 (323), 73–114 (Russian); English translation in Russian Math. Surveys **53** (1998), no. 5, 937–974.
[7] V.V. Fedorchuk, V.V. Filippov, *A manifold with noncoinciding dimensions*, Mat. Sb. **183** (1992), no. 9, 29–44 (Russian); English translation in Russian Acad. Sci. Sb. Math. **77** (1994), no. 1, 25–36.
[8] A.V. Karasev, *On inductive dimension of closed subsets of some nonmetrizable manifolds*, Vestnik Mosk. Univ. (1997), no. 5, 11–14 (Russian); English translation in Moscow Univ. Math. Bull. **52** (1997), no. 5, 11 - 13.
[9] Yu.M. Smirnov, *Some relations in dimension theory*, Mat. Sb. **29** (1951), 157–172 (Russian).
[10] Y. Sternfeld, *Dimension of subsets of product spaces*, Proc. Amer. Math. Soc. **82** (1981), no. 3, 452–454.
[11] W. Weiss, *Versions of Martin's Axiom*, in: K. Kunen, J. Vaughan, ed., Handbook of set-theoretic topology, North-Holland, 1984, 827-886.

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