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TOPOLOGICAL ENTROPY OF MONOTONE MAPS AND CONFLUENT MAPS ON REGULAR CURVES

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Abstract. In [10], G. T. Seidler proved that the topological entropy of every homeomorphism on a regular curve is zero. In [4], L. S. Efremova and E. N. Markhrova proved that the topological entropy of every monotone map on a dendrite which satisfies some special condition is zero. In [3], N. Chinen proved that the topological entropy of every monotone map on any dendrite is zero. In this paper, we generalize these results. In fact, we investigate the topological entropy of confluent maps on regular curves. As å corollary, we show that the topological entropy of every monotone map on any regular curve is zero.

1. Introduction

Recently a wide literature on the dynamical properties of some special maps on some 1-dimensional continua has developed. Examples of continua studied include graphs, dendrites, fractal sets and regular curves (see [3], [4], [6], [7] and [10]). In [10], G. T. Seidler proved that the topological entropy of every homeomorphism on a regular curve is zero. In [4], L. S. Efremova and E. N. Markhrova proved that the topological entropy of every monotone map on a dendrite which satisfies some special condition is zero. In [3], N. Chinen proved that the topological entropy of every monotone map on any dendrite is zero.

This paper is devoted to the study of monotone and confluent dynamical systems defined on regular curves. We investigate the

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topological entropy of confluent maps on regular curves. As a corollary, we show that the topological entropy of every monotone map on any regular curve is zero. Note that this result contains all results above. The notions of monotone maps and confluent maps are important in continuum theory (see [8]).

All spaces considered in this paper are assumed to be separable metric spaces. *Maps* are continuous functions. By a *compactum* we mean a compact metric space. A *continuum* is a nonempty connected compactum. An onto map $f : X \to Y$ of compacta is *monotone* if for each $y \in Y$ the fiber $f^{-1}(y)$ is connected. It is well known that f is monotone if and only if for each subcontinuum Z of Y the set $f^{-1}(Z)$ is also connected. An onto map $f : X \to Y$ is *confluent* if for each subcontinuum Z of Y and each component A of $f^{-1}(Z)$, $f(A) = Z$. Clearly, every monotone map is confluent. A map $f : X \to Y$ is *open* if for each open set U of X, $f(U)$ is also an open set of Y. It is well known that every onto open map is confluent. A continuum X is a *regular curve* if for each $x \in X$ and each open neighborhood U of x in X, there is an open neighborhood V of x in U such that the boundary set $Bd(V)$ of V is a finite set. Clearly, each regular curve is a *Peano curve* (= 1-dimensional locally connected continuum). A continuum X is a *dendrite* (= 1-dimensional compact AR) if X is a locally connected continuum which contains no simple closed curve. It is well known that every dendrite is a regular curve. There are many regular curves which are not locally dendrites (= 1-dimensional compact ANR). Many *fractal sets* are regular curves which are not locally dendrites, for example, Sierpinski triangle (see [2] and [5]).

For a map $f : X \to X$ of a compactum X, we define the topological entropy $h(f)$ as follows (see [11]): Let n be a natural number and $\epsilon > 0$. A subset F of X is an (n, ϵ)-*spanning set* for f if for each $x \in X$, there is $y \in F$ such that

$$\max\{d(f^i(x), f^i(y))|\ 0 \le i \le n-1\ \} \le \epsilon.$$

Let $r_n(\epsilon)$ be the smallest cardinality of any (n, ϵ)-spanning set for f. Put

$$r(\epsilon) = \limsup_{n\to\infty}\ (1/n) \log r_n(\epsilon),$$

and

$$h(f) = \lim_{\epsilon\to 0} r(\epsilon).$$

It is well known that $h(f)$ is equal to the topological entropy which was defined by Adler, Konheim and McAndrew (see [1]).

The next theorem is the main result of this paper.

Theorem 1.1. *Let X be a regular curve. If $f : X \to X$ is a confluent map such that for some natural number $k \geq 1$ such that the number of components of $f^{-1}(y)$ is less than or equal to k for each $y \in X$, then $h(f) \leq \log k$.*

Proof. Let $\epsilon > 0$ and n a natural number. Since X is a regular curve, there is a finite open cover $\mathcal{A}$ of X such that for each $A \in \mathcal{A}$, $Bd(A)$ is a finite set and diam$A < \epsilon$. Let $B = \bigcup\{Bd(A)|\ A \in \mathcal{A}\}$ $(\subset X)$ and let $L = |B|(= L_{\mathcal{A}})$, where $|B|$ denotes the cardinality of the set B. Note that $L < \infty$. For a compactum E, Comp(E) denotes the set of all components of E. Let $a \in B$, $1 \leq i \leq n-1$ and $D \in \text{Comp}((f^i)^{-1}(a))$. If D contains no element of B, choose a point $c = c(a, i, D)$ of D. If D contains an element of B, choose a point $b \in B$ and put $c(a, i, D) = b$.

Consider the set

$$F = B \bigcup \{c(a, i, D)|\ a \in B, 1 \leq i \leq n-1, D \in \text{Comp}((f^i)^{-1}(a))\}.$$

By the assumption, we see that for each subcontinuum Z of X, the number of components of $f^{-1}(Z)$ is less than or equal to k. Hence we see that

$$|F| \leq L(n) = \sum_{j=0}^{n-1} k^i L.$$

We shall show that F is an (n, ϵ)-spanning set for f. Let x be any point of X. Consider the point $x_{n-1} = f^{n-1}(x)$ of X. Choose a subcontinuum C of X such that C is irreducible with respect to the property that C contains $x_{n-1} = f^{n-1}(x)$ and $C \cap B$ is not empty, that is, if H is any proper subcontinuum of C such that $x_{n-1} \in H$, then $H \cap B = \phi$. Note that $C \subset Cl(A)$ for some $A \in \mathcal{A}$. Put $C_{n-1} = C$ and $x_i = f^i(x)$ for $0 \leq i \leq n-1$. Consider the component C'_{n-2} of $f^{-1}(C_{n-1})$ containing x_{n-2}. Since f is confluent, $f(C'_{n-2}) = C_{n-1}$. Consider two cases (i) and (ii) as follows.

Case (i): C'_{n-2} contains a point of B. Take an irreducible subcontinuum C_{n-2} of C'_{n-2} between x_{n-2} and B.

Case (ii): C'_{n-2} contains no point of B. Put $C_{n-2} = C'_{n-2}$.

Consider the component C'_{n-3} of $f^{-1}(C_{n-2})$ containing x_{n-3}. By considering the above two cases, we obtain a continuum C_{n-3} containing x_{n-3} as above. We continue this procedure. Hence by induction, we can obtain a sequence of continua C_i $(i = n-1, n-2, .., 0)$. By the construction of continua C_i $(i = n-1, n-2, .., 0)$, we see that for each i, $x_i \in C_i \subset Cl(A_i)$ for some $A_i \in \mathcal{A}$ and $f(C_i) \subset C_{i+1}$. If C_0 contains a point of B, take a point $y \in C_0 \cap B \subset F$. If C_0 contains no point of B, then by the construction of C_i there is some $1 \leq j \leq n-1$ such that C_j contains an element a of B, and for $1 \leq s \leq j$ the component D of $(f^s)^{-1}(C_j)$ containing x_{j-s} contains no point of B. This implies that C_0 is the component of $(f^j)^{-1}(C_j)$ containing x_0. Since f is confluent and hence f^j is also confluent, we see that $a \in C_j = f^j(C_0)$. Then there is a component D of $(f^j)^{-1}(a)$ such that $D \subset C_0$. Hence we can choose a point $y = c(a, j, D) \in C_0 \cap F$ for $a \in B \cap C_j$ and $D \in \mathrm{Comp}((f^j)^{-1}(a))$.

By the construction of C_i $(1 \leq i \leq n-1)$, we can easily see that for each i, the two points x_i and $f^i(y)$ are contained in $Cl(A_i)$ for some $A_i \in \mathcal{A}$. This implies that the set F is an (n, ϵ)-spanning set of f. Then we see

$$\begin{aligned} r(\epsilon) = \limsup_{n\to\infty}\ (1/n) \log r(n,\epsilon) &\leq \limsup_{n\to\infty}\ (1/n)\log L(n) \\ &\leq \limsup_{n\to\infty}\ (1/n)(\log nk^{n-1}L) \\ = \lim_{n\to\infty}\ (1/n)(\log L + \log n &+ (n-1)\log k) = \log k. \end{aligned}$$

Hence

$$h(f) = \lim_{\epsilon\to 0} r(\epsilon) \leq \log k.$$

This completes the proof. □

Corollary 1.2. *Let X be a regular curve. If $f : X \to X$ is a monotone map, then the topological entropy $h(f)$ is zero.*

Proof. Since f is a monotone map, we can take $k = 1$ in the statement of the above theorem. Hence $0 \leq h(f) \leq \log 1 = 0$. □

Remark. In [9, Theorem 3.6], J. Nikiel proved that if $\{X_n, f_n\}$ is an inverse sequence such that all spaces X_n are totally regular continua and all the bonding maps $f_n : X_{n+1} \to X_n$ are monotone maps, then the inverse limit $X = \text{invlim}(X_n, f_n)$ is a totally regular continuum. A continuum X is *totally regular* if for any countable subset Q of X, each $x \in X$ and $\epsilon > 0$, there is an open neighborhood U of x in X such that $\text{diam}(U) < \epsilon$, $Bd(U)$ is finite and $Bd(U) \cap Q = \phi$. Clearly, each dendrite is totally regular. Hence we know that if $f : X \to X$ is a monotone map of a dendrite X, then the inverse limit (X, f) induced by the map f is also a dendrite. Since the shift homeomorphism $\tilde{f} : (X, f) \to (X, f)$ of f has the same topological entropy as f, by [10] $h(f) = h(\tilde{f}) = 0$. Note that Sierpinski triangle is not totally regular. In general, if $f : X \to X$ is a monotone map of a regular continuum X, we do not know whether the inverse limit (X, f) is also a regular continuum or not. Also, when f is confluent and not monotone, (X, f) may be not locally connected.

Examples.

(1) Let $I = [0, 1]$ and let $0 = x_0 < x_1 < ... < x_k = 1$ be $k + 1$ points of I. Let $f : I \to I$ be the map defined by $f(x) = f_i(x)$ for $x \in [x_i, x_{i+1}]$, where $f_i : [x_i, x_{i+1}] \to I$ is a homeomorphism for each $i = 0, 1, .., k - 1$. Then f is an open map and hence confluent map. Note that $\max\{|\text{Comp}(f^{-1}(y))| \mid x \in X\} \leq k$ and $h(f) = k$.

(2) There is a homeomorphism $f : \mu^1 \to \mu^1$ of the Menger universal curve μ^1 such that $h(f) = \infty$. Note that μ^1 is a Peano curve (1-dimensional locally connected continuum) and μ^1 is not a regular curve. Let $g : C \to C$ be a homeomorphism of a Cantor set $C \subset \mu^1$ such that $h(g) = \infty$ and C is a Z-set of μ^1. Then the map g can be extended to a homeomorphism $f : \mu^1 \to \mu^1$. Then $h(f) = \infty$. Hence, in the statement of theorem we can not replace the condition "*regular continuum*" with the condition "*Peano curve*".

(3) Let X be a compactum. Choose an inverse sequence

$$\underline{X} = \{X_n, p_{n,n+1} \mid n = 1, 2, ...,\}$$

of compact polyhedra X_n and bonding maps $p_{n,n+1} : X_{n+1} \to X_n$ such that $X_1 = \{*\}$ is a one point set and $X = \text{invlim } \underline{X}$. Let $p_n : X \to X_n$ be the natural projection.

Now, consider the infinite mapping telescope

$$T(\underline{X}) = \bigcup_{n=1}^{\infty} M(p_{n,n+1}),$$

where $M(p_{n,n+1})$ denotes the mapping cylinder of $p_{n,n+1}$. That is, in the topological sum $X_n \cup (X_{n+1} \times [1/(n+1), 1/n])$, $M(p_{n,n+1})$ is obtained by identifying points $(x, 1/n) \in X_{n+1} \times \{1/n\}$ and $p_{n,n+1}(x) \in X_n$ for each $x \in X_{n+1}$ and then $T(\underline{X})$ is obtained by identifying each point of $X_n \times \{1/n\}$ in $M(p_{n-1,n})$ and the corresponding point of X_n in $M(p_{n,n+1})$. Put

$$Z(\underline{X}) = X \cup T(\underline{X}).$$

Define a function $\mu : Z(\underline{X}) \to I = [0,1]$ by $\mu([x,t]) = t$ if $[x,t] \in T(\underline{X})$ and $\mu(x) = 0$ if $x \in X$. Also, define a retraction $\phi_t : Z(\underline{X}) \to \mu^{-1}([t,1])$ $(t \in I)$ by $\phi_t(z) = [p_{q(t),n}(x), t]$ for $z = [x,s] \in \mu^{-1}((0,t])$ and $x \in X_n$, $\phi_t(z) = [p_{q(t)}(x), t]$ for $z = x \in X$, and $\phi_t(z) = z$ for $z \in \mu^{-1}([t,1])$, where $q(t)$ is the natural number such that $1/q(t) \leq t < 1/(q(t)-1)$. The topology of $Z(\underline{X})$ is defined by taking as an open base, all open sets of $T(\underline{X})$ and all the sets of the form $\phi^{-1}_{1/n}(U) \cap \mu^{-1}([0,1/n))$, where U is an open set of X_n $(n \geq 1)$. Then $Z(\underline{X})$ is a compact AR.

Let X be a Cantor set. Choose an inverse sequence $\underline{X} = \{X_n, p_{n,n+1} |\ n = 1, 2, ...,\}$ such that $X_1 = \{*\}$, X_n is a finite set for each n and $X = \mathrm{invlim}\underline{X}$. Then $D = Z(X)$ is a dendrite. In fact, we may assume that D is the infinite binary tree (see [5, p.12]).

Let Y_i and $*_i \in Y_i$ be copies of D and $* = X_1 \in D$ for each $i = 1, 2, ...$ Let $* = 1 \in I = [0,1]$. Put $Y = \bigvee_{i=1}^{\infty}(Y_i, *_i) \bigvee (I, *)$, which is a one point union of the pointed spaces $(Y_i, *_i)$ and $(I, *)$. Note that we may assume that $\lim_{i\to\infty} diam(Y_i) = 0$. Then Y is also a dendrite. Let $g_i : Y_i \to Y_{i-1} (i \geq 2)$ be the natural homeomorphism and $\mu : Y_1 = Z(X) \to I$ be the map as above. Consider the map $f : Y \to Y$ as follows.

$$f(x) = \begin{cases} g_i(x) & \text{if } x \in Y_i \ (i \geq 2) \\ \mu(x) & \text{if } x \in Y_1 \\ x & \text{if } x \in I \end{cases}$$

Then $f : Y \to Y$ is an open map and hence it is a confluent map. Note that $f^{-1}(0)$ $(0 \in I \subset Y)$ is a Cantor set. Hence there is no $k(< \infty)$ for f as in the above theorem. Also we can easily see that $h(f) = 0$.

Problem 1.3. *Let X be a regular curve. If $f : X \to X$ is an onto map such that for some $k \geq 1$, the number of components of $f^{-1}(y)$ is less than or equal to k, is it true that $h(f) \leq \log k$?*

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