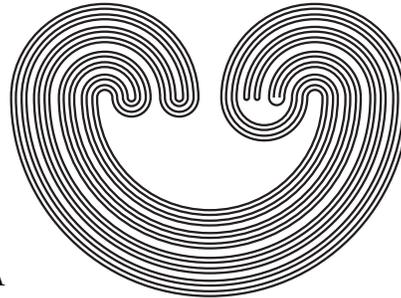


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SEMI-EBERLEIN SPACES

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ABSTRACT. We investigate the class of compact spaces which are embeddable into a power of the real line \mathbb{R}^κ in such a way that $c_0(\kappa) = \{f \in \mathbb{R}^\kappa : (\forall \varepsilon > 0) |\{\alpha \in \kappa : |f(\alpha)| > \varepsilon\}| < \aleph_0\}$ is dense in the image. We show that this is a proper subclass of the class of Valdivia, even when restricted to Corson compacta. We prove a preservation result concerning inverse sequences with semi-open retractions. As a corollary we obtain that retracts of Cantor or Tikhonov cubes belong to the above class.

1. INTRODUCTION

This study is motivated by results on the class of Valdivia compact spaces, i.e. compact spaces embeddable into \mathbb{R}^κ in such a way that the Σ -product $\Sigma(\kappa) = \{f \in \mathbb{R}^\kappa : |\text{suppt}(f)| \leq \aleph_0\}$ is dense in the image. This class was first considered by Argyros, Mercourakis and Negrepontis in [2] (the name *Valdivia compact* was introduced in [5]) and then studied by several authors, see e.g. [23, 24, 5, 11, 12, 13, 14]. Several important results on Valdivia compacta were established by Kalenda and we refer to his survey article [10] for further references. Probably the most remarkable property of a Valdivia compact space is the existence of “many retractions”, namely if K is Valdivia compact and $f: K \rightarrow Y$ is a continuous

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map then there exists a “canonical” retraction $r: K \rightarrow K$ such that $r[K]$ is Valdivia of the same weight as $f[K]$ and $f = fr$ holds. This property implies that the Banach space $C(K)$ has a *projectional resolution of the identity*, a useful property from which one can deduce e.g. the existence of an equivalent locally uniformly convex norm (see [6, Chapter VII] or [7, Chapter 6]).

In this note we consider compact spaces which have a better embedding than Valdivia: namely, compact spaces embeddable into \mathbb{R}^κ in such a way that $c_0(\kappa)$ is dense in the image. We call such spaces *semi-Eberlein*. This class of spaces clearly contains all Eberlein compacta. We show that not all Corson compacta are semi-Eberlein.

An obstacle in proving results parallel to the class of Valdivia compacta is the fact that, in contrast to Σ -products, $c_0(\kappa)$ is not countably compact. As a rule, obvious modifications of arguments concerning Valdivia compacta cannot be applied to semi-Eberlein spaces.

We state some simple characterizations of semi-Eberlein compacta involving families of open F_σ sets and properties of spaces of continuous functions with the topology of pointwise convergence on a dense set. We prove a preservation theorem involving inverse sequences whose bonding maps are semi-open retractions with a metrizable kernel. We show that semi-Eberlein compacta do not contain P-points. This immediately implies that the linearly ordered space $\omega_1 + 1$ is not semi-Eberlein. Finally, we show that the Corson compact space of Todorčević [21] (see also [22]) is not semi-Eberlein. We finish by formulating several open questions.

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2. NOTATION AND DEFINITIONS

All topological spaces are assumed to be completely regular. By a “map” we mean a continuous map. Fix a set S . As mentioned in the introduction, $\Sigma(S)$ denotes the Σ -product of S copies of \mathbb{R} , i.e. the set of all functions $x \in \mathbb{R}^S$ such that $\text{suppt}(x) := \{s \in S: x(s) \neq 0\}$ (the support of x) is countable. $c_0(S)$ denotes the subspace of $\Sigma(S)$

consisting of all functions $x \in \mathbb{R}^S$ such that for every $\varepsilon > 0$ the set $\{s \in S: |x(s)| > \varepsilon\}$ is finite. Given $T \subseteq S$ we denote by pr_T the canonical projection from \mathbb{R}^S onto \mathbb{R}^T . Note that $\text{pr}_T[\Sigma(S)] = \Sigma(T)$ and $\text{pr}_T[c_0(S)] = c_0(T)$.

Given two maps $f: X \rightarrow Y$ and $g: X \rightarrow Z$ we denote by $f\Delta g$ their diagonal product, i.e. $f\Delta g: X \rightarrow Y \times Z$ is defined by $(f\Delta g)(x) = \langle f(x), g(x) \rangle$.

A compact space K is *Eberlein compact* if $K \subseteq c_0(S)$ for some S or equivalently K is embeddable into a Banach space with the weak topology (by Amir, Lindenstrauss [1]).

Let T be a set, a family \mathcal{A} of subsets T is called *adequate* [20] provided 1) $\{t\} \in \mathcal{A}$ for each $t \in T$; and 2) $A \in \mathcal{A}$ iff $M \in \mathcal{A}$ for any finite $M \subseteq A$. Every adequate family is a compact 0-dimensional space, when identifying sets with their characteristic functions lying in the Cantor cube $\{0, 1\}^T$. Any space homeomorphic to an adequate family of sets is called *adequate compact*. A collection of sets \mathcal{U} is *T_0 separating* on K if for every $x, y \in K$ there is $U \in \mathcal{U}$ with $|U \cap \{x, y\}| = 1$.

A compact space K is *Valdivia compact* [5] if for some κ , K embeds into \mathbb{R}^κ so that $\Sigma(\kappa)$ is dense in the image. Given $f \in \mathbb{R}^\kappa$ and $S \subseteq \kappa$ denote by $f \upharpoonright S$ the function $(f \upharpoonright S) \wedge 0_{\kappa \setminus S}$, i.e. $(f \upharpoonright S)(\alpha) = f(\alpha)$ if $\alpha \in S$ and $(f \upharpoonright S)(\alpha) = 0$ otherwise. Let us recall an important factorization property of Valdivia compact spaces (see e.g. [14]): assuming $K \subseteq \mathbb{R}^\kappa$ is such that $\Sigma(\kappa) \cap K$ is dense in K , for every map $f: K \rightarrow Y$ there exists $S \subseteq \kappa$ such that $|S| \leq w(Y)$ and $f(x) = f(x \upharpoonright S)$ for every $x \in K$. In particular, $x \upharpoonright S \in K$ whenever $x \in K$. Applying this result to projections $\text{pr}_T: K \rightarrow \mathbb{R}^T$ we see that K is the limit of a continuous inverse sequence of smaller Valdivia compacta whose all bonding mappings are retractions.

3. SEMI-EBERLEIN COMPACTA

A compact space K will be called *semi-Eberlein* if for some set S there is an embedding $K \subseteq \mathbb{R}^S$ such that $c_0(S) \cap K$ is dense in K . Clearly, every adequate compact space is semi-Eberlein and every semi-Eberlein compact space is Valdivia compact. Since there are examples of adequate Corson compacta which are not Eberlein [20], not every semi-Eberlein Corson compact space is Eberlein.

Various examples of adequate Corson compacta which are not even Gul'ko compacta one could find in [17]. The linearly ordered space $\omega_1 + 1$ is not semi-Eberlein (see below), so not every Valdivia compact space is semi-Eberlein.

Below are two simple characterizations of semi-Eberlein compacta, one in terms of families of open F_σ sets and the other one in terms of spaces of continuous functions. Both results are similar to known analogous characterizations of Eberlein compacta.

Proposition 3.1. *Let K be a compact space. The following properties are equivalent.*

- (a) K is semi-Eberlein.
- (b) There exists a T_0 separating collection \mathcal{U} consisting of open F_σ subsets of K such that $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$ and the set

$$\{p \in K : (\forall n \in \omega) |\{U \in \mathcal{U}_n : p \in U\}| < \aleph_0\}$$

is dense in K .

- (c) There exist a set $S = \bigcup_{n \in \omega} S_n$ and an embedding $h: K \rightarrow [0, 1]^S$ such that the set of all $p \in K$ with the property that $S_n \cap \text{suppt}(h(p))$ is finite for every $n \in \omega$, is dense in K .

Proof. (a) \implies (b) We assume $K \subseteq \mathbb{R}^S$ and $c_0(S) \cap K$ is dense in K . For each $r \in \mathbb{Q} \setminus \{0\}$ and $s \in S$ define

$$U_{s,r} = \{x \in K : x(s) > r\} \quad \text{if } r > 0$$

and

$$U_{s,r} = \{x \in K : x(s) < r\} \quad \text{if } r < 0.$$

Then $U_{s,r}$ is an open F_σ set and $\mathcal{U} = \{U_{s,r} : s \in S, r \in \mathbb{Q} \setminus \{0\}\}$ is T_0 separating on K . Finally, setting $\mathcal{U}_r = \{U_{s,r} : s \in S\}$, we have $\mathcal{U} = \bigcup_{r \in \mathbb{Q} \setminus \{0\}} \mathcal{U}_r$ and for every $p \in c_0(S) \cap K$ there are only finitely many $s \in S$ with $|p(s)| > |r|$, i.e. the set $\{U \in \mathcal{U}_r : p \in U\}$ is finite.

(b) \implies (c) For each $U \in \mathcal{U}$ fix a continuous function $f_U: K \rightarrow [0, 1]$ such that $U = f_U^{-1}[(0, 1]]$. Let h be the diagonal product of $\{f_U\}_{U \in \mathcal{U}}$. Then $h: K \rightarrow [0, 1]^{\mathcal{U}}$ is such that $U \in \text{suppt}(h(p))$ iff $p \in U$; therefore $\text{suppt}(h(p)) \cap \mathcal{U}_n$ is finite for every $n \in \omega$ and for every $p \in D$, where D is the dense set of points described in (b).

(c) \implies (a) We assume $K \subseteq [0, 1]^S$ and $S = \bigcup_{n \in \omega} S_n$ is as in (c). Let D consist of all $p \in K$ such that $\text{suppt}(p) \cap S_n$ is finite for every $n \in \omega$. Fix $s \in S$ and let n be minimal such that $s \in S_n$. Define $h_s: [0, 1] \rightarrow [0, 1]$ by $h_s(t) = t/n$. Then $h = \prod_{s \in S} h_s$ is an

embedding of $[0, 1]^S$ into itself. Now, if $p \in D$ and $\varepsilon > 0$ then $h(p)(s) > \varepsilon$ only if $s \in S_n$ and $1/n > \varepsilon$; thus $\{s \in S: |h(p)(s)| > \varepsilon\}$ is finite. Hence $c_0(S) \cap h[K]$ is dense $h[K]$. \square

We denote by $A(\kappa)$ the one-point compactification of the discrete space of size κ . Given a dense subset D of a space K we denote by $\tau_p(D)$ the topology of pointwise convergence on D on the space $C(K)$.

Proposition 3.2. *Let K be a compact space. Then K is semi-Eberlein iff there exist a dense set $D \subseteq K$ and a copy of $A(\kappa)$ in $\langle C(K), \tau_p(D) \rangle$ which separates the points of K .*

Proof. Assume K is semi-Eberlein, $K \subseteq \mathbb{R}^\kappa$ and $D = K \cap c_0(\kappa)$ is dense in K . Let

$$L = \{\text{pr}_\alpha \upharpoonright K : \alpha < \kappa\} \cup \{0\},$$

where $\text{pr}_\alpha: \mathbb{R}^\kappa \rightarrow \mathbb{R}$ is the α -th projection. We may assume that $\text{pr}_\alpha \upharpoonright K \neq 0$ for every $\alpha < \kappa$, shrinking κ if necessary. We claim that $\langle L, \tau_p(D) \rangle$ is homeomorphic to $A(\kappa)$ and 0 is the only accumulation point of L . Fix a basic neighborhood V of 0 in $\langle C(K), \tau_p(D) \rangle$. Then

$$V = \{x \in C(K) : (\forall i < k) |x(d_i)| < \varepsilon\}$$

for some $d_0, \dots, d_{k-1} \in D$ and $\varepsilon > 0$. By the fact that $d_i \in c_0(\kappa)$ for $i < k$, there is a finite set $S \subseteq \kappa$ such that $|d_i(\alpha)| < \varepsilon$ for $i < k$ and for every $\alpha \in \kappa \setminus S$. Since $\text{pr}_\alpha(d_i) = d_i(\alpha)$, we see that $\text{pr}_\alpha \upharpoonright K \in V$ for $\alpha \in \kappa \setminus S$.

Now assume that $A(\kappa)$ embeds into $\langle C(K), \tau_p(D) \rangle$ for some dense set $D \subseteq K$ and its image separates the points of K . We may assume that $L = \{f_\alpha: \alpha < \kappa\} \subseteq C(K)$ is such that $L \cup \{0\}$ endowed with the topology $\tau_p(D)$ is homeomorphic to $A(\kappa)$, 0 is the accumulation point of L and L separates the points of K . Denote by \mathcal{N} the (countable) collection of all nonempty open rational intervals $v \subseteq \mathbb{R}$ such that $0 \notin \text{cl} v$. For each $v \in \mathcal{N}$ define $U_\alpha^v = f_\alpha^{-1}[v]$. Let $\mathcal{U}_v = \{U_\alpha^v: \alpha < \kappa\}$. Then $\mathcal{U} = \bigcup_{v \in \mathcal{N}} \mathcal{U}_v$ is T_0 separating and consists of open F_σ sets. Fix $x \in D$ and $v \in \mathcal{N}$. There is a finite set $S \subseteq \kappa$ such that $f_\alpha(x) \notin \text{cl} v$ for $\alpha \in \kappa \setminus S$ (because $\{f \in C(K): f(x) \in \mathbb{R} \setminus \text{cl} v\}$ is a neighborhood of 0 in $\tau_p(D)$). If $\alpha \notin S$ then $x \notin U_\alpha^v$. Thus each \mathcal{U}_v is point-finite on D . By Proposition 3.1, K is semi-Eberlein. \square

Remark 3.3. Every compact space L can be embedded into $\langle C(K), \tau_p(D) \rangle$ for some compact space K and a dense set $D \subseteq K$, in such a way that the image of L separates the points of K . Indeed, assuming $L \subseteq [0, 1]^\kappa$, consider first $K_0 = \beta\kappa$, the Čech-Stone compactification of the discrete space κ , and let $D_0 = \kappa$. Then $[0, 1]^\kappa$ embeds naturally into $\langle C(K_0), \tau_p(D_0) \rangle$, since every bounded function on κ extends uniquely to K_0 . Now define an equivalence relation \sim on K_0 by

$$x \sim x' \iff (\forall f \in L) f(x) = f(x').$$

Let K be the quotient space and let $q: K_0 \rightarrow K$ be the quotient map. Then K is Hausdorff and L naturally embeds into $\langle C(K), \tau_p(D) \rangle$, where $D = q[D_0]$. Clearly, after this embedding L separates the points of K .

Remark 3.4. It is known that if K is a compact such that there exists a compact $L \subseteq C_p(K)$ which separates points of K then K is an Eberlein compactum. This fact has no generalization to semi-Eberlein compacta: we give an example which shows that in Proposition 3.2 the compact $A(\kappa)$ cannot be replaced by any other (even metrizable) compact space. Take K to be the Čech-Stone compactification of ω and let $D = \omega$. Then the Cantor set $L = \{0, 1\}^\omega$ is embedded naturally into $\langle C(K), \tau_p(D) \rangle$. Clearly, this Cantor set L separates the points of K . $\beta\omega$ is not a continuous image of any Valdivia compact space [10].

4. A PRESERVATION THEOREM

It is clear that the class of semi-Eberlein compacta is stable under arbitrary products and under closed subsets which have a dense interior. It is not hard to see that the one-point compactification of any topological sum of semi-Eberlein compacta is semi-Eberlein. By [11] every non-Corson Valdivia compact space has a two-to-one map onto a non-Valdivia space. Thus, the class of semi-Eberlein compacta is not stable under continuous images.

It has been shown in [13] that there exists a compact connected Abelian topological group of weight \aleph_1 which is not Valdivia compact. As every compact group is an epimorphic (and therefore open) image of a product of compact metric groups, this shows that there exists a semi-Eberlein compact space (namely some product of

compact metric groups) which has an open map onto a non-Valdivia space. On the other hand, it has been shown in [14] that “small” 0-dimensional open images as well as retracts of Valdivia compact spaces are Valdivia, where “small” means “of weight $\leq \aleph_1$ ”. Another result from [14] states that the class of Valdivia compacta is stable under limits of certain inverse sequences with retractions [14]; in particular a limit of a continuous inverse sequence of metric compact spaces whose all bonding mappings are retractions is Valdivia compact (in fact this characterizes Valdivia compact spaces of weight $\leq \aleph_1$). Below we prove a parallel preservation theorem for semi-Eberlein spaces. As an application we show that retracts of Cantor or Tikhonov cubes are semi-Eberlein, which improves a result from [14]. Note that $\omega_1 + 1$ is an example of a non-semi-Eberlein space which is the limit of a continuous inverse sequence of metric compacta whose all bonding mappings are retractions.

Recall that a map $f: X \rightarrow Y$ is called *semi-open* if for every nonempty open set $U \subseteq X$ the image $f[U]$ has nonempty interior.

Lemma 4.1. *Assume $f: X \rightarrow Y$ is a retraction, i.e. $Y \subseteq X$ and $f \upharpoonright Y = \text{id}_Y$. Assume $g: X \rightarrow [0, 1]^\omega$ is such that $f\Delta g$ is one-to-one. Then Y is G_δ in X and for every T_0 separating family \mathcal{U} on Y there exists a countable family \mathcal{V} of open F_σ sets in X such that $\mathcal{U} \cup \mathcal{V}$ is T_0 separating on X and $Y \cap \bigcup \mathcal{V} = \emptyset$.*

Proof. Fix a metric d on $[0, 1]^\omega$ and define $\varphi(x) = d(g(x), g(f(x)))$. Then $\varphi(x) = 0$ iff $g(x) = g(f(x))$ which is equivalent to $x = f(x)$, by the fact that $f\Delta g$ is one-to-one. Thus $Y = \varphi^{-1}(0)$ is G_δ in X . Now, fix a countable T_0 separating open family \mathcal{V}_0 on $[0, 1]^\omega$ and define $\mathcal{V} = \{g^{-1}[V] \setminus Y : V \in \mathcal{V}_0\}$. Clearly, \mathcal{V} consists of open F_σ sets and if $g(x) \neq g(x')$ then there is $V \in \mathcal{V}$ such that $|\{x, x'\} \cap V| = 1$. Since $f\Delta g$ is one-to-one, $\mathcal{U} \cup \mathcal{V}$ is T_0 separating on X . \square

A map $f: X \rightarrow Y$ has a *metrizable kernel* if there exists a map $g: X \rightarrow [0, 1]^\omega$ such that $f\Delta g$ is one-to-one.

Theorem 4.2. *Assume $\mathbb{S} = \langle K_\alpha; r_\alpha^\beta; \kappa \rangle$ is a continuous inverse sequence such that*

- (1) K_0 is semi-Eberlein,
- (2) each $r_\alpha^{\alpha+1}$ is a semi-open retraction with a metrizable kernel.

Then $\varprojlim \mathbb{S}$ is semi-Eberlein.

Proof. Let $K = \varprojlim \mathbb{S}$. We may assume that each K_α is a subspace of K and each projection $r_\alpha: K \rightarrow K_\alpha$ is a retraction (see [4, Prop. 4.6]); therefore $r_\alpha^\beta = r_\alpha \upharpoonright K_\beta$ for $\alpha < \beta$. We construct inductively families of open F_σ sets $\mathcal{U}_\alpha = \bigcup_{n \in \omega} \mathcal{U}_\alpha^n$ and dense sets $D_\alpha \subseteq K_\alpha$ such that:

- (a) \mathcal{U}_α^n is point-finite on $D_\alpha \subseteq K_\alpha$;
- (a') each $U \in \mathcal{U}_\alpha$ is of the form $r_\alpha^{-1}[U']$ for some open F_σ subset of K_α ;
- (b) $\alpha < \beta \implies \mathcal{U}_\alpha^n \subseteq \mathcal{U}_\beta^n$ and $K_\alpha \cap \bigcup (\mathcal{U}_\beta^n \setminus \mathcal{U}_\alpha^n) = \emptyset$ for every $n \in \omega$;
- (c) if γ is a limit ordinal then $\mathcal{U}_\gamma^n = \bigcup_{\alpha < \gamma} \mathcal{U}_\alpha^n$ and $D_\gamma = \bigcup_{\alpha < \gamma} D_\alpha$;
- (d) $D_{\alpha+1} = (r_{\alpha+1}^\alpha)^{-1}[D_\alpha]$.
- (e) \mathcal{U}_α is T_0 separating on K_α .

We start by finding a suitable family $\mathcal{V}_0 = \bigcup_{n \in \omega} \mathcal{V}_0^n$ on K_0 , using Proposition 3.1(b). Let D_0 be a dense set with the property that every \mathcal{V}_0^n is point-finite on D_0 . Define $\mathcal{U}_0^n = \{r_0^{-1}[V]\}_{V \in \mathcal{V}_0^n}$. Clearly (a), (a') and (e) hold.

The successor stage is taken care of by Lemma 4.1: we use the fact that $r_{\alpha+1}^\alpha$ is semi-open to deduce that $(r_{\alpha+1}^\alpha)^{-1}[D_\alpha]$ is dense in $K_{\alpha+1}$. Note also that if $V \subseteq K_{\alpha+1}$ is disjoint from K_α then $r_{\alpha+1}^{-1}[V] \cap K_\alpha = \emptyset$, which ensures that condition (b) is satisfied.

It remains to check the limit stage. Fix a limit ordinal $\delta \leq \kappa$. Clearly, D_δ defined by (c) is dense and \mathcal{U}_δ defined in (c) consists of open F_σ sets and satisfies (a'). Fix $x \in D_\delta$ and $n \in \omega$. Then $x \in D_\alpha$ for some $\alpha < \delta$ and thus by induction hypothesis x belongs to only finitely many elements of \mathcal{U}_α^n . By the second part of (b), $x \notin U$ for any $U \in \mathcal{U}_\beta^n \setminus \mathcal{U}_\alpha^n$. It follows that each \mathcal{U}_δ^n is point-finite on D_δ .

Now fix $x \neq x'$ in K_δ . Then $r_\alpha(x) \neq r_\alpha(x')$ for some $\alpha < \delta$ (by the continuity of the sequence) and hence there is $U \in \mathcal{U}_\alpha$ such that e.g. $r_\alpha(x) \in U$ and $r_\alpha(x') \notin U$. By (a') we have $U = r_\alpha^{-1}[U']$. Note that $U' = U \cap K_\alpha$, because $r_\alpha \upharpoonright K_\alpha = \text{id}_{K_\alpha}$. Thus $r_\alpha(x) \in U'$, $r_\alpha(x') \notin U'$ and hence $x \in U$ and $x' \notin U$. It follows that \mathcal{U}_δ is T_0 separating.

Thus the construction can be carried out. By Proposition 3.1, each K_α , and therefore also K , is semi-Eberlein. \square

Corollary 4.3. *Every retract of a Cantor or Tikhonov cube is semi-Eberlein.*

Proof. Let K be a retract of a Cantor or Tikhonov cube. Then $K = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle K_\alpha; r_\alpha^\beta; \kappa \rangle$ is a continuous inverse sequence such that K_0 is a compact metric space and each $r_\alpha^{\alpha+1}$ is an open retraction with a metrizable kernel. In both cases Theorem 4.2 applies.

In the case of Cantor cubes this is Haydon–Koppelberg decomposition theorem [16, Corollary 2.8]. In fact this result is about *projective Boolean algebras* which, via Stone duality, correspond to retracts of Cantor cubes.

In the case of Tikhonov cubes, one needs to refer to Chapter 2 of Schepin’s work [19]. Since the quoted result is not proved explicitly in [19] and we could not find a suitable bibliographic reference for it, we briefly sketch the proof, giving references to appropriate results from [19]. For a full and self-contained proof we refer to [15].

Assume $r: [0, 1]^\kappa \rightarrow K$ is a retraction. Following Shchepin, we say that a set $S \subseteq \kappa$ is *r-admissible* if $x \upharpoonright S = x' \upharpoonright S$ implies $r(x) \upharpoonright S = r(x') \upharpoonright S$ for every $x, x' \in [0, 1]^\kappa$. This property is equivalent to the following: if $f \in C(K)$ depends on S , i.e. $x \upharpoonright S = x' \upharpoonright S \implies f(x) = f(x')$, then $fr \in C([0, 1]^\kappa)$ also depends on S . Define $K_S = \text{pr}_S[K]$ and let $p_S: K \rightarrow K_S$ be defined by $p_S = \text{pr}_S \upharpoonright K$, where $\text{pr}_S: [0, 1]^\kappa \rightarrow [0, 1]^S$ is the projection. If $S \subseteq \kappa$ is *r-admissible* then it is easy to see that the map p_S is a retraction; moreover, by Lemma 1 in [19, Chapter 2], p_S is open. Observe that the union of any family of *r-admissible* sets is *r-admissible*. By Lemma 3 in [19, Chapter 2], every countable subset of κ can be enlarged to a countable *r-admissible* set. For each $\alpha < \kappa$ choose a countable *r-admissible* set $S_\alpha \subseteq \kappa$ such that $\alpha \in S_\alpha$. Define $A_\alpha = \bigcup_{\xi < \alpha} S_\xi$. Then each A_α is *r-admissible* and hence each $p_\alpha := p_{A_\alpha}$ is an open retraction. Given $\alpha \leq \beta < \kappa$, let p_α^β be the unique map such that $p_\alpha = p_\alpha^\beta p_\beta$. Then $\mathbb{S} = \langle K_\alpha; p_\alpha^\beta; \kappa \rangle$, where $K_\alpha = K_{A_\alpha}$, is a continuous inverse sequence such that each $p_\alpha^{\alpha+1}$ is an open retraction with a metrizable kernel (since S_α is countable) and $K = \varprojlim \mathbb{S}$. \square

5. SEMI-EBERLEIN SPACES HAVE NO P-POINTS

Recall that a point $p \in X$ is called a *P-point* if p is not isolated and $p \in \text{int} \bigcap_{n \in \omega} U_n$ for every sequence $\{U_n : n \in \omega\}$ of neighborhoods of p . The simplest example of a compact space with a P-point is $\omega_1 + 1$, considered as a linearly ordered space. In this section we observe that semi-Eberlein spaces do not have P-points, which implies in particular that $\omega_1 + 1$ is not semi-Eberlein. Using this observation and a simple forcing argument, we show that the Corson compact of Todorćević defined in [21] is not semi-Eberlein (Example 5.5 below).

A net $\{x_\sigma\}_{\sigma \in \Sigma}$ in a topological space X will be called σ -*bounded* if Σ is σ -directed and for every countable set $S \subseteq \Sigma$ there is $\tau \in \Sigma$ such that $\text{cl}\{x_\sigma : \sigma \geq \tau\} \cap \text{cl}\{x_\sigma : \sigma \in S\} = \emptyset$. In case where $\{x_\sigma\}_{\sigma \in \Sigma}$ converges to $p \in X$, this just means that $p \notin \text{cl}\{x_\sigma : \sigma \in S\}$ for any countable $S \subseteq \Sigma$.

Lemma 5.1. *For every set S , there are no σ -bounded nets in $c_0(S)$ which converge in \mathbb{R}^S .*

Proof. Suppose $\{x_\sigma\}_{\sigma \in \Sigma} \subseteq c_0(S)$ is a σ -bounded net which converges to $p \in \mathbb{R}^S$. Then $\text{suppt}(p)$ is uncountable, because $\{x_\sigma : \sigma \in \Sigma\} \subseteq \Sigma(S)$ and Σ -products of real lines have countable tightness. Choose $\{s_n : n \in \omega\} \subseteq \text{suppt}(p)$ so that $|p(s_n)| > \varepsilon$ for some fixed $\varepsilon > 0$. Using the fact that $p = \lim_{\sigma \in \Sigma} x_\sigma$, for each $n \in \omega$ we find $\sigma_n \in \Sigma$ such that $|x_{\sigma_n}(s_n)| > \varepsilon$ for every $\sigma \geq \sigma_n$. Let $\tau \in \Sigma$ be such that $\sigma_n < \tau$ for every $n \in \omega$. Then $|x_\tau(s_n)| > \varepsilon$ for every $n \in \omega$ and consequently $x_\tau \notin c_0(S)$, a contradiction. \square

Theorem 5.2. *Semi-Eberlein compact spaces do not have P-points.*

Proof. Assume $K \subseteq \mathbb{R}^\kappa$, $c_0(\kappa) \cap K$ is dense in K and suppose that $p \in K$ is a P-point. Then $p \notin c_0(\kappa)$. Let Σ be a fixed base at p . Consider Σ as a (σ -directed) poset with reversed inclusion. For each $u \in \Sigma$ choose $x_u \in u \cap c_0(\kappa)$. If $S \subseteq \Sigma$ is countable then there is $v \in \Sigma$ such that $v \subseteq K \setminus \{x_u : u \in S\}$ and therefore $p \notin \text{cl}\{x_u : u \in S\}$. Thus $\{x_u\}_{u \in \Sigma}$ is a σ -bounded net in $c_0(\kappa)$ which converges to p . By Lemma 5.1, we get a contradiction. \square

Corollary 5.3. *The linearly ordered space $\omega_1 + 1$ is not semi-Eberlein.*

Actually, we can strengthen Theorem 5.2 by saying that if K is semi-Eberlein then no forcing notion can force a P-point in K . More precisely, if K is a compact space, \mathbb{P} is a forcing notion and G is a \mathbb{P} -generic filter then by K^G we denote the compact space in the G -generic extension which consists of all ultrafilters in $CL(K)$, where $CL(K)$ denotes the lattice of closed subsets of K defined in the ground model. It is not hard to check that we get the same space if we take, instead of $CL(K)$, any of its sublattices which is a closed base for K . Indeed, if $p \neq q$ are ultrafilters in $CL(K)$ and \mathbb{L} is a sublattice of $CL(K)$ which is a closed base then, by compactness, there are disjoint $a, b \in \mathbb{L}$ such that $a \in p$ and $b \in q$. This shows that the map $h: \text{Ult}(CL(K)) \rightarrow \text{Ult}(\mathbb{L})$, defined by $h(p) = p \cap \mathbb{L}$, is one-to-one ($\text{Ult}(\mathbb{L})$ denotes the space of all ultrafilters over \mathbb{L}). Clearly, h is a continuous surjection and, when defined in a fixed generic extension, it shows that $\text{Ult}(\mathbb{L})$ is homeomorphic to $\text{Ult}(CL(K))$.

It follows that $[0, 1]^G$ is the same as the usual unit interval in the G -generic extension, since it can be described as the space of ultrafilters over the lattice generated by all closed rational intervals. A similar fact is true for any cube $[0, 1]^\kappa$.

Next, observe that K is dense in K^G . Indeed, $K \subseteq K^G$, because being an ultrafilter in a lattice is absolute. A basic open set in K^G is of the form $a^- = \{p \in K^G: a \notin p\}$, where a is an element of $CL(K)$ from the ground model. Now, if $a^- \neq \emptyset$ then there is $b \in CL(K)$ such that $a \cap b = \emptyset$ and therefore in the ground model there is $p \in \text{Ult}(CL(K))$ such that $b \in p$; consequently $K \cap a^- \neq \emptyset$.

Finally, if K, L are compact spaces such that $K \subseteq L$ then K^G is the closure of K in L^G . Indeed, the fact $K \subseteq L$ is encoded in the lattice epimorphism $h: \mathbb{L} \rightarrow \mathbb{K}$, defined by $h(a) = a \cap K$, where $\mathbb{K} = CL(K)$ and $\mathbb{L} = CL(L)$. In any generic extension, h induces an embedding of K^G into L^G and since K is dense in K^G and K^G is closed, we have $K^G = \text{cl}_{L^G}(K)$.

In particular, if $K \subseteq [0, 1]^\kappa$ then K^G can be regarded as the closure of K in $([0, 1]^\kappa)^G$, where $([0, 1]^\kappa)^G$ is the Tikhonov cube $[0, 1]^\kappa$ defined in the G -extension. Clearly, K^G is semi-Eberlein if K is so. Thus:

Proposition 5.4. *No forcing notion can introduce a P-point in a semi-Eberlein space. More precisely, if K is a semi-Eberlein space, \mathbb{P} is a forcing notion and G is a \mathbb{P} -generic filter then K^G has no P-points.*

This observation leads to examples of Corson compact spaces which are not semi-Eberlein.

Example 5.5 (Todorčević). There exists a Corson compact space which is not semi-Eberlein. In fact the Corson compact space of [21] (see also [22, p. 287]) is such an example. Let us recall the construction of this space. For a tree T , let $P(T)$ be the set of all initial branches of T (i.e. linearly ordered sets $x \subseteq T$ such that $t \in x \ \& \ s < t \implies s \in x$) with the topology induced from the Cantor cube $\{0, 1\}^T$. When T has no uncountable branches, $P(T)$ is an example of a Corson compact space. The tree T used in [21] is the set of all subsets of some fixed stationary and costationary set $A \subseteq \omega_1$ which are closed in ω_1 . Thus, in particular, such a tree T is Baire (see [22, Lemma 9.12]) and therefore forcing with T does not collapse ω_1 (see [9, Theorem 15.6], where a Baire forcing notion is called \aleph_0 -distributive). A T -generic filter $G \subseteq T$ gives an uncountable branch which is obviously a P-point in $P(T)^G$. Hence, $P(T)$ is not semi-Eberlein.

Remark 5.6. It is shown in [21] that the Corson compactum $P(T)$ has no dense metrizable subspaces. Evidently, for each tree the family of all its chains is an adequate family. Note that the adequate (and hence semi-Eberlein) Corson compactum built on this tree T also has no dense metrizable subspaces [18]. This shows that a semi-Eberlein compact which is Corson does not have to be Eberlein or even Gul'ko compact (since Gul'ko compacta contain dense completely metrizable subspaces [8]).

6. QUESTIONS

There are several open questions concerning Valdivia compacta; one may ask parallel questions for semi-Eberlein spaces. Below are questions specific for semi-Eberlein compacta.

Question 6.1. *Can a semi-Eberlein compact space have weak P-points?*

Question 6.2. *Is the class of semi-Eberlein spaces stable under closed G_δ sets?*

Even for adequate compacta the same question seems to be open.

Question 6.3. *Let X be an adequate compact and $F \subseteq X$ is a closed G_δ set. Is F a semi-Eberlein compact space?*

As a relevant result, we mention that Bell [3] gave an example of a centered (continuous image of adequate) compact X and a closed G_δ set $Z \subseteq X$ such that Z is not homeomorphic to any centered compact. However Z in his example is an Eberlein compact.

Question 6.4. *Does there exist a semi-Eberlein space K which does not embed into any cube $[0, 1]^\kappa$ so that $\sigma(\kappa) \cap K$ is dense in K ? ($\sigma(\kappa)$ is the small sigma-product of κ copies of \mathbb{R})*

Question 6.5. *Assume K is semi-Eberlein, not Eberlein. Does K have a non-semi-Eberlein image?*

Question 6.6. *Assume K is semi-Eberlein and $f: K \rightarrow L$ is a retraction or an open surjection such that L has densely many G_δ points. Is L semi-Eberlein?*

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