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RHYTHMIC FUNCTIONS AND IP RECURRENCE

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Abstract. Various classes of functions arising in the study of a problem in Diophantine approximation are studied. Chief among these are so-called rhythmic functions, as defined analytically by J. van der Corput, and IP recurrent functions, as defined analytically by H. Furstenberg and B. Weiss. Equivalent definitions for these classes are given in the language of topological dynamics, and strong parallels are observed in consequence. A recently developed alternative method for attacking the underlying Diophantine approximation problem is mentioned, with new results thereby obtained.

1. Variations on a theme

Recall that a sequence $(x_n)_{n\in\mathbf{N}}$ is *equidistributed modulo* 1 if it hits, modulo one, each subinterval of $[0,1)$ with the expected frequency. That is, if for every α,β with $0\leq\alpha<\beta<1$, $\lim_N \frac{|\{n:1\leq n\leq N,\alpha<x_n<\beta\}|}{N}=\beta-\alpha$. A well known theorem of H. Weyl ([W1], [W2]) asserts that if $p(x)$ is a real polynomial with at least one irrational coefficient (other than the constant term) then the sequence $\big(p(n)\big)_{n\in\mathbf{N}}$ is equidistributed modulo 1. In particular, this implies the weaker result that for such polynomials p, the set $\{p(n):n\in\mathbf{Z}\}$ is dense mod 1 in the unit interval.

So, for example, one is assured that there exists some integer x such that the fractional part of πx^2 lies between $\frac{1}{4}$ and $\frac{1}{3}$. Or to put it another way, there exist integers x and y such that $\frac{1}{4}<\pi x^2-y<\frac{1}{3}$, that is to say, the inequality $\frac{1}{4}<\pi x^2-y<\frac{1}{3}$ is solvable in integers. This latter formulation suggests an extension

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in which *systems* of inequalities are considered. For example, do there exist integers x, y and z such that both $\frac{1}{4} < \pi x^2 - y < \frac{1}{3}$ and $\frac{2}{5} < \sqrt{2}xy^3 - z < \frac{1}{2}$? J. van der Corput considered a general class of such systems of inequalities, and succeeded in showing that, when they do have integer solutions, there are in fact "many" such solutions.

Theorem A ([VdC]). *Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be real numbers and let $m, n \in \mathbf{N}$. For each $k = 1, \ldots, n$, let p_k be a polynomial of $m + k - 1$ unknowns. If the system*

$$\begin{aligned}
\alpha_1 &< p_1(x_1, \ldots, x_m) - x_{m+1} < \beta_1 \\
\alpha_2 &< p_2(x_1, \ldots, x_m, x_{m+1}) - x_{m+2} < \beta_2 \\
\alpha_3 &< p_3(x_1, \ldots, x_m, x_{m+1}, x_{m+2}) - x_{m+3} < \beta_3 \\
&\vdots \\
\alpha_n &< p_n(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n-1}) - x_{m+n} < \beta_n
\end{aligned}$$

has an integer valued solution $(a_1, \ldots, a_{m+n})$ then the set

$$\big\{(s_1, \ldots, s_m) : \exists \text{ a solution } (s_1{+}a_1, s_2{+}a_2, \ldots, s_m{+}a_m, s_{m+1}, \ldots, s_{m+n})\big\}$$

intersects any large enough m-dimensional cube in $\mathbf{Z}^m$.

Let us elaborate on the sense of "many" implied by the conclusion of this theorem. A subset S of $\mathbf{Z}^m$ is said to be *syndetic* if it intersects any large enough m-dimensional cube. Consider now the concrete system mentioned above in variables x, y and z. This system corresponds to the case $m = 1$, $n = 2$ of Theorem A. A 1-dimensional cube is just an interval, so a syndetic subset of $\mathbf{Z}$ is precisely one with "bounded gaps", that is, a subset that intersects any long enough interval. Theorem A therefore tells us that if the system $\frac{1}{4} < \pi x^2 - y < \frac{1}{3}$, $\frac{2}{5} < \sqrt{2}xy^3 - z < \frac{1}{2}$ has an integer valued solution (x, y, z) (and one may check that $(3, 28, 93134)$ is such a solution) then the set of integers x, for which there are integers y, z with (x, y, z) a solution, is syndetic, i.e., has bounded gaps.

We will be taking Theorem A as a point of departure for an examination of several combinatorial notions of largeness together with their relationships to each other and to various parallel concepts in topological dynamics. Syndeticity is one such notion, and we invite the reader now to convince himself, as a motivational exercise indicative of the nature of such relationships, that given a compact metric space X, a minimal automorphism T of X (i.e. there exist no nontrivial closed, T-invariant sets $Y \subset X$), and a point x contained in an open set $U \subset X$, the set $\{n : T^n x \in U\}$ is syndetic. A converse holds as well: if for some $x \in X$ it is the case that the set of returns of x under T to any of its neighborhoods is syndetic, then T acts minimally on the orbit closure $\overline{\{T^n x : n \in \mathbf{Z}\}}$.

Another combinatorial notion of largeness, similar to but stronger than that of syndeticity, is that of being IP*. Briefly (more formal definitions will be offered below), a subset of an additive abelian group is said to be IP* if it intersects non-trivially the set of finite sums of any sequence in that group. We leave to the reader the easy exercise that in $\mathbf{Z}$, this is a strictly stronger property than that of syndeticity. (The crucial fact is that any set containing arbitrarily long intervals contains the set of finite sums of some sequence.) Like syndeticity, the IP* notion has connections with dynamics, this time dealing with the notion of *distality.* Unlike syndeticity, the IP* property is not shift invariant. The even numbers, for example, form an IP* subset of $\mathbf{Z}$, while the odd numbers do not.

Our interest here in the IP* property originates in regard to [FW], in which H. Furstenberg and B. Weiss established a result that upgrades in Theorem A (at least in the special case $m = 1$ and $\alpha_i < 0 < \beta_i$, with p_i vanishing at zero, $1 \leq i \leq n$) the syndeticity condition to the stronger condition of being IP*. A further extension, having a proof that is at once much simpler and yet perhaps less interesting in terms of connections to other areas of mathematics, was given recently in [BHM1]. It replaces syndeticity by the still stronger condition of being "VIP*." (A VIP* set intersects, in particular, any polynomial image of a set of finite sums of a sequence, however let us put off for the moment the complete and somewhat technical definition of VIP*, as it features very little in our presentation anyway.)

The curious fact serving as impetus for this investigation is that, although the arguments given in [VdC], [FW] and to a lesser extent [BHM1] are plainly distinct in their details, there are rather strong parallels in their general outlines, a fact made all the more remarkable in that to all appearances, the proofs arose independently. Each begins with an investigation of a (unique to it) property determining a corresponding family of functions defined modulo 1 having the property that the members of any finite sub-family of functions "recur" (in some unique sense) in concert. This property is then extended to real-valued functions on $\mathbf{Z}^m$, where an algebra of functions possessed of said property is constructed that is sufficiently "function-rich" to obtain the desired result.

Our primary intention here is to elucidate the similarities and differences between these various properties, especially those employed in [VdC] and [FW], in part by transferring them from the analytic/combinatorial language in which they were originally formulated into a language more familiar to the field of topological dynamics, particularly as regards the aforementioned phenomena of minimality and distality. The culminating observation is that the algebras of functions constructed in [VdC] and [FW], though quite differently defined, in fact coincide. What we have to say will have much in common with [F, Chapter 9], however the special nature of the systems dealt with here preclude, in most cases, quoting theorems from [F] outright; we choose therefore to keep our discussion entirely self-contained, inviting the interested reader to confer with various portions of [F] along the way.

In the final section, we do give two original theorems arising out of the (somewhat trickier, though much more elementary and consequently more efficient) methodology employed in [BHM1], and it does not appear to us that these results are readily obtainable from the methods of [VdC] or [FW]. Nevertheless, this paper is for the most part intended as a survey (not really of results, as we mention few, so much as of concepts and their interconnections), the principal thesis of which is that the proofs presented in [FW] and [VdC] are of a seriousness and depth poorly mirrored by the results achieved therein, and that, moreover, this may be fully appreciated only by carefully filling in the "invisible lines" connecting each to the other...lines passing innocuously through a seemingly unrelated field–namely that of topological dynamics. We can effectively argue this point, at least, if we assume following Hardy that the seriousness of a piece of mathematics may be measured in part by the extent to which it connects different *areas* of mathematics. And, to extrapolate only slightly on the British analyst's aesthetic, if we take the levels of abstraction to which one descends in the proof of a result to be a fair gauge of the depth of that *proof*–even if such a trek turns out to have been ultimately unnecessary for the attainment of the mere result in itself.

Presently we turn to details. For the sake of convenience, and because it makes comparison easier in the case of [FW], we will limit ourselves, in fact, to the case examined there. Thus the formulation we shall investigate is as follows:

Theorem B (respectively [VdC], [FW], [BHM1])**.** *Let $\epsilon > 0$ and let $n \in \mathbf{N}$. For each $k = 1, \ldots, n$, let p_k be a polynomial of k unknowns, vanishing at zero. Then the set of $x_1 \in \mathbf{Z}$ such that there exist $x_2, \ldots, x_{n+1} \in \mathbf{Z}$ satisfying the system of equations*

$$\begin{aligned} &|p_1(x_1) - x_2| < \epsilon \\ &|p_2(x_1, x_2) - x_3| < \epsilon \\ &|p_3(x_1, x_2, x_3) - x_4| < \epsilon \\ &\quad\vdots \\ &|p_n(x_1, ..., x_n) - x_{n+1}| < \epsilon \end{aligned} \tag{1}$$

is (respectively syndetic, IP, VIP*).*

2. The various properties

Let us consider the case of equations (1) when $n = 3$. We take $[x]$ to be the greatest integer less than or equal to x, which implies that $[x + \frac{1}{2}]$ is the (or a) closest integer to x. We also let $||x||$ be the distance from x to a nearest integer. Let

$$\begin{aligned} f_1(x) &= p_1(x) \\ f_2(x) &= p_2(x, [p_1(x) + \frac{1}{2}]), \\ f_3(x) &= p_3\Big(x, [p_1(x) + \frac{1}{2}], \big[p_2(x, [p_1(x) + \frac{1}{2}]) + \frac{1}{2}\big]\Big). \end{aligned}$$

Then f_1, f_2 and f_3 are all functions of a single variable x, vanishing at zero.

Suppose now that there is some property of real-valued functions on $\mathbf{Z}$, say $\mathcal{P}$, such that each of the functions f_i possesses $\mathcal{P}$, and such that $\mathcal{P}$ has the following "simultaneous recurrence to $\mathbf{Z}$" property: for any $\epsilon > 0$ and any finite family P of functions, all of which possess $\mathcal{P}$, the set $\{x \in \mathbf{Z} : ||p(x)|| < \epsilon \text{ for all } p \in \mathcal{P}\}$ is "large" (here by large we mean one of {syndetic, IP*, VIP*}). It would follow, in this case, that the set of x_1 for which $||f_i(x_1)|| < \epsilon$, $i = 1, 2, 3$, is large (in the same sense).

Moreover, assuming $\epsilon < \frac{1}{2}$, this implies that, letting

$$\begin{aligned} x_2 &= \big[p_1(x_1) + \frac{1}{2}\big], \\ x_3 &= \Big[p_2\big(x_1, [p_1(x_1) + \frac{1}{2}]\big) + \frac{1}{2}\Big], \\ x_4 &= \Bigg[p_3\Big(x_1, \big[p_1(x_1) + \frac{1}{2}\big], \big[p_2(x_1, [p_1(x_1) + \frac{1}{2}]) + \frac{1}{2}\big]\Big) + \frac{1}{2}\Bigg], \end{aligned}$$

that (x_1, x_2, x_3, x_4) is an integer solution to system (1). Showing the existence of such a property $\mathcal{P}$, then, provides a proof of Theorem B (at least in the $n = 3$ case, which is of course typical) for the corresponding notion of largeness.

To be somewhat more precise in what we require of $\mathcal{P}$, let us denote by $\mathcal{A}$ the smallest family of real valued functions on $\mathbf{Z}$ containing the identity function $x \to x$ and having the property that if $p(x_1, \ldots, x_n)$ is a real polynomial in n unknowns, vanishing at zero, and $f_1, \ldots, f_n$ are all members of $\mathcal{A}$, then the functions

$$x \to [f_1(x) + \frac{1}{2}] \quad \text{and} \quad x \to p\big(f_1(x), f_2(x), \ldots, f_n(x)\big)$$

are again in $\mathcal{A}$. The reader should check that $\mathcal{A}$ is well defined, that any member of $\mathcal{A}$ vanishes at zero, and that given one of our three notions of largeness it is sufficient, for a proof of the corresponding version of Theorem B, to show that (i) every member of $\mathcal{A}$ is possessed of $\mathcal{P}$, and (ii) $\mathcal{P}$ has the simultaneous recurrence to $\mathbf{Z}$ property (for the notion of largeness in question).

Van der Corput's notion of an absolutely rhythmic function.

First we introduce the version of $\mathcal{P}$ employed in the approach of van der Corput. A function $f : \mathbf{Z} \to [0, 1)$ is said to be *rhythmic* if for every $\epsilon > 0$ and $n, l \in \mathbf{N}$ there exists $N \in \mathbf{N}$ such that for any $R \in \mathbf{N}$ there is some k, with $R \leq k < R + N$, such that $||f(n + i) - f(k + i)|| < \epsilon$, $0 \leq i < l$. In other words, a function is rhythmic if every l consecutive values it achieves are syndetically approximated, in order, up to ϵ (mod 1).

A system of functions $\mathbf{Z} \to [0,1)$, $\{f_1, \dots, f_J\}$, $J \in \mathbf{N}$, is said to be rhythmic if for every $\epsilon > 0$ and $n, l \in \mathbf{N}$ there exists $N \in \mathbf{N}$ such that for any $R \in \mathbf{N}$ there is some k, with $R \le k < R + N$, such that $||f_v(n+i) - f_v(k+i)|| < \epsilon$, $0 \le i < l$, $1 \le v \le J$. So (f_v) is rhythmic if every l consecutive values of the J-tuple $(f_1, \dots, f_J)$ it achieves are syndetically approximated, in order, up to ϵ (mod 1).

Van der Corput defines a function $f : \mathbf{Z} \to [0,1)$ to be *absolutely rhythmic* if any rhythmic system remains so upon inclusion of f. That is, if for every rhythmic system $\{f_1, \dots, f_J\}$, the system $\{f_1, \dots, f_J, f\}$ is again rhythmic. Finally, a function $f : \mathbf{Z} \to \mathbf{R}$ is defined to be (absolutely) rhythmic if $n \to \langle f(n) \rangle$ is (absolutely) rhythmic, where $\langle x \rangle = x - [x]$ denotes the fractional part of x.

One cannot help but admire the sensibility of van der Corput's approach, as it follows now immediately that any system composed of absolutely rhythmic functions is rhythmic. In particular, absolutely rhythmic functions vanishing at zero will have the simultaneous recurrence to $\mathbf{Z}$ property for the syndeticity notion of largeness. This, then, is the property $\mathcal{P}$ employed in this case: that of being an absolutely rhythmic function vanishing at zero.

The Furstenberg-Weiss notion of an IP-recurrent function.

We now move to the version of $\mathcal{P}$ used by Furstenberg and Weiss. Let $\mathcal{F}$ denote the family of finite, non-empty subsets of $\mathbf{N}$. An IP system is a function n from $\mathcal{F}$ into an additive abelian semigroup G satisfy $n(\alpha \cup \beta) = n(\alpha) + n(\beta)$ whenever $\alpha \cap \beta = \emptyset$. An IP* set is a set that intersects the range of any IP system non-trivially. It is an easy exercise that any IP* set is syndetic. (So, in particular, the IP* version of Theorem B is stronger than the syndetic version.)

Equivalently, $n : \mathcal{F} \to G$ is an IP system if there exists a sequence (x_i) in G such that $n(\alpha) = \sum_{i \in \alpha} x_i$. An IP system n may be thought of as a homomorphism from the "partial semigroup" $(\mathcal{F}, \cup)$ (partial in that one only allows disjoint unions, that is, the operation is not necessarily defined on all of $\mathcal{F} \times \mathcal{F}$, yet is still associative whenever it is defined) to G.

Indeed, the target set G need not even necessarily be a semigroup; it may be a partial semigroup. Thus we may speak of an IP system $\tau : \mathcal{F} \to \mathcal{F}$. In this case one requires that if $\alpha \cap \beta = \emptyset$ then $n(\alpha \cup \beta) = n(\alpha) \cup n(\beta)$, so that in particular (in order that the latter be defined), $n(\alpha) \cap n(\beta) = \emptyset$. τ is therefore invertible, and the partial subsemigroup $\mathcal{F}^{(1)} = \tau(\mathcal{F})$, having structure isomorphic to that of $\mathcal{F}$, will be referred to as an $\mathcal{F}$-ring.

If $x : \mathcal{F} \to X$ is a function, where X is a metric space, and $\mathcal{F}^{(1)}$ is an $\mathcal{F}$-ring, we write $\operatorname{IP-lim}_{\alpha \in \mathcal{F}^{(1)}} x(\alpha) = x$ if for every neighborhood U of x there exists $\beta \in \mathcal{F}$ such that $x(\alpha) \in U$ for every $\alpha \in \mathcal{F}^{(1)}$ satisfying $\alpha \cap \beta = \emptyset$. Hindman's theorem [H], which states that for every finite partition of $\mathcal{F}$ there exists an $\mathcal{F}$-ring $\mathcal{F}^{(1)}$ in one cell, immediate implies that such an $\mathcal{F}^{(1)}$ and x may be found when X is finite, and it is an easy exercise to see that this is again so when X is compact.

A function f from $\mathbf{Z}$ to a compact metric space X is said to be *IP recurrent* (and we write $f \in IPR$) if for any IP system p in $\mathbf{Z}$ there exists an $\mathcal{F}$-ring $\mathcal{F}^{(1)}$ such that $\operatorname{IP-lim}_{\alpha \in \mathcal{F}^{(1)}} f\big(n + p(\alpha)\big) = f(n)$ for all $n \in \mathbf{Z}$. One immediately sees, by passing successively to a nested sequence of $\mathcal{F}$-rings, that if $f_1, \ldots, f_J$ are IP recurrent then for any IP system p in $\mathbf{Z}$ there exists an $\mathcal{F}$-ring $\mathcal{F}^{(1)}$ such that $\operatorname{IP-lim}_{\alpha \in \mathcal{F}^{(1)}} f_i\big(n + p(\alpha)\big) = f(n)$ for all $n \in \mathbf{Z}$ and all $i = 1, \ldots, J$ simultaneously.

If $f : \mathbf{Z} \to \mathbf{R}$, we write $f \in LIPR$ if for every real λ, $g(n) = \exp(2\pi i \lambda f(n))$ is IPR on the unit circle of $\mathbf{C}$. It follows therefore, by the simultaneous recurrence phenomenon expounded upon in the previous paragraph, that LIPR functions vanishing at zero will have the simultaneous recurrence to $\mathbf{Z}$ property for the IP* notion of largeness. This is the property $\mathcal{P}$ employed in [FW]: that of being an LIPR function vanishing at zero.

Proto-VIP functions.

Finally we move to the approach of [BHM1]. Let $d \in \mathbf{N}$ and suppose $v : \mathcal{F} \to G$ is a function, where G is an abelian group. If for every pairwise disjoint $\alpha_0, \alpha_1, \ldots, \alpha_d \in \mathcal{F}$ one has

$$\sum_{\emptyset \neq B \subset \{\alpha_0, \ldots, \alpha_d\}} (-1)^{|B|} v\Big(\bigcup_{\alpha \in B} \alpha\Big) = 0, \tag{2}$$

then v is said to be a VIP system, and the least d satisfying this condition is called the *degree* of the system. A set intersecting non-trivially the range of any VIP system is therefore identified as a VIP* set. Since any IP system is a VIP system (of degree 1), any VIP* set is in particular IP*. Hence the VIP* version of Theorem B is stronger than the IP* version.

One may check that, in a commutative ring, if p is a polynomial of degree d vanishing at zero and n is an IP system, then $v(\alpha) = p\big(n(\alpha)\big)$ will be a VIP system of degree at most d (we remark that this is not nearly the general case, even for $\mathbf{Z}$).

One particularly nice property of VIP systems in compact topological groups (the operation $(x, y) \to xy$ being continuous) is that their IP limits must exist along appropriately chosen IP rings, and must also invariably be equal to the identity of the group (this is just a consequence of equation (2), as there is one more negative term than positive term in the sum). Indeed, it is this observation that motivates the following definition.

A real valued function f on $\mathbf{Z}$ is said to be a *proto-VIP function* if for any VIP system $v : \mathcal{F} \to \mathbf{Z}$ there exists an $\mathcal{F}$-ring $\mathcal{F}^{(1)}$ such that $u : \mathcal{F}^{(1)} \to \mathbf{R}$ defined by $u(\alpha) = f\big(v(\alpha)\big)$ is VIP. Since the identically zero function is the only constant that is VIP, one sees in particular that proto-VIP functions must vanish at zero. Moreover, having passed to an $\mathcal{F}$-ring along which $f\big(v(\alpha)\big)$ is VIP, one immediately sees that $\exp 2\pi i f(v(\alpha))$ is a (multiplicative) VIP system on the unit circle, which means that in passing to a sub-ring $\mathcal{F}^{(1)}$, one may assume that $\text{IP-}\lim_{\alpha \in \mathcal{F}^{(1)}} \exp 2\pi i f(v(\alpha))$ exists and must therefore equal 1 (the identity of the group). This in particular implies that the proto-VIP property has the simultaneous recurrence to $\mathbf{Z}$ property. This, then, is the version of $\mathcal{P}$ used in [BHM1]; that of being proto-VIP.

3. Relationships between the various properties.

Let (X, d) be a compact metric space and suppose $T : X \to X$ is a homeomorphism. If the only closed, T-invariant subsets of X are $\emptyset$ and X, the system (X, T) is said to be minimal. In the general case, if $f \in X$ then the orbit closure $\overline{\{T^n f : n \in \mathbf{Z}\}}$ is a T-invariant subset of X. If the subsystem $\big(\overline{\{T^n f : n \in \mathbf{Z}\}}, T\big)$ is minimal, f is called a *minimal point*. It is well known that f is a minimal point if and only if f is *uniformly recurrent*, that is, if for every neighborhood U of f, the set of k with $T^k f \in U$ is syndetic.

Two points $x, y \in X$ are said to be *proximal* if there exists a sequence (n_k) in $\mathbf{Z}$ such that $d(T^{n_k}x, T^{n_k}y) \to 0$. A system (X, T) for which there exist no two distinct points that are proximal to each other is said to be *distal*. In the general case, a point f not proximal to any other point in its orbit closure, that is, not proximal to any y with $f \neq y \in \overline{\{T^n f : n \in \mathbf{Z}\}}$, is called a *distal point*.

We apply these notions in the compact metric space $X = [0,1)^{\mathbf{Z}}$ (product topology), where the topology on $[0,1)$ is that which identifies 0 with 1. That is, comes from the metric $d(x, y) = ||x - y||$. (With this topology, of course, $[0,1)$ is a topological group, the so-called 1-torus, isomorphic to the unit circle of $\mathbf{C}$.) We let $T : X \to X$ be the shift homeomorphism, defined by $Tf(n) = f(n+1)$.

The following proposition, while elementary and indeed purely classical in conception, is crucial to our entire approach.

Proposition 3.1. *Let $f \in X$. Then f is rhythmic if and only if f is a minimal point under the shift.*

Proof. First suppose that f is rhythmic. We must show that $\overline{\{T^n f : n \in \mathbf{Z}\}}$ is minimal. Suppose not. Then there exists a closed, nontrivial, T-invariant subset $Y \subset \overline{\{T^n f : n \in \mathbf{Z}\}}$

Let $y \in Y$. Since f is not in Y, there exists some neighborhood U of f not intersecting Y. This implies the existence of some $\epsilon > 0$ and $N \in \mathbf{N}$ with $Y \cap \{g \in X : ||g(n) - f(n)|| < \epsilon, -N \leq n \leq N\} = \emptyset$.

Since f is rhythmic, there exists $L \in \mathbf{N}$ such that for all $R \in \mathbf{Z}$ there exists k, with $R \leq k < R+L$, satisfying $||f(k+i) - f(i)|| < \frac{\epsilon}{2}$, $-N \leq i \leq N$. Choose now a neighborhood V of y having the property that for all $h \in V$, $||y(n) - h(n)|| < \frac{\epsilon}{2}$, $-N \leq n \leq L + N$.

Since $y \in \overline{\{T^n f : n \in \mathbf{Z}\}}$, we may choose $R \in \mathbf{Z}$ with $T^R f \in V$, which implies that $||y(n) - f(R+n)|| < \frac{\epsilon}{2}$, $-N \leq n \leq L + N$. Choose now k, with $R \leq k < R + L$, with $||f(k+i) - f(i)|| < \frac{\epsilon}{2}$, $-N \leq i \leq N$. Then for $-N \leq i \leq N$ one has $-N \leq k - R + i \leq L + N$, so that

$$||T^{k-R}y(i) - f(i)|| \leq ||y(k - R + i) - f\big(R + (k - R + i)\big)|| + \cdots$$
$$\cdots + ||f(k+i) - f(i)|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is a contradiction, as $T^{k-R}y \in Y$. Next suppose that f is a minimal point, which implies in particular that every point in its orbit closure is uniformly recurrent. We must show that f is rhythmic.

Suppose not. Then there exist $\epsilon > 0$ and $n, l \in \mathbf{N}$ such that for every $N \in \mathbf{N}$ one may find $R \in \mathbf{N}$ so that for all k with $R \leq k < R+N$, there is some i with $0 \leq i < l$ satisfying $||f(n+i) - f(k+i)|| \geq \epsilon$, which is to say that $||T^n f(i) - T^k(i)|| \geq \epsilon$. In other words, if we let $V = \{g \in X : ||g(i) - T^n f(i)|| < \epsilon, 0 \leq i < l\}$, the set of k for which $T^k f \in V$ fails to be syndetic. This contradicts the fact that $T^n f$ is a uniformly recurrent point. □

One may check that the proof above makes no explicit reference to the target space $[0,1)$. Suppose then that $\{f_1, \ldots, f_J\}$ is a system of functions $\mathbf{Z} \to [0,1)$. We may realize this system as a J-tuple of members of X, that is, a member of X^J, or, as is more convenient for us, as a member of the isomorphic space $X_J = \big([0,1)^J\big)^{\mathbf{Z}}$. The following is now an immediate corollary of the proof of Proposition 3.1.

Corollary 3.2. *Let $S = \{f_1, \ldots, f_J\}$ be a system of functions $\mathbf{Z} \to [0,1)$. Then S is rhythmic if and only if the function $\pi(n) = \big(f_1(n), \ldots, f_J(n)\big)$ is a minimal point of the space $X_J = \big([0,1)^J\big)^{\mathbf{Z}}$ under the shift.*

The following lemma prepares us for the demonstration of certain equivalences to come. Although we do not use its entire strength, we include the complete proof, in part because it involves a technique we shall mimic later.

Lemma 3.3 (cf. [F, Lemmas 8.15, 9.10]). *If (Y, T) is a minimal system and $f \in Y$ is proximal to $g \in Y$, then there exists an IP system $n : \mathcal{F} \to \mathbf{N}$ such that $\underset{\alpha \in \mathcal{F}}{\text{IP-lim}}\ T^{n(\alpha)} f \to g$ and $\underset{\alpha \in \mathcal{F}}{\text{IP-lim}}\ T^{n(\alpha)} g \to g$ simultaneously.*

Proof. We inductively construct a sequence (m_i) in $\mathbf{N}$ and a sequence (ϵ_i) of positive reals with $\epsilon_i \leq 2^{-i}$ such that for all $\alpha \in \mathcal{F}$, upon writing $m_\alpha = \sum_{i \in \alpha} m_i$ one has $d(T^{m_\alpha} f, g) < \epsilon_{\min \alpha}$ and $d(T^{m_\alpha} g, g) < \epsilon_{\min \alpha}$. This will plainly suffice for the proof, letting $n(\alpha) = m_\alpha$. Suppose we have chosen $m_1, \ldots, m_{n-1}$ and $\epsilon_1, \ldots, \epsilon_{n-1}$ consistent with these goals, where $n \in \mathbf{N}$. If $n = 1$ let $\epsilon_n = \frac{1}{2}$, otherwise choose $\epsilon_n < \frac{\epsilon_{n-1}}{2}$ so small that if $d(x, g) < \epsilon_n$ then for all nonempty $\alpha \subset \{1, 2, \ldots, n-1\}$, $d(T^{m_\alpha} x, T^{m_\alpha} g) < \epsilon_{\min \alpha} - d(T^{m_\alpha} g, g)$.

Next choose M such that for all $k \in \mathbf{Z}$ there exists m, with $k \leq m < k+M$, satisfying $d(T^m g, g) < \frac{\epsilon_n}{2}$. Choose $\delta > 0$ such that if $d(x,y) < \delta$ then $d(T^i x, T^i y) < \frac{\epsilon_n}{2}$, $0 \leq i < M$. Since f and g are proximal, there exists $k \in \mathbf{Z}$ such that $d(T^k f, T^k g) < \delta$. Choose now m_n with $k \leq m_n < k+M$ satisfying $d(T^{m_n} g, g) < \frac{\epsilon_n}{2}$. Note also that $d(T^{m_n} f, T^{m_n} g) < \frac{\epsilon_n}{2}$, so that $d(T^{m_n} f, g) < \epsilon_n$.

Now suppose $\emptyset \neq \alpha \subset \{1, \ldots, n\}$. We must show that $d(T^{m_\alpha} f, g) < \epsilon_{\min \alpha}$ and $d(T^{m_\alpha} g, g) < \epsilon_{\min \alpha}$. If $\alpha = \{n\}$ or $\alpha \subset \{1, \ldots, n-1\}$ this is already shown, so suppose $n \in \alpha$ and assume $\beta = \alpha \setminus \{n\}$ is non-empty. Then

$$\begin{aligned} d(T^{m_\alpha} f, g) &= d(T^{m_\beta} T^{m_n} f, g) \\ &\leq d(T^{m_\beta} T^{m_n} f, T^{m_\beta} g) + d(T^{m_\beta} g, g) \\ &< \big(\epsilon_{\min \beta} - d(T^{m_\beta} g, g)\big) + d(T^{m_\beta} g, g) = \epsilon_{\min \beta} = \epsilon_{\min \alpha} \end{aligned}$$

and

$$\begin{aligned} d(T^{m_\alpha} g, g) &= d(T^{m_\beta} T^{m_n} g, g) \\ &\leq d(T^{m_\beta} T^{m_n} g, T^{m_\beta} g) + d(T^{m_\beta} g, g) \\ &< \big(\epsilon_{\min \beta} - d(T^{m_\beta} g, g)\big) + d(T^{m_\beta} g, g) = \epsilon_{\min \beta} = \epsilon_{\min \alpha}. \end{aligned}$$

□

The following is the main result of this section. It states that, for functions into the 1-torus $[0,1)$, absolute rhythmicity and IP recurrence are the same phenomenon.

Theorem 3.4 (cf. [F, Theorem 9.11]). *Let $f \in X$. The following are equivalent:*

(a) f is absolutely rhythmic.
(b) f is a distal point.
(c) f is IP recurrent.

Proof. (c)→(b): Suppose f is IP recurrent. Then in particular the set of returns of f under T to any of its neighborhoods is IP*, so in particular is syndetic. Hence f is uniformly recurrent and the orbit closure of f is miminal. If there is some g in the orbit closure of f, with f and g proximal, then by Lemma 3.3 there exists an IP system n with $\text{IP-}\lim_{\alpha \in \mathcal{F}} T^{n(\alpha)} f \to g$. On the other hand, since f is IP recurrent, by passing to a subsystem we may ensure that $\text{IP-}\lim_{\alpha \in \mathcal{F}} T^{n(\alpha)} f = f$. Therefore $f = g$ and f is a distal point.

(b)→(c): Suppose that f is a uniformly recurrent distal point. Suppose further that f fails to be IP recurrent. Then there exists some IP system $n : \mathcal{F} \to \mathbf{Z}$ such that $\text{IP-}\lim_{\alpha\in\mathcal{F}} T^{n(\alpha)}f = g \neq f$. By passing to a subring we may in fact assume that $\text{IP-}\lim_{\alpha\in\mathcal{F}} T^{n(\alpha)}g = h$ exists as well. By distality, we have $g \neq h$. Let W be a closed neighborhood of h not containing g. Pass now to a subring $\mathcal{F}^{(1)}$ for which $T^{n(\alpha)}f$ is never in W, $\alpha \in \mathcal{F}^{(1)}$. However, for some $\alpha \in \mathcal{F}^{(1)}$, we will have $T^{n(\alpha)}g \in W$. Let U be a neighborhood of g with $T^{n(\alpha)}U \subset W$. For some $\beta \in \mathcal{F}^{(1)}$ disjoint from α, we will have $T^{n(\beta)}f \in U$, which implies that $T^{n(\alpha\cup\beta)}f \in W$, a contradiction.

(a)→(b): Let f be absolutely rhythmic. Suppose that f is proximal to some g in its orbit closure. We must show that $f = g$. As the orbit closure of f is minimal and contains g, g is itself uniformly recurrent and hence rhythmic by Proposition 3.1. It follows that the system $\{f, g\}$ is rhythmic, from which one may conclude by Corollary 3.2 that the function $\pi(n) = \big(f(n), g(n)\big)$ has minimal orbit closure in X_2.

We may now use compactness together with the fact that f and g are proximal to find a sequence (n_i) with $d(T^{n_i}f, T^{n_i}g) \to 0$ and $T^{n_i}f \to \gamma$ for some $\gamma \in X$. Then clearly $T^{n_i}g \to \gamma$ as well, which implies that $T^{n_i}\pi \to \Gamma \in X_2$, where $\Gamma(n) = \big(\gamma(n), \gamma(n)\big)$. Hence Γ lies in the orbit closure of π, and since this orbit closure is minimal, π must also lie in the orbit closure of Γ. But this obviously necessitates $f = g$.

(b)→(a): Suppose f is distal. Then f is IP recurrent, as was shown above. We must show that f is absolutely rhythmic. Accordingly, let $\{f_1, \ldots, f_J\}$ be a rhythmic system. Then by Corollary 3.2, $\pi(n) = \big(f_1(n), \ldots, f_J(n)\big)$ defines a minimal point of X_J under the shift. We must show that $\{f_1, \ldots, f_J, f\}$ is a rhythmic system. Again by Corollary 3.2, it suffices to show that $\tau(n) = \big(f_1(n), \ldots, f_J(n), f(n)\big)$ defines a minimal point of X_{J+1} under the shift.

Let Y be a closed subset of the orbit closure of τ. We inductively construct sequences (m_i) of natural numbers and (ϵ_i) of positive reals with $\epsilon_i \leq 2^{-i}$ such that for $\alpha \in \mathcal{F}$, writing $m_\alpha = \sum_{i\in\alpha} m_i$, one has $d(T^{m_\alpha}\tau, Y) < \epsilon_{\min\alpha}$.

Accordingly, let $y \in Y$ and suppose we have chosen $m_1, \dots, m_{n-1}$, $\epsilon_1, \dots, \epsilon_{n-1}$ consistent with these goals, where $n \in \mathbf{N}$. If $n = 1$ let $\epsilon_n = \frac{1}{2}$, otherwise choose $\epsilon_n < \frac{\epsilon_{n-1}}{2}$ so small that if $d(x, y) < \epsilon_n$ then $d(T^{m_\beta}x, T^{m_\beta}y) < \epsilon_{\min\beta}$ for all non-empty $\beta \subset \{1, \dots, n-1\}$. Then choose m_n such that $d(T^{m_n}\tau, y) < \epsilon_n$.

We must show that for every non-empty $\alpha \subset \{1, \dots, n\}$ one has $d(T^{m_\alpha}\tau, Y) < \epsilon_{\min\alpha}$. If $\alpha = \{n\}$ or $n \notin \alpha$, this has been shown, so we assume that $n \in \alpha$ and that $\beta = \alpha \setminus \{n\}$ is non-empty. We then have $d(T^{m_\alpha}\tau, Y) \leq d(T^{m_\beta}T^{m_n}\tau, T^{m_\beta}y) < \epsilon_{\min\beta} = \epsilon_{\min\alpha}$.

Letting now $n(\alpha) = m_\alpha$, n is an IP system, and by passing to a subring $\mathcal{F}^{(1)}$ we may assume that $\text{IP-}\lim_{\alpha\in\mathcal{F}^{(1)}} T^{n(\alpha)}\tau = \omega$ exists. Moreover, $\omega \in Y$ necessarily. Passing to a sub-ring again, we may assume without loss of generality that $\text{IP-}\lim_{\alpha\in\mathcal{F}^{(1)}} T^{n(\alpha)}\omega = h$ exists. Moreover (cf. the proof of (b)$\rightarrow$(c)), $\omega = h$, which implies in particular that τ and ω are proximal.

Recall that $\tau(n) = \big(\pi(n), f(n)\big)$. Similarly, write $\omega(n) = \big(\kappa(n), g(n)\big)$. Then in particular, g and f are proximal, so by hypothesis, $g = f$. Also κ and π are proximal in X_J, with κ in the orbit closure of π, which is minimal. Thus by Lemma 3.2 there exists an IP system $r : \mathcal{F} \rightarrow \mathbf{N}$ such that $\text{IP-}\lim_{\alpha\in\mathcal{F}} T^{r(n)}\kappa = \pi$. Plainly then $\text{IP-}\lim_{\alpha\in\mathcal{F}} T^{r(n)}\omega = \tau$, which in particular implies that τ is in the orbit closure of the minimal point ω. Hence τ is a minimal point. □

What the previous theorem did for functions into $[0, 1)$, the following immediate corollary does for functions into $\mathbf{R}$.

Corollary 3.5. *Let* $f : \mathbf{Z} \rightarrow \mathbf{R}$.

a. f is absolutely rhythmic if and only if $\langle f \rangle \in X$ is IP recurrent.

b. f is LIPR if and only if for all real λ, λf is absolutely rhythmic.

We mention in closing this section that there is no inclusion relation between the proto-VIP notion and the notions of rhythmicity and/or IP recurrence. Indeed, if f is proto-VIP and g agrees with f on a VIP* set E, g must be proto-VIP as well, though clearly, with complete freedom on E^c, one can easily make g pathological enough not to achieve even rhythmicity. On the other hand one may easily choose absolutely rhythmic or IP recurrent functions f, vanishing at zero, for which $\{f(n) : 0 \neq n \in \mathbf{Z}\}$ is a rationally independent set, which plainly precludes f being proto-VIP.

4. The algebras R and $\mathcal{L}$

Having defined his notion of absolute rhythmicity, van der Corput proceeds to construct an algebra R of absolutely rhythmic functions containing, in particular, the family $\mathcal{A}$ of the introduction. This provides, of course, a proof of the syndetic version of Theorem B. Similarly, Furstenberg and Weiss construct an algebra $\mathcal{L}$ of LIPR functions containing $\mathcal{A}$, thus providing a proof of the IP* version of Theorem B.

For a function $f : \mathbf{Z} \to \mathbf{R}$, we put $\Delta f(x) = f(x+1) - f(x)$. The respective definitions follow.

Definition 4.1. Let R be the smallest set of functions $\mathbf{Z} \to \mathbf{R}$ having the following properties.

(a) $0 \in R$.

(b) If $f, g \in R$ then $(f + g) \in R$.

(c) If φ is a bounded function $\mathbf{Z} \to \mathbf{Z}$ such that for all real c, $c\varphi$ is absolutely rhythmic, and $f \in R$, then $\varphi f \in R$.

(d) If $\Delta f \in R$ then $f \in R$.

Definition 4.2. Let $\mathcal{L}$ be the smallest set of functions $\mathbf{Z} \to \mathbf{R}$ having the following properties.

(i) The constants are in $\mathcal{L}$,

(ii) If $f, g \in \mathcal{L}$ then $(f + g) \in \mathcal{L}$.

(iii) If φ is a finite-valued IPR function $\mathbf{Z} \to \mathbf{R}$ and $f \in R$, then $\varphi f \in \mathcal{L}$.

(iv) If $\Delta f \in \mathcal{L}$ then $f \in \mathcal{L}$.

Here at last is the equivalence we have been working toward.

Theorem 4.3. $\mathcal{L} = R$.

Proof. It is obvious that any IPR function $f : \mathbf{Z} \to \mathbf{R}$ is LIPR, for if $T^{n(\alpha)} f \to f$ pointwise in $\mathbf{R}$, then trivially $\exp\bigl(2\pi i \lambda f(k+n(\alpha))\bigr) \to \exp\bigl(2\pi i \lambda f(k)\bigr)$ for all k. A partial converse holds: any bounded LIPR function g is IPR. To see this, pick $\lambda > 0$ small enough that $f(n) = \frac{1}{2} + \lambda g(n)$ satisfies $\frac{1}{4} \leq f(n) \leq \frac{3}{4}$ for all n. f is IP recurrent on the torus $[0, 1)$ by definition, but since $f(n)$ is bounded away from 0 and 1 it is easily seen that f is actually IP recurrent on $\mathbf{R}$, from which it follows that $g(n) = \bigl(f(n) - \frac{1}{2}\bigr)\lambda^{-1}$ is IPR as well.

Hence one sees quite easily that $\mathcal{L}$ meets the axioms of R. In particular, it meets axiom (c) by virtue of the fact that the functions φ appearing there are LIPR by Corollary 3.5 b. and finite valued, hence IPR–now apply axiom (iii). We may thus conclude that $R \subset \mathcal{L}$.

That $\mathcal{L} \subset R$ is not quite so apparent. R certainly meets axioms (ii) and (iv), as these are just (b) and (d). It meets axiom (i) by (a) and (d). As for (iii), let φ be a finite valued IPR function $\mathbf{Z} \to \mathbf{R}$ and let $f \in \mathcal{L}$. We must show that $\varphi f \in \mathcal{L}$.

Write $\varphi(n) = \sum_{i=1}^{r} c_i 1_{E_i}(n)$, where $\{E_i : 1 \leq i \leq r\}$ is a partition of $\mathbf{N}$ and the c_i are real numbers. From that fact that φ is IPR it follows easily that each 1_{E_i} is IPR, hence LIPR, which implies that $c1_{E_i}$ is absolutely rhythmic for every real c. Moreover 1_{E_i} is integer valued. If only we knew that $c_i f \in R$ for each i, we would have by axiom (c) that $f c_i 1_{E_i} \in R$ by (c) and thus $\varphi f = \sum_{i=1}^{r} f c_i 1_{E_i} \in R$ by (b). Unfortunately that $cf \in R$ for all real c is not immediate from the axioms; it is true, however (R is in fact an algebra), and may be proved in the following way.

Let R_0 consist of the zero function, and let R_1 consist of all the constant functions. Having defined R_{n-1}, let R_n consist of all those functions $f : \mathbf{Z} \to \mathbf{R}$ such that one of the following holds:

1. $f(n) = g(n) + h(n)$, where $g, h \in R_{n-1}$.
2. $f(n) = \varphi(n)g(n)$, where φ is a bounded function $\mathbf{Z} \to \mathbf{Z}$ such that for all real c, $c\varphi$ is absolutely rhythmic, and $g \in R_{n-1}$.
3. $\Delta f(n) \in R_{n-1}$,

Plainly, $R = \bigcup_{n=0}^{\infty} R_n$. One may now easily show, by induction on n, that if $f \in R_n$ and $c \in \mathbf{R}$ then $cf \in R_n$. □

And so, van der Corput and Furstenberg-Weiss arrive, via different paths, at the very same algebra of functions. The proofs are accordingly of approximately the same depth, and it is perhaps only because of their different formulation that Furstenberg and Weiss recognized the stronger IP*property, which was surely not even looked for by van der Corput.

The work is hardly done at this stage, of course. One must still show that the necessary functions (eg. the algebra $\mathcal{A}$ of the introduction) belong to this algebra, and for that matter that the members of the algebra are in fact possessed of the appropriate property (are LIPR or absolutely rhythmic). These are by no means trivial concerns.

The reward for this difficult work seems to be that the algebra obtained is actually much richer than is needed for the result at hand; unfortunately, it is not entirely clear how one might put this added strength to use. Perhaps one might allow p_1, in the formulation of Theorem B, to be any member of R, and to choose each p_k, $k = 2, \ldots, n$, having the property that for any functions $f_1, \ldots, f_k \in R$, the composition $x \to p_k\big(f_1(x), \ldots, f_k(x)\big)$ lies in R. But these are aesthetically awkward conditions.

5. Advantages of the proto-VIP method; two new results

We have seen how, in pursuit of proofs of two versions of Theorem B, convergent paths of considerable sophistication and depth were taken by remarkably talented mathematicians to arrive at what is, in essence, the same solution. By comparison, the proof via the proto-VIP property is remarkably straightforward; it seems almost paradoxical that the result achieved, in spite of this consideration, fully generalizes the others. Moreover, we intend to compound this apparent paradox by offering in this section further evidence of the strength of the proto-VIP methodology in the form of two further extensions of Theorem B that can be achieved with minimal additional effort.

Perhaps, though, a disclaimer is in order. It is the feeling of the author that at least some of this relative strength is illusary; as mentioned at the end of the previous section, the algebra $R/\mathcal{L}$ achieved does not yield itself especially well to a reformulation of Theorem B that would utilize its full strength. This is the sense in which we say that the results achieved in [VdC] and [FW] do not fully reflect the quality of the mathematics utilized in achieving them. The methods of [BHM1], by contrast, are more "clever" than "sophisticated." They are "tricky" rather than "deep." Their advantage lies in their efficiency. Indeed, the algebra of proto-VIP functions constructed there is rather sparse, consisting of essentially all that is necessary and nothing more; the so-called *admissible generalized polynomials.* Briefly, these include the smallest set $\mathcal{S}$ of functions $\mathbf{Z} \to R$ such that:

(1) The identity function $n \to n$ is in $\mathcal{S}$.
(2) If $f, g \in \mathcal{S}$ and $c \in \mathbf{R}$ then $\{cf, f+g, fg\} \subset \mathcal{S}$.
(3) If $f \in \mathcal{S}$ and $0 < k < 1$ then $n \to [f(n) + k]$ is in $\mathcal{S}$.

These clearly include the set $\mathcal{A}$ of the introduction, and it is a surprisingly trivial matter to prove, inductively, that they are proto-VIP. Another eight lines or so is all it takes to establish that proto-VIP functions on $\mathbf{R}$ recur simultaneously to $\mathbf{Z}$, and that essentially completes the proof of the VIP* version of Theorem B.

Two refinements.

The aesthetic disadvantages regarding absolutely rhythmic functions cited at the end of the previous section do not apply to proto-VIP functions. Indeed, it is a simple and natural matter to define the notion for real-valued functions on $\mathbf{Z}^k$...or for that matter between arbitrary abelian groups: if $f : G \to H$, then f is proto-VIP if for every VIP system $v : \mathcal{F} \to G$, there exists an $\mathcal{F}$-ring $\mathcal{F}^{(1)}$ such that $f \circ v$, restricted to $\mathcal{F}^{(1)}$, is a VIP system into H. Moreover, it is immediate that the composition of proto-VIP functions is again proto-VIP. The following extension of the VIP* version of Theorem B, then, is immediate.

Theorem 5.1. *Let $\epsilon > 0$ and let $m \in \mathbf{N}$. For each $k = 1, \ldots, n$, let p_k be a proto-VIP function $\mathbf{Z}^k \to \mathbf{R}$. Then the set of x_1 such that there exist $x_2, \ldots, x_{n+1}$ satisfying the system of equations*

$$\begin{aligned} &|p_1(x_1) - x_2| < \epsilon \\ &|p_2(x_1, x_2) - x_3| < \epsilon \\ &|p_3(x_1, x_2, x_3) - x_4| < \epsilon \\ &\quad\vdots \\ &|p_n(x_1, ..., x_n) - x_{n+1}| < \epsilon \end{aligned} \tag{1}$$

is VIP*.

For our next application, which is the primary original result of this paper, we introduce a related notion, that of an *FVIP-system.* A function $v : \mathcal{F} \to G$, where G is an additive abelian group, is said to be a simple FVIP system if there exist $d \in \mathbf{N}$ and sequences $(y_i)_{i=1}^\infty \subset G$, $(n_i^{(j)})_{i=1}^\infty \subset \mathbf{Z}$ such that

$$v(\alpha) = \sum_{\gamma \subset \alpha, |\gamma| = d} n_{i_1}^{(1)} n_{i_2}^{(2)} \cdots n_{i_{d-1}}^{(d-1)} y_{i_d}.$$

An FVIP system is then the sum of finitely many simple FVIP systems.

FVIP systems are VIP systems in particular, but of a special form that makes them useful for questions related to recurrence in ergodic theory and hence in density Ramsey theory. To wit:

Theorem 5.2 ([BHM2]). *Let $A \subset \mathbf{Z}$ be of positive upper Banach density. That is, assume that*

$$d^*(A) = \limsup_{N-M\to\infty} \frac{|A\cap\{M+1,\ldots,N\}|}{N-M} > 0.$$

Then for any FVIP system $v : \mathcal{F} \to \mathbf{Z}$, there exists an $\mathcal{F}$-ring $\mathcal{F}^{(1)}$ such that $\{v(\alpha) : \alpha \in \mathcal{F}^{(1)}\} \subset A - A$.

FVIP systems are of value to us here for another reason as well; one can easily define a notion of a proto-FVIP function by analogy with the notion of proto-VIP. Namely, a function $f : G \to H$ of abelian groups is proto-FVIP if for any FVIP system $v : \mathcal{F} \to G$ there exists an $\mathcal{F}$-ring $\mathcal{F}^{(1)}$ such that $f \circ v : \mathcal{F}^{(1)} \to H$ is FVIP. We now have:

Theorem 5.3 ([BHM2]). *Any admissible generalized polynomial is proto-FVIP.*

We can now give the following extension of Theorem B.

Theorem 5.4. *Let $\epsilon > 0$, let $m \in \mathbf{N}$ and suppose $A \subset \mathbf{Z}$ with $d^*(A) > 0$. For each $k = 1, \ldots, n$, let p_k be a polynomial of k unknowns, vanishing at zero. Then the system of equations*

$$\begin{aligned} &|p_1(x_1) - x_2| < \epsilon \\ &|p_2(x_1, x_2) - x_3| < \epsilon \\ &|p_3(x_1, x_2, x_3) - x_4| < \epsilon \\ &\qquad \vdots \\ &|p_n(x_1, ..., x_n) - x_{n+1}| < \epsilon \end{aligned}$$

has a solution in integers, with $\{x_1, \ldots, x_{n+1}\} \subset A - A$. Moreover, x_1 may be chosen from the range of an arbitrary FVIP system.

Proof. Let

$$\begin{aligned} f_1(x) &= p_1(x) \\ f_2(x) &= p_2(x, [f_1(x) + \frac{1}{2}]), \\ f_3(x) &= p_3(x, [f_1(x) + \frac{1}{2}], [f_2(x) + \frac{1}{2}]) \\ &\vdots \\ f_n(x) &= p_n(x, [f_1(x) + \frac{1}{2}], [f_2(x) + \frac{1}{2}], \ldots, [f_{n-1}(x) + \frac{1}{2}]). \end{aligned}$$

Each $f_i(x)$ is an admissible real-valued generalized polynomial, and each function $g_i(x) = [f_i(x) + \frac{1}{2}]$ is an admissible integer-valued generalized polynomial, $1 \leq i \leq n$. Hence these are all proto-FVIP functions. Let $v : \mathcal{F} \to \mathbf{Z}$ be an arbitrary FVIP system. There exists an $\mathcal{F}$-ring $\mathcal{F}^{(1)}$ such that $f_i\big(v(\alpha)\big)$ and $g_i\big(v(\alpha)\big)$ are FVIP systems, $1 \leq i \leq n$. Put $g_0(x) = x$. By iteration of Theorem 5.2 and the fact that any VIP system on the torus $[0, 1)$ converges to zero along some subring, we may assume without loss of generality that for all $\alpha \in \mathcal{F}^{(1)}$, $||f_i\big(v(\alpha)\big)|| < \epsilon$, $1 \leq i \leq n$ and $g_i\big(v(\alpha)\big) \in A - A$, $0 \leq i \leq n$. Picking some $\alpha \in \mathcal{F}^{(1)}$, put $x_{i+1} = g_i\big(v(\alpha)\big)$, $0 \leq i \leq n$. Then

$$\epsilon > ||f_i\big(v(\alpha)\big)|| = |f_i\big(v(\alpha)\big) - g_i\big(v(\alpha)\big)| = |p_i(x_1, x_2, \ldots, x_i) - x_{i+1}|.$$

□

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