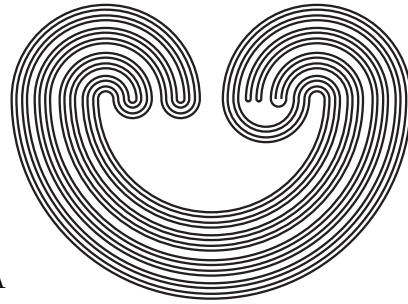


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THE PROCESS OF FINDING f' FOR AN ENTIRE FUNCTION f HAS INFINITE TOPOLOGICAL ENTROPY

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ABSTRACT. Let $(H(\mathbb{C}), \rho)$ be the metric space of all entire functions f where the metric ρ induces the topology of uniform convergence on compact subsets of the complex plane. Let $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ be the linear mapping that assigns to each f its derivative, $D(f) = f'$. We show in this note that there exists a compact subset of $H(\mathbb{C})$, say K , that is invariant under D , and D restricted to K has infinite topological entropy.

1. INTRODUCTION

Let $(H(\mathbb{C}), \rho)$ be the metric space of all entire functions f where the metric ρ induces the topology of uniform convergence on compact subsets of the complex plane. Let $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ be the linear mapping that assigns to each f its derivative, $D(f) = f'$. At first sight, finding f' does not seem to be a complex dynamical process. But, it is! In order to show this we recall the concept of topological entropy. The entropy of a mapping has three options: It is either i) zero, ii) a positive real number or iii) infinite. The entropy helps us to distinguish between simple and complex dynamics. Positive or infinite entropy means the dynamical system induced by the mapping is complicated. In section 3 we produce a subset of $H(\mathbb{C})$, say K , that is invariant under D , $D(K) = K$. Then we prove that the entropy of $D|_K : K \rightarrow K$ is infinite.

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In order to calculate the entropy of $D|_K$ we study first some dynamical properties of the shift mapping defined in the Hilbert Cube Q , $\sigma : Q \rightarrow Q$.

2. THE ENTROPY OF $\sigma : Q \rightarrow Q$

Let I denote the unit interval in the real line and Q be the Hilbert Cube,

$$Q = \prod_{i=0}^{\infty} I = \{\hat{t} = (t_0, t_1, t_2, \dots) : t_i \in I\}.$$

The metric in Q is given by

$$\hat{d}(\hat{t}, \hat{s}) = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i |t_i - s_i|.$$

Now consider the *shift mapping* $\sigma : Q \rightarrow Q$ given by the formula

$$\sigma(\hat{t}) = \sigma(t_0, t_1, t_2, \dots) = (t_1, t_2, t_3, \dots).$$

Note that σ is a continuous mapping.

Let (X, d) be a metric space and $f : X \rightarrow X$ be a continuous mapping. Given a point x in X , the *orbit of x under f* is the set $o(x, f) = \{x, f^1(x), f^2(x), \dots\}$, where $f^1 = f$ and for each $n \geq 2$, $f^n = f \circ f^{n-1}$. It is said that x is a *periodic point* of f if there exists a positive integer n , $n \in \mathbb{N}$, such that $f^n(x) = x$. Let us denote by $P(f)$ the set of all periodic points of f . The mapping f is *transitive* on X if for each pair of nonempty open subsets of X , say U and W , there exist a point $x \in U$ and an $n \in \mathbb{N}$ such that $f^n(x) \in W$, and f is *sensitive* on X if there exists $\varepsilon > 0$ such that for each $x \in X$ and each $\delta > 0$, there exist $y \in X$ with $d(x, y) < \delta$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \varepsilon$. According to R. Devaney (see [5] and [1]) the mapping f is *chaotic* on X provided that $P(f)$ is a dense set in X and f is transitive and sensitive on X .

The proof of the next proposition is not difficult, so we leave it to the reader.

Proposition 2.1. *The mapping $\sigma : Q \rightarrow Q$ is chaotic (in the sense proposed by Devaney) on Q .*

Now we calculate the topological entropy of $\sigma : Q \rightarrow Q$.

Definition 2.2. Let X be a compact topological space and let $f : X \rightarrow X$ be a continuous map. If α is an open cover of X , let $N(\alpha)$ denote the number of sets in a finite subcover of α with smallest cardinality. If α and β are two open covers of X , let $\alpha \vee \beta = \{A \cap B \mid A \in \alpha, B \in \beta\}$ and $f^{-1}(\alpha) = \{f^{-1}(A) \mid A \in \alpha\}$. For an open cover α and $n \in \mathbb{N}$ let

$$\bigvee_{i=0}^{n-1} f^{-i}(\alpha) = \alpha \vee f^{-1}(\alpha) \vee f^{-2}(\alpha) \vee \cdots \vee f^{-(n-1)}(\alpha)$$

and

$$\text{ent}(f, \alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} f^{-i}(\alpha)\right).$$

The topological entropy of f is defined by

$$\text{ent}(f) = \sup \{\text{ent}(f, \alpha) \mid \alpha \text{ is an open cover of } X\}.$$

The following proposition contains a result that is already known. We refer the reader to [6] for a detailed proof.

Proposition 2.3. Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two mappings defined on compact topological sets. Let $h : X \rightarrow Y$ be a homeomorphism. If the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array},$$

then $\text{ent}(f) = \text{ent}(g)$.

For the sake of completeness we supply the proof of the next lemma. It is, with slight changes, the same as that given for proposition 8 of chapter VIII of [3].

Lemma 2.4. Let $k \in \mathbb{N}$. If there exist k pairwise disjoint non-empty closed subsets of X , $A_1, A_2, \dots, A_k$, such that

$$A_1 \cup A_2 \cup \cdots \cup A_k \subset f(A_1) \cap \cdots \cap f(A_k),$$

then $\text{ent}(f) \geq \log k$.

Proof. Let $O_1, O_2, \dots, O_k$ be k pairwise disjoint open subsets of X such that $A_i \subset O_i$, $1 \leq i \leq k$. Let $O_{k+1} = X \setminus (A_1 \cup A_2 \cup \cdots \cup A_k)$. The collection $\alpha = \{O_1, O_2, \dots, O_{k+1}\}$ is an open cover of X .

Let $n \in \mathbb{N}$. The set

$$\Gamma = \{(x_1, x_2, \dots, x_n) \mid x_i \in \{1, 2, \dots, k\}\}$$

has cardinality k^n . For each element in Γ , say $(x_1, x_2, \dots, x_n) = \bar{x}$, the set

$$E_{\bar{x}} = \{p \in X \mid p \in A_{x_1}, f(p) \in A_{x_2}, \dots, f^{n-1}(p) \in A_{x_n}\}$$

is not empty. Each point in this set is contained in a unique element of the cover $\alpha \vee f^{-1}(\alpha) \vee \dots \vee f^{-(n-1)}(\alpha)$, namely

$$O_{x_1} \cap f^{-1}(O_{x_2}) \cap \dots \cap f^{-(n-1)}(O_{x_n}).$$

It follows that

$$N(\alpha \vee f^{-1}(\alpha) \vee \dots \vee f^{-(n-1)}(\alpha)) \geq k^n.$$

Hence $\text{ent}(f, \alpha) \geq \log k$, and therefore $\text{ent}(f) \geq \log k$. $\square$

Proposition 2.5. *The entropy of $\sigma : Q \rightarrow Q$ is infinite.*

Proof. Let $k \in \mathbb{N}$. Consider the following $k+1$ subsets of Q , $A_0, A_1, \dots, A_k$, defined in this way: For each $0 \leq i \leq k$, let

$$A_i = \left\{ \hat{t} = (t_0, t_1, t_2, \dots) \in Q : t_0 = \frac{i}{k} \right\}.$$

It readily follows that

- i) Each A_i is a closed subset of Q .
- ii) $A_i \cap A_j = \emptyset$ provided that $i \neq j$.
- iii) For each i , $\sigma(A_i) = Q$.

Therefore, by lemma 2.4, $\text{ent}(\sigma) \geq \log(k+1)$. Thus the topological entropy of σ is infinite. $\square$

3. A SUBSET OF $H(\mathbb{C})$ THAT IS INVARIANT UNDER D

An *entire function*, $f : \mathbb{C} \rightarrow \mathbb{C}$, is a function which is analytic on the whole complex plane. If f is an entire function, then f has a power series expression

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with infinite radius of convergence. Let $H(\mathbb{C})$ denote the set of all entire functions. Notice that $H(\mathbb{C})$ is a vector space. Sometimes we refer to $f \in H(\mathbb{C})$ as a point in $H(\mathbb{C})$. Given f and g in $H(\mathbb{C})$ and $n \in \mathbb{N}$, we define

$$\rho_n(f, g) = \sup \{|f(z) - g(z)| : |z| \leq n\}$$

and

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}.$$

It is known (see chapter VII in [4]) that ρ is a metric defined in $H(\mathbb{C})$, and $(H(\mathbb{C}), \rho)$ is a complete space. For the proof of the next useful lemma we refer the reader to lemma 1.7, chapter VII, of [4].

Lemma 3.1. *If $\varepsilon > 0$ is given, then there exist a $\delta > 0$ and a positive integer n such that for f and g in $H(\mathbb{C})$, $\rho_n(f, g) < \delta$ implies $\rho(f, g) < \varepsilon$.*

Let $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ be the linear mapping defined by $D(f) = f'$. Now we produce a compact subset of $H(\mathbb{C})$, say K , and a homeomorphism $h : Q \rightarrow K$. The goal is to show that K is invariant under D and the entropy of $D|_K : K \rightarrow K$ is infinite.

For each $\hat{t} = (t_0, t_1, t_2, \dots) \in Q$, let $h(\hat{t})$ be the analytic function whose series expression is

$$t_0 + t_1 z + t_2 \frac{z^2}{2!} + t_3 \frac{z^3}{3!} + \dots = \sum_{i=0}^{\infty} \frac{t_i}{i!} z^i.$$

Since for each $i \geq 0$ we have that $0 \leq \frac{t_i}{i!} \leq \frac{1}{i!}$, the radius of convergence of $\sum_{i=0}^{\infty} \frac{t_i}{i!} z^i$ is infinite. Therefore $h(\hat{t}) \in H(\mathbb{C})$. Notice that $h(\hat{1}) = \exp(z)$, where $\hat{1} = (1, 1, 1, \dots)$.

It is easy to see that h is injective. Let $K = h(Q)$.

Proposition 3.2. *$h : Q \rightarrow K$ is continuous.*

Proof. Let $\hat{t} = (t_0, t_1, t_2, \dots) \in Q$ and $\varepsilon > 0$. By lemma 3.1, there exist $N \in \mathbb{N}$ and $\gamma > 0$ such that for φ and ψ in $H(\mathbb{C})$, $\rho_N(\varphi, \psi) < \gamma$ implies $\rho(\varphi, \psi) < \varepsilon$. Consider $k \in \mathbb{N}$ such that $\sum_{i=k+1}^{\infty} \frac{N^i}{i!} < \frac{\gamma}{2}$. Let

$$\hat{s} = (s_0, s_1, s_2, \dots) \in Q, h(\hat{s}) = g \text{ and } h(\hat{t}) = f.$$

For each $z \in \mathbb{C}$ with $|z| \leq N$, we have that

$$|f(z) - g(z)| \leq |f(w) - g(w)| = \left| \sum_{i=0}^{\infty} \frac{t_i - s_i}{i!} w^i \right|$$

for some $w \in \mathbb{C}$ with $|w| = N$. Hence

$$|f(z) - g(z)| \leq \sum_{i=0}^{\infty} \frac{|t_i - s_i|}{i!} N^i < \sum_{i=0}^k \frac{|t_i - s_i|}{i!} N^i + \frac{\gamma}{2}.$$

Let $\delta = \frac{\gamma}{2^{k+1}(k+1)N^k} > 0$. If $\widehat{d}(\widehat{t}, \widehat{s}) = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i |t_i - s_i| < \delta$, then for each $0 \leq i \leq k$,

$$|t_i - s_i| < \frac{2^i \gamma}{2^{k+1}(k+1)N^k} \leq \frac{\gamma}{2(k+1)N^k},$$

and

$$\frac{|t_i - s_i|}{i!} N^i \leq |t_i - s_i| N^i < \frac{\gamma N^i}{2(k+1)N^k} \leq \frac{\gamma}{2(k+1)}.$$

Therefore $\sum_{i=0}^k \frac{|t_i - s_i|}{i!} N^i < \sum_{i=0}^k \frac{\gamma}{2(k+1)} = \frac{\gamma}{2}$. Thus if $\widehat{d}(\widehat{t}, \widehat{s}) < \delta$, then $\rho_N(f, g) < \gamma$, and $\rho(h(\widehat{t}), h(\widehat{s})) < \varepsilon$. $\square$

Since Q is a compact space, K is a compact subset of $H(\mathbb{C})$. Note that K is also a metric space, then the inverse mapping $h^{-1} : K \rightarrow Q$ is continuous. Therefore $h : Q \rightarrow K$ is a homeomorphism.

Notice that K is invariant under D . Actually it is easy to check that for each $\widehat{t} \in Q$, $D(h(\widehat{t})) = h(\sigma(\widehat{t}))$. That is the following diagram commutes:

$$\begin{array}{ccccc} Q & & \xrightarrow{\sigma} & & Q \\ h \downarrow & & \xrightarrow{\quad} & & \downarrow h \\ K & & D|_K & & K \end{array}.$$

This implies that the maps σ and $D|_K$ have the same entropy. By propositions 2.3 and 2.5, the proof of the next theorem is complete.

Theorem 3.3. *The topological entropy of $D|_K : K \rightarrow K$ is infinite.*

4. FINAL REMARKS

We devote this section to some remarks.

1. Using proposition 2.1 and that $D(h(\hat{t})) = h(\sigma(\hat{t}))$, $\hat{t} \in Q$, it is not difficult to show that $D|_K$ is chaotic in the sense of Devaney on K . It is interesting that although $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ is a linear mapping, there exists an invariant subset where it is chaotic. It is known that a linear mapping is not chaotic on any invariant subset if the space where it is defined is finite dimensional. We refer the reader to [2] where he can find not only a proof of the previous assertion but also an example (maybe the first one to appear in the literature) of a linear and chaotic mapping. Actually it is not difficult to show that D is chaotic on the whole space $H(\mathbb{C})$.

2. Let (X, d) be a metric space not necessarily compact. It is possible to define the topological entropy of a continuous mapping $f : X \rightarrow X$ (see definition 7.10 in [6]). The only condition on f is that it must be uniformly continuous. This new definition coincides with the one we gave in definition 2.2 if X is a compact metric space.

In order to see that the mapping $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ fits in this new case we recall the following theorem. The reader can find the proof in the chapter VII of [4].

Theorem 4.1. *If $\{f_n\}$ is a sequence in $H(\mathbb{C})$ such that*

$$\lim_{n \rightarrow \infty} \rho(f_n, f) = 0,$$

then $\lim_{n \rightarrow \infty} \rho((f_n)', f') = 0$.

It follows that $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ is continuous.

Rest only to show that D is uniformly continuous. Note that D is continuous in the constant function $\varphi(z) = 0$. Therefore, given $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$\rho(f, g) = \rho(f - g, 0) < \delta,$$

then

$$\rho(f', g') = \rho(f' - g', 0) = \rho((f - g)', 0) < \varepsilon.$$

Hence, $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ is uniformly continuous.

Since the topological entropy of $D|_K : K \rightarrow K$ is infinite, the topological entropy of $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ is infinite.

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