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NOTES ON TIDY SUBGROUPS OF LOCALLY COMPACT TOTALLY DISCONNECTED GROUPS

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Dedicated to Professor W.W. Comfort

Abstract. Willis established a structure theory of locally compact totally disconnected groups. An important feature of his theory is the notion of a tidy subgroup. In this note we provide results regarding these subgroups.

In recent years, George Willis developed a structure theory of locally compact totally disconnected groups (see [3,4]). An important feature of his theory is the notion of a tidy subgroup, whose definition is recalled:

Definition: (Willis) Let G be a totally disconnected locally compact group and let f be a bicontinuous automorphism of G. Let U be a compact open subgroup of G and set

$$U_+ = \bigcap_{n=0}^{\infty} f^n(U) \text{ and } U_- = \bigcap_{n=0}^{\infty} f^{-n}(U).$$

Then U is said to be *tidy* for f if it satisfies:

T1 $U = U_+U_- = U_-U_+$, and

T2 $\bigcup_{n=0}^{\infty} f^n(U_+)$ and $\bigcup_{n=0}^{\infty} f^{-n}(U_-)$ are closed in G.

Willis has shown the existence of tidy subgroups and has described properties that are possessed by these subgroups. In this note, we provide additional observations regarding tidy subgroups.

Throughout this paper, G will denote a locally compact totally disconnected group and f a bicontinuous automorphism of G. Let U be a compact open subgroup of G. We define $U(r) = \bigcap_{n=0}^{r} f^n(U)$ for any non-negative integer r.

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Let U be an arbitrary compact open subgroup of G. The first step in constructing a tidy subgroup is to find a compact open subgroup $V \subseteq U$ such that V satisfies T1. More precisely, there exists an integer r such that $U(r)$ satisfies T1 [3, Lemma 1]. The following proposition provides an alternate proof showing the existence of an integer r such that $U(r)$ satisfies T1.

Proposition 1. *Let U be a compact open subgroup of G. Then there exists an integer r such that $U(r)$ satisfies T1.*

Proof. For each $n \geq 0$, $U \cap f(U(n))$ is a compact open subgroup of $f(U(n))$. Hence the index $[f(U(n)) : U \cap f(U(n))]$ is finite. Since $U(n) \subseteq U(m)$, when $m \leq n$, we have $[f(U(m)) : U \cap f(U(m))] \geq [f(U(n)) : U \cap f(U(n))]$. This nonincreasing sequence of integers $\{[f(U(n)) : U \cap f(U(n))] : n = 0, 1, 2, \cdots\}$ is bounded below by $[f(U_+) : U_+]$. Thus there exists an integer r such that

$$[f(U(r)) : U \cap f(U(r))] = [f(U(n)) : U \cap f(U(n))] \text{ for } n \geq r.$$

Let $V = U(r)$. Then V is a compact open subgroup and the associated sequence $\{[f(V(n)) : V \cap f(V(n))] : n = 0, 1, 2, \cdots\}$ is constant. Let m be an arbitrary non-negative integer. Then $f(V_+)(V \cap f(V(m)))$ is an open neighborhood of $f(V_+)$. Since $\{f(V(n)) \colon n = 0, 1, 2, \cdots\}$ is a decreasing sequence of compact sets with $\bigcap_{n\geq 0} f(V(n)) = f(V_+)$, there exists an integer $n \geq m$ such that $f(V(n)) \subseteq f(V_+)(V \cap f(V(m)))$. Therefore we have the following injection map:

$$\frac{f(V(n))}{V \cap f(V(n))} \to \frac{f(V_+)(V \cap f(V(m)))}{V \cap f(V(m))} \to \frac{f(V(m))}{V \cap f(V(m))}$$

Since $[f(V(n)) : V \cap f(V(n))] = [f(V(m)) : V \cap f(V(m))]$, $[f(V(m)) : V \cap f(V(m))] = [f(V_+)(V \cap f(V(m))) : V \cap f(V(m))]$. Therefore $f(V(m)) = f(V_+)(V \cap f(V(m))) = f(V_+)V(m+1)$ for any $m \geq 0$. By induction, we have

$$(*)\ f^j(V) = f^j(V_+)V(j) \text{ for } j = 1, 2, 3, \cdots.$$

Let $y \in V$ and for all $j = 1, 2, 3, \cdots$, define $C_j =$

$$\{z \in V_+ : f^j(y) \in f^j(z)V(j)\}.$$

The C_j is a compact subset of V_+ and $C_j \neq \varnothing$ by (*). Since $C_{j+1} \subseteq C_j$, $\bigcap_{j\geq 1} C_j \neq \varnothing$. Choose z in this intersection. For each j, $f^j(z^{-1}y) \in V(j) \subseteq V$. Therefore $z^{-1}y \in f^{-j}(V)$ for $j = 1, 2, 3, \cdots$. Thus $z^{-1}y \in V_-$, so $y \in zV_-$. Hence $V \subseteq V_+V_-$. By a similar argument, $V \subseteq V_-V_+$. Therefore V satisfies T1. $\square$

Remark. The last paragraph of the above proof is based on Willis' proof of the result in [3].

Let U be a compact open subgroup of G. Since $f(U) \supseteq f(U_+)$, we have $[f(U) : U \cap f(U)] \geq [f(U_+)(U \cap f(U)) : U \cap f(U)] = [f(U_+) : U_+]$, with equality if and only if U satisfies T1 [4, p. 145]. Proposition 1 shows the existence of an integer r such that $U(r)$ satisfies T1. If we apply the above condition to $U(r)$, then $[f(U(r)) : U(r) \cap f(U(r))] = [f(U(r)_+) : U(r)_+]$. Since $U(r)_+ = U_+$, the above equality is of the form $[f(U(r)) : U(r) \cap f(U(r))] = [f(U_+) : U_+]$, which can be written as $[f(U(r)) : U \cap f(U(r))] = [f(U_+) : U_+]$.

Corollary 2. *Let U be a compact open subgroup of G. Then $U(r)$ satisfies T1 if and only if $[f(U(r)) : U \cap f(U(r))] = [f(U(n)) : U \cap f(U(n))]$ for all $n \geq r$. In this case $[f(U(r)) : U \cap f(U(r))] = [f(U_+) : U_+]$.*

The following lemma by Willis displays an important dynamical property of the automorphism f:

Lemma 3 [3, Lemma 2]**.** *Let U be a compact open subgroup of G satisfying T1. Suppose that $w \in G$ satisfies $f^m(w) \in U$ and $f^n(w) \in U$, where $m \leq n$. Then $w = yz$, where $f^k(y) \in U$ for $k \leq m$ and $k \geq n$, and $f^k(z) \in U$ for $m \leq k \leq n$.*

Remark. For $k \leq m$ and $k \geq n$, $f^k(w) \in U$ if and only if $f^k(z) \in U$. For $m \leq k \leq n$, $f^k(w) \in U$ if and only if $f^k(y) \in U$. In particular, if $f^k(w) \notin U$ for some $m < k < n$, then $f^k(y) \notin U$.

Given a compact open subgroup U of G, we define $U_{++} = \bigcup_{n=0}^{\infty} f^n(U_+)$. The following lemma gives necessary and sufficient conditions for U_{++} to be closed.

Lemma 4 [1, Proposition 2]**.** *Let U be a compact open subgroup of G. Then U_{++} is closed if and only if $U_{++} \cap U \subseteq f^l(U_+)$ for some $l \geq 0$.*

Willis has shown [3, Lemma 3] that if U satisfies T1, then U_{++} is closed if and only if $U_{++} \cap U = U_+$. Moreover, [3, Lemma 3] also shows that U satisfies T2 (that is, U is tidy) if and only if U_{++} is closed.

Let V be a compact open subgroup of G satisfying T1. Define $\mathcal{L}_V = \{g \in G : f^n(g) \in V$ for all but finitely many integers $n\}$ and let L_V be the closure of $\mathcal{L}_V$. Then $\mathcal{L}_V \subseteq V_{++}$, $\mathcal{L}_V$ is an $f-$ invariant subgroup of G, and $V_+ \cap V_- \subseteq \mathcal{L}_V$. The condition defining $\mathcal{L}_V$ is equivalent to the existence of integers m and n such that $f^m(g) \in V_+$ and $f^n(g) \in V_-$ [3, p. 347].

The following corollary of Willis is asserted in [3]. Here we include a proof of this result.

Corollary 5 [3, Corollary to Lemma 3]**.** *Let U be a compact open subgroup of G satisfying T1. Then U is tidy if and only if $\mathcal{L}_U = U_+ \cap U_-$.*

Proof. Suppose U is a tidy subgroup. By Lemma 4, $U_{++} \cap U \subseteq f^l(U_+)$ for some $l \geq 0$. Let $x \in \mathcal{L}_U$. Then there exists an integer n such that for $m \geq n$, $f^m(x) \in U_-$. In particular, $f^{n+l}(x) \in U_- \subseteq U$. Thus $f^{n+l}(x) = f^l(y)$ for some $y \in U_+$. Therefore $f^n(x) = y \in U_+ \cap U_-$. Hence $x \in U_+ \cap U_-$.

Conversely, suppose that $\mathcal{L}_U \subseteq U_+ \cap U_-$ and $U_{++} \cap U \nsubseteq U_+$. Then there exists an $x \in U_+$ such that $f^n(x) \in U \backslash U_+$ for some $n > 1$. Also, $f^m(x) \notin U$ for some m, where $0 < m < n$. (Otherwise, if $x \in U_+$ and $f(x), \cdots, f^n(x) \in U$, then $f^n(x) \in U_+$.) By Lemma 3, $x = yz$, where $y \in \mathcal{L}_U$ and $f^k(z) \in U$ for $0 \leq k \leq n$. Since $y \in \mathcal{L}_U \subseteq U_+ \cap U_-$, $f^k(y) \in U$ for any integer k. In particular, $f^m(y) \in U$. Then $f^m(x) = f^m(y)f^m(z) \in U$, which is a contradiction. Hence $U_{++} \cap U \subseteq U_+$. Therefore, U is tidy. □

Proposition 6. *Let U be a compact open subgroup of G satisfying T1. Then U is tidy if and only if $\mathcal{L}_U$ is closed.*

Proof. If U is tidy, then $\mathcal{L}_U = U_+ \cap U_-$, so it is closed.

Conversely, suppose $\mathcal{L}_U$ is closed. If U is not tidy, then there exists $x \in \mathcal{L}_U \backslash (U_+ \cap U_-)$. We may assume that $x \in U_+$, $f^n(x) \in U_-$ for some $n > 1$ and $f^m(x) \notin U$ for some m, where $0 < m < n$. (Otherwise, if $x \in U_+$ and $f(x), \cdots, f^n(x) \in U$, then $f^n(x) \in U_+ \cap U_-$, so $x \in U_+ \cap U_-$.) Since $\mathcal{L}_U$ is an $f-$ invariant subgroup, the sequence $\{x,\ xf^n(x),\ xf^n(x)f^{2n}(x),\ \cdots\}$ is contained in $\mathcal{L}_U \cap U$. Because U is compact, $\mathcal{L}_U \cap U$ is compact. Therefore the sequence $\{x, xf^n(x), xf^n(x)f^{2n}(x), \cdots\}$ has an accumulation point $w \in \mathcal{L}_U \cap U$.

Now $w = \lim_{k_i \to \infty} x f^n(x) f^{2n}(x) \cdots f^{k_i n}(x)$. For a fixed positive integer j, consider the product:

$$f^{-jn+m}(x) f^{n-jn+m}(x) f^{2n-jn+m}(x) \cdots f^{k_i n-jn+m}(x).$$

Since each term in the above product is in U except $f^m(x)$, the product is not in U. Therefore $f^{-jn+m}(w) \notin U$, so $w \notin \mathcal{L}_U$, which is a contradiction. □

Remark. The above proof is adapted from the proof of [3, Lemma 3].

Proposition 7. *Let U be a compact open subgroup of G. Then U_{++} is compact if and only if U_+ is $f-$ invariant.*

Proof. Suppose $U_{++} = \bigcup_{n=0}^{\infty} f^n(U_+)$ is compact. Since U_{++} is a countable union of compact subgroups, by the Baire Category Theorem, $f^k(U_+)$ must be open for some k. Thus $f^n(U_+)$ is open for all $n \geq 0$. Therefore there exists an l such that $U_{++} = f^l(U_+) = f^n(U_+)$ for all $n \geq l$. Hence $f(U_+) = U_+$. Conversely, if $f(U_+) = U_+$, then $U_{++} = U_+$ is compact. □

Corollary 8. *Let U be a tidy subgroup of G such that U_{++} is compact. Then for any other tidy subgroup V, V_{++} is also compact.*

Proof. Since U and V are both tidy subgroups of G, $U \cap V$ is also a tidy subgroup of G [3, Lemma 10]. Since $(U \cap V)_{++} \subseteq U_{++}$, $(U \cap V)_{++}$ is compact. Therefore we may assume that $U \subseteq V$. By [3, Lemma 11], $V_+ \cap U = U_+$. Since U is an open subgroup, U_+ is an open subgroup of V_+. Hence the index $[V_+ : U_+]$ is finite. Since f is an automorphism, $[f(V_+) : f(U_+)] = [V_+ : U_+]$. From the equation $[f(V_+) : f(U_+)][f(U_+) : U_+] = [f(V_+) : V_+][V_+ : U_+]$, we can conclude that $f(V_+) = V_+$. Therefore V_{++} is also compact. □

Let U be a compact open subgroup of G satisfying T1. Willis has shown that $\mathcal{L}_U$ is a relatively compact subgroup of G [3, Lemma 6]. Here we use his idea to present a different proof. Define $\epsilon = (f(U_+) \backslash U_+) \cap \mathcal{L}_U$ (see [3, p. 347]). Since $f(U_+)$ is compact and $U_+ = U \cap f(U_+)$ is an open subgroup of $f(U_+)$, ϵ is contained in finitely many cosets $\{U_+ g_1, \cdots, U_+ g_m\}$, where each $g_i \in \epsilon$. Since each $g_i \in \mathcal{L}_U$, there exists a least positive integer p such that $f^p(g_i) \in U_-$ for $i = 1, \cdots, m$.

Proposition 9. $\mathcal{L}_U \subseteq f^{p+1}(U_+)U_-$.

Proof. Let $z \in \mathcal{L}_U$. If $z \in U_+ \cap U_-$, then clearly $z \in f^{p+1}(U_+)U_-$. Therefore, we may assume that $z \in \mathcal{L}_U \backslash (U_+ \cap U_-)$. Since $z \in \mathcal{L}_U \backslash (U_+ \cap U_-)$, there exists an $x \in \epsilon$ such that $z = f^l(x)$ [3, p. 347]. To see this, we may first assume that $z \in U_- \backslash U_+$. (If $z \in \mathcal{L}_U \backslash (U_+ \cap U_-)$, then $f^m(z) \in U_-$ for some m and $f^m(z) \notin U_+$. Otherwise, $f^m(z) \in U_+ \cap U_-$ which would imply that $z \in U_+ \cap U_-$. Now take $f^m(z)$ as z.) Because $z \in \mathcal{L}_U$, $f^{-n}(z) \in U_+$ for some $n > 0$. Choose the least positive integer k such that $f^{-k}(z) \in U_+$. Then $f^{-k+1}(z) = x \in (f(U_+)\backslash U_+) \cap \mathcal{L}_U$. Then $x = yg_i$, where $y \in U_+$ and $g_i \in \mathcal{L}_U$. Therefore $z = f^l(yg_i)$, where $y \in U_+ \cap \mathcal{L}_U$ and $g_i \in \mathcal{L}_U$.

We consider the following cases.

(i) $l < 0$: Then $z = f^l(yg_i) \in f^{l+1}(U_+) \subseteq U_+ \subseteq f^{p+1}(U_+)U_-$.

(ii) $0 \leq l \leq p$: Then $z = f^l(yg_i) \in f^{l+1}(U_+) \subseteq f^{p+1}(U_+) \subseteq f^{p+1}(U_+)U_-$.

(iii) $l > p$: If $f^l(y) \in f(U_+)$, then $f^l(y) \in f^{p+1}(U_+)$. Since $f^l(g_i) \in U_-$, $z = f^l(y)f^l(g_i) \in f^{p+1}(U_+)U_-$. Therefore, we may assume that $f^l(y) \notin f(U_+)$. Since $f^l(y) \notin f(U_+)$, there exists an integer $0 < k < l$ such that $f^k(y) \in \epsilon = (f(U_+)\backslash U_+) \cap \mathcal{L}_U$. Let $z' = f^k(y) \in \epsilon$. Then $z = f^{l-k}(z')f^l(g_i)$. By repeating this process, one can show that $f^{l-k}(z') \in f^{p+1}(U_+)U_-$. Since $f^l(g_i) \in U_-$, we can conclude that $z \in f^{p+1}(U_+)U_-$. □

Corollary 10 [3, Lemma 6]**.** *$L_U = \overline{\mathcal{L}_U}$ is a compact $f-$ invariant subgroup of G.*

Given two compact open subgroups U and V of G, both satisfying T1, the following proposition provides a method to compare the size of $\mathcal{L}_U$ and $\mathcal{L}_V$.

Proposition 11. *Let U and V be compact open subgroups of G both satisfying T1. If $U_+ \cap U_- \subseteq V$, then $\mathcal{L}_U \subseteq \mathcal{L}_V$.*

Proof. Suppose $U_+ \cap U_- \subseteq V$. Since $\lim_{n\to\infty} f^{-n}(U_+) = U_+ \cap U_-$, there exists a positive integer p such that $f^{-p}(U_+) \subseteq V$. Then for any $n \geq 0$, $f^{-n}(f^{-p}(U_+)) = f^{-p}(f^{-n}(U_+)) \subseteq f^{-p}(U_+) \subseteq V$. Therefore, $f^{-p}(U_+) \subseteq f^n(V)$ for any $n \geq 0$. Hence $f^{-p}(U_+) \subseteq V_+$.

If we apply the above argument to f^{-1}, then there exists a positive integer q such that $f^q(U_-) \subseteq V_-$.

Let $x \in \mathcal{L}_U$. Then there exists integers m and n such that $f^m(x) \in U_+$ and $f^n(x) \in U_-$. Hence $f^{-p}(f^m(x)) = f^{-p+m}(x) \in V_+$ and $f^q(f^n(x)) = f^{q+n}(x) \in V_-$. Therefore $x \in \mathcal{L}_V$. □

In [4], Willis shows how to construct a tidy subgroup in three steps. Given an arbitrary compact open subgroup U of G, the first step is to find a compact open subgroup $V \subseteq U$ such that V satisfies T1. This step is Proposition 1. The second step is to identify a particular compact $f-$invariant subgroup L of G. Here we take $L = L_V$. The third step uses L and V to produce a tidy subgroup W. Define

$$V' = \{v \in V : lvl^{-1} \in VL \text{ for all } l \in L\}.$$

Then $W = V'L$ is a tidy subgroup of G.

In [5], Willis describes a new procedure for constructing tidy subgroups. This new procedure differs from the above construction at step 2. As before, start with an arbitrary compact open subgroup U of G, and define V as above. The second step is to define a compact $f-$ invariant subgroup of G. Define

$$\mathcal{K} = \{l \in G : \lim_{j\to\infty} f^j(l) = e \text{ and } \{f^{-j}(l)\}_{j\geq 0} \text{ is bounded}\}.$$

Then $K = \overline{\mathcal{K}}$ is an $f-$ invariant compact subgroup of L_V [5, Lemma 2.2] and $L_V = (V \cap L_V)K$ [5, Lemma 2.3]. The third step uses K and V to produce a tidy subgroup. Define

$$V'' = \{v \in V : lvl^{-1} \in VK \text{for all } l \in K\}.$$

Then $V''K$ is a compact open subgroup of G and is the same tidy subgroup W produced earlier [5, Proposition 2.1]. The benefit of this construction is that $\mathcal{K}$ does not depend on any compact open subgroup.

Proposition 12. *Let V be a compact open subgroup of G satisfying T1. Then V is tidy if and only if $V \supseteq \mathcal{K}$. In particular, if $\mathcal{K} = \{e\}$, then every compact open subgroup satisfying T1 is tidy.*

Proof. If V is tidy, then $V \supseteq L_V \supseteq \mathcal{K}$. Conversely, if $V \supseteq \mathcal{K}$, then $L_V = (V \cap L_V)K \subseteq V$. Therefore, V is tidy. □

Corollary 13. *If $\mathcal{K}$ is closed, then every compact open subgroup of G contains a tidy subgroup.*

Proof. Suppose $\mathcal{K}$ is closed. Then $f^{-1}|_{\mathcal{K}}$ is a contraction (see [2]). Then for any compact subset C of $\mathcal{K}$, the sequence $\{(f^{-1})^n(C)\}_{n\leq 0} = \{f^n(C)\}_{n\geq 0}$ converges to the identity [2, Proposition 2.1]. In particular, $f^n(\mathcal{K})$ converges to the identity. In this case, $\mathcal{K} = \{e\}$, so the above proposition applies. □

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