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CONNECTEDNESS PROPERTIES OF WHITNEY LEVELS

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ABSTRACT. It is shown that δ-connectedness is not a Whitney reversible property. This answers in the negative a question posed by Sam B. Nadler, Jr. in 1978.

A topological property $\mathcal{P}$ is said to be:

(a) a *Whitney property* provided that if a continuum X has property $\mathcal{P}$, so does $\mu^{-1}(t)$ for each Whitney map μ for $C(X)$ and each $t \in [0, \mu(X))$ ([6, p. 165]);
(b) a *Whitney reversible property* provided that whenever X is a continuum such that $\mu^{-1}(t)$ has property $\mathcal{P}$ for all Whitney maps μ for $C(X)$ and all $t \in (0, \mu(X))$, then X has property $\mathcal{P}$ ([8, p. 235]).

A continuum X is said to be:

(c) δ-*connected* provided that for every two points of X there exists an irreducible continuum between them which is hereditarily decomposable ([5, p. 90]);
(d) λ-*connected* provided that for every two points of X there exists an irreducible continuum between them which is of type λ (that is, each of its indecomposable subcontinua has empty interior) ([5, p. 85]).

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† Sadly, Professor Janusz J. Charatonik passed away on July 11, 2004.

(Note that in some papers, in particular in Sam B. Nadler, Jr.'s monograph [7, (0.30), p. 16], the name "λ-connected" is used in the sense of "δ-connected." See [3, p. 118] for an explanation.)

Answering a question in [6, Section 6, p. 179] (cf. [7, Question 14.36, p. 432]), the second named author has proved in [2, Construction 5.1, p. 387, and Remark 6, p. 389] that δ-connectedness is not a Whitney property. A similar assertion for λ-connectedness is not known; see [2, Question 7, p. 390] and compare [4, Question 51.3, p. 280].

In [7, Question 14.57, p. 464], (compare [4, Question 51.4, p. 280]), Nadler asks if (i) δ-connectedness is a Whitney reversible property, and if (ii) λ-connectedness is a Whitney reversible property. In this paper we present an example of a continuum showing a negative answer to (i). The question concerning (ii) remains open.

All considered spaces are assumed to be metric. We denote by $\mathbb{N}$ the set of all positive integers, and by $\mathbb{R}$ the space of reals. A *continuum* means a compact connected space, and a *mapping* means a continuous function. Given two points a and b in a Euclidean space, we denote by $\overline{ab}$ the straight line segment joining these points.

A continuum is said to be *decomposable* provided that it can be represented as the union of two of its proper subcontinua. Otherwise, it is said to be *indecomposable*. A continuum is said to be *hereditarily decomposable (hereditarily indecomposable)* provided that each of its non-degenerate subcontinua is decomposable (indecomposable). A continuum X is said to be *irreducible* (between points a and b of X) provided that no proper subcontinuum of X contains these points. Points a and b are then called *points of irreducibility of X*. An irreducible continuum X is said to be of *type λ* if each indecomposable subcontinuum of X has empty interior.

Given a continuum X, we let 2^X denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H (see e.g., [7, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X. We will write $A = \operatorname{Lim} A_n$ to denote that the (closed) sets tend to A with respect to the Hausdorff metric.

A *Whitney map* for $C(X)$ is a mapping $\mu : C(X) \to [0, \infty)$ such that:

(0.1) $\mu(A) < \mu(B)$ for every two $A, B \in C(X)$ such that $A \subset B$ and $A \neq B$;

(0.2) $\mu(A) = 0$ if and only if A is a singleton.

For the concept and existence of a Whitney map, see [4, Section 13, p. 105-110]. For each $t \in [0, \mu(X)]$ the preimage $\mu^{-1}(t)$ is called a *Whitney level*. It is known that each Whitney level is a continuum, see [4, p. 159].

The reader is referred to monographs [4] and [7] for definitions and basic properties of other notions used in the paper.

To present the needed example of continuum X showing that δ-connectedness is not a Whitney reversible property, we start with some auxiliary constructions.

Construction 1. In the Cartesian coordinates (x, y) in the plane, let S_0 be the standard $\sin \frac{1}{x}$-curve, that is,

(1.1) $S_0 = (\{0\} \times [-1, 1]) \cup \{(x, \sin \frac{\pi}{x}) \in \mathbb{R}^2 : x \in (0, 1]\}$.

We will call $(1, 0)$ the *end point* of S_0, and $\{0\} \times [-1, 1]$ the *limit segment* of S_0.

Take a sequence of local maxima of S_0, that is, a sequence of points p_n determined by $p_n = (x_n, 1) \in S_0$ with $x_{n+1} < x_n$ for each $n \in \mathbb{N}$. Thus, $(0, 1) = \lim p_n$. Further, in the segment $\{0\} \times [1, 2]$ take a sequence of points $q_n = (0, y_n)$ such that the numbers y_n form a decreasing sequence tending to 1

(1.2) $\lim y_n = 1 < \cdots < y_{n+1} < y_n < \cdots < y_1 = 2$.

Thus, $(0, 1) = \lim q_n$.

For each $n \in \mathbb{N}$, let S_n be a homeomorphic copy of S_0 situated in the rectangle $[0, 1] \times [1, 2]$ in such a way that:

(1.3) p_n is the end point, and $L_n = \{0\} \times [y_{2n}, y_{2n-1}]$ is the limit segment, of S_n, for each n;

(1.4) $S_n \cap S_0 = \{p_n\}$ for each n;

(1.5) $S_m \cap S_n = \emptyset$ for $m, n \in \mathbb{N}$ with $m \neq n$;

(1.6) $\operatorname{Lim} S_n = \{(0, 1)\}$.

Define

(1.7) $X_1 = (\{0\} \times [1, 2]) \cup S_0 \cup \bigcup\{S_n : n \in \mathbb{N}\}$

and observe that X_1 is a continuum having two arc components, and that the limit segments of S_0 and of all S_n are components of the set of non-local connectedness of X_1.

Construction 2. Recall that the *pseudo-arc* means an arc-like hereditarily indecomposable continuum (see e.g., [9, 1.23, p. 13]). Let X_1 be the continuum defined by (1.7). For each $n \in \{0\} \cup \mathbb{N}$, let L_n stand for the limit segment, and M_n stand for the non-compact arc component of S_n. Thus, $L_0 = \overline{(0,-1)(0,1)}$ and $L_n = \overline{q_{2n}q_{2n-1}}$ for any $n \in \mathbb{N}$. Hence, $S_n = L_n \cup M_n$ for each integer $n \geq 0$, and $L_0, L_1, L_2, \ldots$ are components of the set of points at which X_1 is not locally connected.

Note that each M_n is locally compact. Since for each locally compact, noncompact metric space M an arbitrary continuum P can be a remainder of a compactification of M, (see [1, Theorem, p. 35]), it is possible to replace L_n by a copy P_n of the pseudo-arc in such a way that $\operatorname{diam}(P_n) = \operatorname{diam}(L_n)$ and that, if for each $n \in \{0\} \cup \mathbb{N}$, the symbol M'_n denotes a one-to-one copy of the non-compact arc component M_n of S_n, then $P_n = \operatorname{cl}(M'_n) \setminus M'_n$ for $n \in \{0, 1, 2, \ldots\}$. Since the pseudo-arc is a plane continuum, the construction can be made in such a way that all the inserted pseudo-arcs $P_0, P_1, P_2, \ldots$ lie in the plane $\{(x, y, z) \in \mathbb{R}^3 : x = 0\}$. It follows that

(2.1) for each $n \in \mathbb{N}$, the end points q_{2n} and q_{2n-1} of L_n belong to the pseudo-arc P_n.

Denote by X_2 the continuum obtained from X_1 by the replacement described above, that is,

(2.2) $X_2 = \bigcup\{(M'_n \cup P_n) : n \in \{0\} \cup \mathbb{N}\} \cup \bigcup\{\overline{q_{2n+1}q_{2n}} : n \in \mathbb{N}\}$,

where $\overline{q_{2n+1}q_{2n}} \subset \{0\} \times [1, 2]$ is the segment joining the point $q_{2n+1} \in P_{n+1}$ with $q_{2n} \in P_n$. Observe that X_2 is located in the half-space $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0\}$ and that, if $S'_n = M'_n \cup P_n$ denotes the compactification of the ray M'_n having P_n as its remainder, then

$$\operatorname{Lim} P_n = \operatorname{Lim} S'_n = \{(0, 1, 0)\},$$

whence it follows that

(2.3) for each $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that $S'_n \subset B(\varepsilon)$ for each $n > n(\varepsilon)$

where $B(\varepsilon)$ means the ball of radius $\frac{\varepsilon}{2}$ centered at $(0, 1, 0)$.

Construction 3. Recall that, given a continuum B, the *cone over* B is defined as the quotient space $\operatorname{Cone}(B) = (B \times [0, 1])/(B \times \{1\})$.

The set $B \times \{0\}$ is called the *base* of the cone, and the point which corresponds to $B \times \{1\}$ is called its *vertex*.

Let X_2 be the continuum defined above in Construction 2. For each $n \in \{0\} \cup \mathbb{N}$ let C_n be a cone over the pseudo-arc P_n such that

(3.1) $C_n \cap X_2 = P_n$ is the base of C_n;
(3.2) $\operatorname{diam}(C_n) = \operatorname{diam}(P_n)$;
(3.3) $C_m \cap C_n = \emptyset$ for $m \neq n$.

The needed continuum X will be obtained by attaching all the cones C_n to the continuum X_2. It can be geometrically realized in the space $\mathbb{R}^3$ as follows. Put $v_0 = (-2, 0, 0)$ and let C_0 be the geometric cone with the vertex v_0 and the base P_0. For each $n \in \mathbb{N}$ let $d_n = \operatorname{diam}(P_n)$ and put $v_n = (-d_n, \frac{1}{2}(y_{2n} + y_{2n-1}), 0)$, where y_{2n} and y_{2n-1} are the y-coordinates of the end points of L_n (see (1.2)). Then consider C_n as the geometric cone with the vertex v_n and the base P_n. Note that each C_n such defined is located in the half-space $\{(x, y, z) \in \mathbb{R}^3 : x \leq 0\}$ and observe that the cones C_n satisfy conditions (3.1)-(3.3). So, define

(3.4) $X = X_2 \cup \bigcup\{C_n : n \in \{0\} \cup \mathbb{N}\}$,

and note that X is a continuum.

Recall that a continuum W is said to be *continuum-chainable* provided that for each $\varepsilon > 0$ and every two distinct points $p, q \in X$ there is a finite sequence of subcontinua $\{A_1, \ldots, A_k\}$ of X such that $\operatorname{diam}(A_i) < \varepsilon$, $p \in A_1$, $q \in A_k$ and $A_i \cap A_{i+1} \neq \emptyset$ for each index $i < k$.

The main result of this paper is the following theorem.

Theorem 4. *There exists a continuum X having the following properties.*

(4.1) *X is not δ-connected;*
(4.2) *X is λ-connected;*
(4.3) *X has two arc-components;*
(4.4) *X is continuum-chainable;*
(4.5) *for each Whitney map $\mu : C(X) \to [0, \infty)$ and for each $t \in (0, \mu(X))$, the Whitney level $\mu^{-1}(t)$ is arcwise connected.*

Proof: The continuum X is defined by (3.4). We have to show that it has the properties (4.1)-(4.5).

1) Let $p = v_0$ and $q = p'_0$ be the vertex of the cone C_0 and the end point of S'_0, respectively. Then each irreducible continuum between p and q contains indecomposable subcontinua (contained in the union $\bigcup\{P_n : n \in \{0\} \cup \mathbb{N}\}$), so X is not δ-connected.

2) For each $n \in \{0\} \cup \mathbb{N}$, the ray M'_n approximates the pseudo-arc $P_n = \mathrm{cl}(M'_n) \setminus M'_n$ according to Construction 2. Thus, each P_n (and therefore each indecomposable subcontinuum of X) has empty interior. Hence, X is λ-connected.

3) Indeed, it follows from the constructions 1-3 that the two arc components of X are

$$A^+ = \bigcup\{M'_n : n \in \{0\} \cup \mathbb{N}\} \subset \{(x, y, z) \in \mathbb{R}^3 : x > 0\},$$

$$A^- = C_0 \cup \bigcup\{(C_n \cup \overline{q_{2n+1}q_{2n}}) : n \in \mathbb{N}\} \subset \{(x, y, z) \in \mathbb{R}^3 : x \le 0\}.$$

4) To see that X is continuum-chainable, take two distinct points $p, q \in X$ and $\varepsilon > 0$. If both p and q belong to the same arc component of X, the argument is obvious. So, let $p \in A^-$ and $q \in A^+$. Choose $n > n(\varepsilon)$ according to assertion (2.3) of Construction 2 and note that, by (2.1) and (2.3),

$$q_{2n-1} \in P_n \subset S'_n \subset B(\varepsilon).$$

Let $p'_n \in S'_n$ be the end point of S'_n and let D^+ be an arc from p'_n to q contained in the arc component A^+ of X. Further, let D^- be an arc from p to q_{2n-1} contained in A^-. Then the union $U = D^- \cup S'_n \cup D^+$ is a continuum joining p and q. Since $\mathrm{diam}(S'_n) < \varepsilon$ and since each of the arcs D^- and D^+ can be represented as a finite union of subarcs of diameter less than ε, we conclude that all the conditions of the definition of continuum chainability are satisfied. Hence, (4.4) is shown.

5) Conditions (4.4) and (4.5) coincide with conditions (a) and (d) of [4, Theorem 33.4, p. 248], and thereby they are equivalent.

The proof is complete. □

Theorem 4 implies, by the definition of a Whitney reversible property, the following corollary.

Corollary 5. *δ-connectedness is not a Whitney reversible property.*

Remark 6. Since by (4.2) the continuum X is λ-connected, the described example does not answer the other part of [4, Question 54.4, p. 280].

References

[1] J. M. Aarts and P. van Emde Boas, "Continua as remainders in compact extensions," *Nieuw Arch. Wisk. (3)* **1**5 (1967), 34–37.

[2] Włodzimierz J. Charatonik, "Some Counterexamples Concerning Whitney Levels," *Bull. Polish Acad. Sci. Math.* **31** (1983), no. 9-12, 385–391 (1984).

[3] Charles L. Hagopian, "Mapping Theorems for Plane Continua," *Topology Proc.* **3** (1978), no. 1, 117–122 (1979).

[4] Alejandro Illanes and Sam B. Nadler, Jr., *Hyperspaces: Fundamentals and Recent Advances*. Monographs and Textbooks in Pure and Applied Mathematics, 216. New York: Marcel Dekker, Inc., 1999.

[5] B. Knaster and S. Mazurkiewicz, "Sur un problème concernant les transformations continues," *Fund. Math.* **21** (1933), 85–90.

[6] J. Krasinkiewicz and Sam B. Nadler, Jr., "Whitney properties," *Fund. Math.* **98** (1978), no. 2, 165–180.

[7] Sam B. Nadler, Jr., *Hyperspaces of Sets: A Text with Research Questions*. Monographs and Textbooks in Pure and Applied Mathematics, 49. New York-Basel: Marcel Dekker, Inc., 1978.

[8] ______, "Whitney-reversible properties," *Fund. Math.* **109** (1980), no. 3, 235–248.

[9] ______, *Continuum Theory: An Introduction*. Monographs and Textbooks in Pure and Applied Mathematics, 158. New York: Marcel Dekker, Inc., 1992.

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