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CONTINUOUS SELECTIONS AND PURELY TOPOLOGICAL CONVEX STRUCTURES

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Abstract. In 1983, Frank Deutsch and Petar Kenderov give some necessary and sufficient conditions for convex-valued multifunctions to have continuous approximations. Inspired by Deutsch and Kenderov's result, we introduce and characterize coherent multifunctions, investigating the relationship between lower semi-continuity and coherence. We then interpolate the lemmas behind the well-known Michael results on continuous selections. In doing so, we define a suitable and quite natural convex structure on every topological space, not just on metrizable ones. We produce a selection theorem stronger than Michael's selection theorems, both the convex-valued and the zero-dimensional version, in general considered as two independent cases in the literature.

1. Introduction

A multifunction (or set-valued map) is a map assigning to each element of a set X a (possibly empty) subset of another set Y. A selection for a given multifunction is a function whose graph is contained in the one of the multifunction. The existence of a selection for an arbitrary multifunction is merely equivalent to the axiom of

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choice. The existence of "nice" selections for "nice" multifunctions can be a much more interesting and difficult question. In particular, the existence of continuous selections is strictly connected to the possibility of extending continuous functions.

In the mid 1950's, with a sequence of three papers ([19], [20], [22]), Ernest Michael gathers together the first systematic account of the theory of selections, connecting the selection problem (the problem of determining necessary conditions for a multifunction to admit a continuous selection) to several other topological problems. Among the results presented in these papers, two theorems stand out for both their connection to extension problems and their contribution in the setting of paracompact spaces. The first of these theorems, known as the convex version of Michael's selection theorem [19, Theorem 3.2], characterizes paracompact spaces via the existence of continuous selections for suitable multifunctions. The second one, the finite-dimensional version of Michael's selection theorem [20, Theorem 1.2], deals with multifunctions of finite-dimensional paracompact domain and has applications in particular in the zero-dimensional case.

Following Michael, many topologists, in recent years, have taken an interest in a variety of selection problems and worked out different versions of finite-dimensional selection and approximation theorems. Many ([2], [29], [33], [1], [32], [13], [14], [15], [16], [34]) have been working basically with the same tools and notions as Michael (lower semi-continuous multifunctions, covering dimension, filtrations of set-valued maps, (equi-)LC^n families and C^n families), even if, during the last decade, a new approach exploiting the notions of CW-complex and the one of extension dimension has been gaining more and more ground (see [4], [5], [8], [9], [10]).

In spite of this abundant literature, not many attempts have been made to unify the seminal selection theorems recalled above into a more general one. There are not so many attempts either to get rid of the rigid and not purely topological requirement of convexity which affects all the results where no limitation on the dimension of the domain space is given. A few interesting results are due to Carl P. Pixley and Michael himself (see [23], [24], [25], [26], [28]), but do not concern both these challenges at the same time. In [23], in particular, Michael generalizes the convex version of his selection theorem by weakening the hypothesis on the range space: instead of Banach, the range is assumed to be only complete metric with a suitable defined "convex structure." An improvement to this result is due to D. W. Curtis (see [6]).

The present paper is an attempt to both unify the convex version and the zero-dimensional version of Michael's Selection Theorem and define a "purely topological" convex structure on the range space.

Michael's proofs are essentially based on the possibility of getting a uniformly convergent sequence of continuous functions, the limit function being the required selection. Such a possibility is guaranteed once the multifunction in question admits a continuous ϵ-approximation, for every $\epsilon > 0$ (see [21], [3]). Some necessary and sufficient conditions for the existence of continuous approximations are in turn given, in 1983, by Frank Deutsch and Petar Kenderov [7] (see also [3]).

Following Deutsch and Kenderov, we introduce the notion of *coherence* and investigate the relationship between lower semi-continuity and coherence. By interpolating the lemmas behind Michael's results, we bring more light on where and how the paracompactness of the domain space on one side, and the metric/norm on range space on the other, are really used. As a consequence, we can work with an assumption stronger than paracompactness: we introduce and characterize n-paracompact spaces. At the same time, we are able to define a suitable and quite natural convex structure on any topological space, not just on metrizable ones.

Our main theorem (see section 11) is to be considered not only as a selection theorem stronger than both Michael's results (convex versions and zero-dimensional version), but also as a way of introducing new classes of topological spaces (namely, n-paracompact spaces and formally n-convex spaces) which can be of some independent interest.

2. Preliminaries

Let $\mathbf{V}$ denote the class of all sets: $X \in \mathbf{V}\setminus\{\emptyset\}$ is an abbreviation for *"X is a nonempty set"* (see chapter 8 in [17]). $P(X)$ stands for the power set of a set X.

Let Top denote the class of all topological spaces. For $X \in Top$ and $x \in X$, $\mathcal{N}(x)$ denotes the family of all open neighborhoods of

x; if $E \in P(X)$, $\overline{E}$ and $int(E)$ stand for the closure and the interior of E in X, respectively.

Let $X, Y \in \mathbf{V}\backslash\{\emptyset\}$. A *multifunction*, or *set-valued map*, T from X to Y, denoted by $T : X \Rightarrow Y$ (see [3, chapter 6]), is a function from X to $P(Y)$. Clearly, a function $f : X \to Y$ is a multifunction. Y^X will denote the sets of all functions from X to Y.

The set $gr(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is the *graph* of T, while the domain, $dom(T)$, and the range, $range(T)$, of T are the sets $\{x \in X : T(x) \neq \emptyset\}$ and $\{y \in Y : \exists x \in X \text{ with } y \in T(x)\}$, respectively. In particular, $dom(T) = X$ means that T takes only nonempty values.

For $U \in P(Y)$, $T^{-1}(U) = \{x \in X : T(x) \cap U \neq \emptyset\}$ is the *inverse image* of U under T. The *inverse* of T is $T^{-1} : Y \Rightarrow X$ defined by $T^{-1}(y) = T^{-1}(\{y\}) = \{x \in X : y \in T(x)\}$, where $y \in Y$. It is easy to check that $dom(T^{-1}) = range(T)$.

T is *lower semi-continuous* (l.s.c.) if $T^{-1}(U)$ is open in X whenever U is open in Y; T has *open fibers* if $T^{-1}(y)$ is open in X, whenever $y \in Y$. For $\{U_i : i \in I\} \subseteq P(Y)$, $T^{-1}(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} T^{-1}(U_i)$. Consequently, if T has open fibers, then T is l.s.c.. The converse is not true in general[1]

If $Z \in \mathbf{V}\backslash\{\emptyset\}$ and $S : Y \Rightarrow Z$, then $S \circ T : X \Rightarrow Z$ is defined by $(S \circ T)(x) = \bigcup_{y \in T(x)} S(y)$, where $x \in X$. Note that $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$ [3, Example 6.1.3].

Let (X, d) be a metric space. For $x \in X$ and $A \in P(X)$, $S_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ is the open ball with center x and radius $\epsilon > 0$; $d(x, A) = inf\{d(x, a) : a \in A\}$ is the distance from x to A and $S_d(A, \epsilon) = \bigcup_{x \in A} S_d(x, \epsilon)$ is the ϵ-enlargement of A.

If $X \in Top$, (Y, d) is a metric space, $T : X \Rightarrow Y$ is nonempty valued and $\epsilon > 0$, then $\{T^{-1}(S_d(y, \epsilon)) : y \in Y\}$ is a cover for X.

3. Michael's selections and coherence

Let $X, Y \in \mathbf{V}\backslash\{\emptyset\}$ and $T : X \Rightarrow Y$ be nonempty valued. $f \in Y^X$ is a *selection* for T if for every $x \in X$, $f(x) \in T(x)$.

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[1]Every continuous strictly monotone real-valued function is an l.s.c. multifunction with no open fibers.

We will consider the following partial order on the set of all multifunctions from X into Y.

Definition 3.1. Let $X, Y \in \mathbf{V}\backslash\{\emptyset\}$ and $T_1, T_2 : X \Rightarrow Y$. We say that T_1 is a *submultifunction* of T_2, and we write $T_1 \leq T_2$, if $gr(T_1) \subseteq gr(T_2)$ (equivalently, $\forall x \in X$, $T_1(x) \subseteq T_2(x)$).

For a function f and a nonempty valued multifunction T, $f \leq T$ means that f is a selection for T. Theorems 3.2 and 3.3 reproduce the well-known results of Michael.

Theorem 3.2 (**Theorem 1 in** [21]). *Let X be a paracompact*[2] *space, Y be a Banach space, and $T : X \Rightarrow Y$ be nonempty closed convex valued. If T is l.s.c., then T has a continuous selection.*

Theorem 3.3 (**Theorem 2 in** [21]). *Let X be an ultraparacompact*[3] *space, Y be a complete metric space, and $T : X \Rightarrow Y$ be nonempty closed valued. If T is l.s.c., then T has a continuous selection.*

Michael's results follow from two approximation lemmas ([21, Lemma 1 and Lemma 2]), thanks to which every l.s.c. nonempty (convex) valued multifunction has, if not a continuous selection, at least a continuous approximating function. If $X \in \mathbf{V}\backslash\{\emptyset\}$ and (Y, d) is a metric space, $f \in Y^X$ is an *ϵ-approximation* ($\epsilon > 0$) for a nonempty valued $T : X \Rightarrow Y$ if for each $x \in X$, $d(f(x), T(x)) < \epsilon$.

Rather than with approximations, we can work always with selections; we need to introduce the following notion.

Definition 3.4. Let $X \in \mathbf{V}\backslash\{\emptyset\}$ and (Y, d) be a metric space. Let $\epsilon \geq 0$ and $T : X \Rightarrow Y$ be nonempty valued. The multifunction $\epsilon T : X \Rightarrow Y$, defined by

$$\epsilon T(x) = \begin{cases} S_d(T(x), \epsilon) & \text{if } \epsilon > 0, \\ T(x) & \text{if } \epsilon = 0 \end{cases}$$

where $x \in X$, is called the *ϵ-enlargement of T*.

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[2] $X \in Top$ is *paracompact* if it is Hausdorff and each of its open covers has an open locally finite refinement.

[3] $X \in Top$ is *ultraparacompact* if each of its open covers has a disjoint open refinement [30], equivalently, if it is paracompact and strongly zero-dimensional (strongly zero-dimensional means $Ind(X) = 0$; [12, section 7.1]).

Remark. Let $X \in \mathbf{V}\backslash\{\emptyset\}$, Y be a metric space, and $F, T : X \Rightarrow Y$ be nonempty valued.

(1) $\forall\epsilon, \delta > 0$, $(\delta\epsilon)T \neq \delta(\epsilon T) \leq (\delta + \epsilon)T$.
(2) If $F \leq T$, then $\forall\epsilon \geq 0$, $\epsilon F \leq \epsilon T$.
(3) If $F \leq \epsilon T$ for some $\epsilon \geq 0$, then $\forall\delta \geq 0$, $\delta F \leq (\delta + \epsilon)T$.

Lemma 3.5. *Let $X \in \mathbf{V}\backslash\{\emptyset\}$ ($X \in Top$, resp.), (Y, d) be a metric space, $T : X \Rightarrow Y$ be nonempty valued, and $f \in Y^X$. For every $\epsilon > 0$, the following are equivalent:*

(a) *f is a selection (continuous selection, resp.) for ϵT;*
(b) *f is an ϵ-approximation (continuous ϵ-approximation, resp.) for T.*

Stated in terms of enlargements, Michael's approximation lemmas become two selection theorems:

Lemma 3.6 (Lemma 1 in [21]). *Let X be a paracompact space, $(Y, \|\cdot\|)$ be a normed linear space, and $T : X \Rightarrow Y$ be nonempty convex valued. If T is l.s.c., then for every $\epsilon > 0$, there exists a continuous $f \in Y^X$ such that $f \leq \epsilon T$.*

Lemma 3.7 (Lemma 2 in [21]). *Let X be an ultraparacompact space, Y be a metric space, and $T : X \Rightarrow Y$ be nonempty valued. If T is l.s.c., then for every $\epsilon > 0$, there exists a continuous $f \in Y^X$ such that $f \leq \epsilon T$.*

The following is due to Deutsch and Kenderov [7].

Proposition 3.8 (Proposition 6.6.2 in [3]). *Let X be a paracompact space, $(Y, \|\cdot\|)$ be a normed linear space, and $T : X \Rightarrow Y$ be nonempty convex valued. The following are equivalent:*

(a) $\forall\epsilon > 0$, $\exists$ *a continuous $f \in Y^X$ such that $f \leq \epsilon T$;*
(b) $\forall\epsilon > 0$, $\forall x \in X$, $\exists$ $W \in \mathcal{N}(x)$ *such that* $\bigcap_{z \in W}(\epsilon T)(z) \neq \emptyset$.

Although never explicitly considered in the literature, a zero-dimensional version of this result is also true.

Proposition 3.9. *Let X be an ultraparacompact space, (Y, d) be a metric space, and $T : X \Rightarrow Y$ be nonempty valued. The following are equivalent:*

(a) $\forall\epsilon > 0$, $\exists$ *a continuous $f \in Y^X$ such that $f \leq \epsilon T$;*

(*b*) $\forall \epsilon > 0$, $\forall x \in X$, $\exists\, W \in \mathcal{N}(x)$ *such that* $\bigcap_{z \in W} (\epsilon T)(z) \neq \emptyset$.

Proof: $(a) \Rightarrow (b)$: Fix $\epsilon > 0$ and $x \in X$. Let $f \in Y^X$ be continuous such that $f \leq \frac{\epsilon}{2} T$. By the continuity of f, there exists $W \in \mathcal{N}(x)$ such that $f(W) \subseteq (\frac{\epsilon}{2} f)(x)$. Hence, for every $z \in W$, $d(f(z), f(x)) < \frac{\epsilon}{2}$ and $d(f(z), T(z)) < \frac{\epsilon}{2}$. Thus, for every $z \in W$, $d(f(x), T(z)) < \epsilon$, so that $f(x) \in \bigcap_{z \in W} (\epsilon T)(z) \neq \emptyset$.

$(b) \Rightarrow (a)$: Fix $\epsilon > 0$ and for every $x \in X$, choose $W_x \in \mathcal{N}(x)$ such that $\bigcap_{z \in W_x} (\epsilon T)(z) \neq \emptyset$. Since X is ultraparacompact, the cover $\{W_x : x \in X\}$ has a disjoint open refinement $\mathcal{V}$. For $V \in \mathcal{V}$, let $x_V \in X$ be such that $V \subseteq W_{x_V}$ and fix $y_V \in \bigcap_{z \in W_{x_V}} (\epsilon T)(z) \subseteq \bigcap_{z \in V} (\epsilon T)(z)$. The function $f : X \to Y$, defined by $f(x) = y_V$ if $x \in V$, is then a continuous selection for ϵT. □

Inspired by the last two results, we introduce the notion of "coherence."

Definition 3.10. Let $X \in Top$, $Y \in \mathbf{V} \backslash \{\emptyset\}$, and $T : X \Rightarrow Y$ be nonempty valued. We say that T is *coherent at* x if there exists $W \in \mathcal{N}(x)$ such that $\bigcap_{w \in W} T(w) \neq \emptyset$. We say that T is *coherent* if it is coherent at each $x \in X$.

Remark 3.11. (*b*) of both Proposition 3.8 and Proposition 3.9 is the statement that "for every $\epsilon > 0$, ϵT is coherent."

Lemma 3.12. *Let* $X \in Top$, $Y \in \mathbf{V} \backslash \{\emptyset\}$ *and* $S, T : X \Rightarrow Y$. *If* $S \leq T$ *and* S *is coherent, then* T *is coherent. In particular, if* Y *is a metric space and* $\epsilon \geq \eta \geq 0$, ηT *coherent implies* ϵT *coherent.*

4. Open fibers, lower semi-continuity and coherence

Proposition 4.1. *Let* $X, Y \in Top$ *and* $T : X \Rightarrow Y$ *be nonempty valued. If* T *has open fibers, then* T *is coherent.*

Coherence cannot be weaker than lower semi-continuity. Every strictly monotone real-valued continuous function is l.s.c. but not coherent.

Given a metric space (X, d) and $\epsilon > 0$, let $S_\epsilon^X : X \Rightarrow X$ be defined by $S_\epsilon^X(x) = S_d(x, \epsilon)$. If $x, x' \in X$, with $x \neq x'$, then

$$x' \in S_\epsilon^X(x) \Leftrightarrow d(x, x') < \epsilon \Leftrightarrow x \in S_\epsilon^X(x').$$

Hence, $x' \in S_\epsilon^X(x) \Leftrightarrow x' \in (S_\epsilon^X)^{-1}(x) = \{z \in X : x \in S_\epsilon^X(z)\}$. This proves the following lemma.

Lemma 4.2. *Let (X, d) be a metric space and $\epsilon > 0$. Then $S_\epsilon^X = (S_\epsilon^X)^{-1}$.*

Corollary 4.3. *Let $X \in \mathbf{V} \setminus \{\emptyset\}$, (Y, d) be a metric space, $T : X \Rightarrow Y$ be nonempty valued, and $\epsilon > 0$. Then $(\epsilon T)^{-1} = (S_\epsilon^Y \circ T)^{-1} = T^{-1} \circ S_\epsilon^Y$.*

Proposition 4.4. *Let $X \in Top$, (Y, d) be a metric space, and $T : X \Rightarrow Y$. T is l.s.c. if and only if ϵT has open fibers for every $\epsilon > 0$.*

Corollary 4.5. *Let $X \in Top$, (Y, d) be metric, and $T : X \Rightarrow Y$ nonempty valued. If T is l.s.c., then ϵT is coherent for every $\epsilon > 0$.*

Lemma 4.6. *Let $X, Y \in Top$ and $T : X \Rightarrow Y$ be nonempty valued. For every $x \in X$, the following are equivalent:*

(1) *T is coherent at x;*
(2) *$\exists\, y \in Y \ : \ x \in int(T^{-1}(y))$;*
(3) *$\exists\, y \in T(x) \ : \ x \in int(T^{-1}(y))$;*
(4) *$(intT^{-1})^{-1}(x) \neq \emptyset$.*

Proof: Fix $x \in X$. (1) $\Leftrightarrow$ (2) follows from the definition of coherence, while (2) $\Leftrightarrow$ (4) and (3) $\Rightarrow$ (2) are trivial. For (2) $\Rightarrow$ (3), notice that $x \in int(T^{-1}(y)) \ \Rightarrow \ x \in T^{-1}(y) \Leftrightarrow \ y \in T(x)$. □

Proposition 4.7. *Let $X, Y \in Top$ and $T : X \Rightarrow Y$ be nonempty valued. The following are equivalent:*

(1) *T is coherent;*
(2) *$\{int(T^{-1}(y)) : y \in Y\}$ is an (open) cover for X;*
(3) *$\forall x \in X, \exists\, y \in T(x) : x \in int(T^{-1}(y))$;*
(4) *$(intT^{-1})^{-1}$ is nonempty valued.*

Proposition 4.8. *Let $X, Y \in Top$ and $T : X \Rightarrow Y$ be nonempty valued. If T is coherent, then $(intT^{-1})^{-1}$ is a nonempty valued l.s.c. submultifunction of T.*

Proof: By Proposition 4.7, $(intT^{-1})^{-1}$ is nonempty valued. For $x \in X$, $y \in (intT^{-1})^{-1}(x) \ \Leftrightarrow \ x \in int(T^{-1}(y)) \Rightarrow \ x \in T^{-1}(y) \Leftrightarrow y \in T(x)$. Hence, $(intT^{-1})^{-1} \leq T$. Finally, if U is open in Y, then $((intT^{-1})^{-1})^{-1}(U) = int(T^{-1}(U))$, which is open in X. □

The example below shows that coherence $\not\Rightarrow$ lower semi-continuity. It also shows that the converse of both Proposition 4.1 and Corollary 4.5 is not in general true.

Example 1 [**A coherent multifunction, not l.s.c.**]. Let $X, Y = \mathbb{R}$ endowed with the Euclidean metric. Let $x_0, x_1 \in X$ be such that $x_0 < x_1$ and $y_0, y_1 \in Y$ be such that $y_0 < y_1$. Let $T : X \Rightarrow Y$ be defined by

$$T(x) = \begin{cases} \{y_0\} & \text{if } x < x_0, \\ \{y_0, y_1\} & \text{if } x_0 \leq x \leq x_1, \\ \{y_1\} & \text{if } x > x_1. \end{cases}$$

T is not l.s.c.; for example, $T^{-1}((y_0 - \frac{y_1-y_0}{3}, y_0 + \frac{y_1-y_0}{3})) = (-\infty, x_1]$ is not open.

On the other hand, $\{int(T^{-1}(y)) : y \in Y\}$ is an (open) cover for X, which is equivalent to the coherence of T by Proposition 4.7.

Having open fibers implies both lower semi-continuity and coherence but, even when combined together, these two requirements do not suffice for a multifunction to have open fibers.

Example 2 [**An l.s.c. coherent multifunction, not open fibers**]. Let $X, Y = [0, +\infty)$ be endowed with standard subspace topology. Choose $x_0, x_1 \in X$, $x_0 < x_1$, and $y \in Y$. Let $T : X \Rightarrow Y$ be defined by

$$T(x) = \begin{cases} [0, y] & \text{if } x_0 \leq x \leq x_1, \\ [0, y) & \text{otherwise.} \end{cases}$$

For every $x \in X$ and every $U \in \mathcal{N}(x)$, $[0, y) \subseteq \bigcap_{z \in U} T(z)$, so that T is coherent. It is also clear that T is l.s.c.: the preimage of an open subset of Y is either $\emptyset$ or X. Since $T^{-1}(y) = [x_0, x_1]$, T does not have open fibers.

5. Coherence and ϵ-continuity: existence of continuous selections implies coherence for enlargements

Definition 5.1. Let $X \in Top$, (Y, d) be a metric space, and $\epsilon > 0$. $f \in Y^X$ is *ϵ-continuous* if for every $x \in X$, there exists $U \in \mathcal{N}(x)$ such that $d(f(z), f(x)) < \epsilon$, whenever $z \in U$. f is *0-continuous* if for every $x \in X$, there exists $U \in \mathcal{N}(x)$ such that $f(z) = f(x)$, whenever $z \in U$ (i.e., if f is locally constant).

Lemma 5.2. *Let $X \in Top$, (Y, d) be metric, $f \in Y^X$, and $\epsilon \geq 0$.*

(1) *If f is ϵ-continuous, then f is η-continuous for every $\eta > \epsilon$.*
(2) *If f is ϵ-continuous, then ϵf is coherent.*
(3) *If ϵf is coherent, then f is (2ϵ)-continuous.*

Proposition 5.3. *Let $X \in Top$, (Y, d) be a metric space, and $f \in Y^X$. The following are equivalent:*

(1) *f is continuous;*
(2) *$\forall \epsilon > 0$, f is ϵ-continuous;*
(3) *$\forall \epsilon > 0$, ϵf is coherent;*
(4) *$\forall \epsilon > 0$, ϵf has open fibers.*

Proof: For (1) $\Leftrightarrow$ (4), apply Proposition 4.4 with f in place of T. □

Proposition 5.4. *Let $X \in Top$, (Y, d) be a metric space, and $f \in Y^X$. Then, f is locally constant if and only if f is coherent.*

Lemma 3.12 and the Remark after Definition 3.4 yield the following.

Proposition 5.5. *Let $X \in Top$ and (Y, d) be a metric space. Let $\epsilon, \delta \geq 0$ and $T : X \Rightarrow Y$ be nonempty valued. If there exists $f \in Y^X$ such that $f \leq \epsilon T$ and δf is coherent, then $(\delta + \epsilon)T$ is coherent.*

Corollary 5.6. *Let $X \in Top$ and (Y, d) be a metric space. Let $\epsilon \geq 0$ and $T : X \Rightarrow Y$ be nonempty valued. If there exists $f \in Y^X$ such that $f \leq \epsilon T$ and ϵf is coherent, then $(2\epsilon)T$ is coherent.*

Corollary 5.7 ((a) ⇒ (b) of Proposition 3.8 and Proposition 3.9). *Let $X \in Top$, (Y, d) be a metric space, and $T : X \Rightarrow Y$ be nonempty valued. If for every $\epsilon > 0$, ϵT has a continuous selection, then for every $\epsilon > 0$, ϵT is coherent.*

Corollary 5.8. *Let $X \in Top$, (Y, d) be a metric space, and $T : X \Rightarrow Y$ be nonempty valued. If T has a continuous selection, then ϵT is coherent for every $\epsilon > 0$.*

Corollary 5.8 tells us that the coherence of each ϵ-enlargement is not only implied by lower semi-continuity, but it is also a necessary condition for the existence of a continuous selection.

Proposition 5.5 and Proposition 5.4 yield the following corollary.

Corollary 5.9. *Let $X \in Top$, (Y, d) be a metric space, and $T : X \Rightarrow Y$ be nonempty valued. For every $\epsilon \geq 0$, if ϵT has a locally constant selection, then ϵT is coherent.*

Using the characterization of continuity stated in Proposition 5.3 and Corollary 5.6, we may complete Proposition 3.8 and Proposition 3.9 as follows:

Proposition 5.10. *Let X be a paracompact (ultraparacompact, resp.) space, $(Y, \|\cdot\|)$ be a normed linear space ((Y, d) be a metric space, resp.), and $T : X \Rightarrow Y$ be nonempty convex valued (nonempty valued, resp.). The following are equivalent:*

(*a*) $\forall \epsilon > 0$, $\exists$ *a continuous* $f \in Y^X$ *such that* $f \leq \epsilon T$*;*
(*b*) $\forall \epsilon > 0$, $\exists f \in Y^X$ *such that* $f \leq \epsilon T$ *and* ϵf *is coherent;*
(*c*) $\forall \epsilon > 0$, ϵT *is coherent.*

Remark 5.11. Neither the paracompactness of X and the convexity of Y on one side, nor the zero-dimensionality of X on the other plays any role in the entire section; we need the assumption on X and Y only to show $(c) \Rightarrow (a)$ (see [3, proofs of Proposition 6.6.2] and Proposition 3.9).

6. Formal convex combinations

In section 7, we will define a convex structure on a generic topological space X. The first step in this direction is to give a meaning to the expression "convex combination of elements from X." The notion we need mustn't involve any linear property; it will be clear in a moment that for our purposes X can be just a set.

Definition 6.1. Let $X \in \mathbf{V} \setminus \{\emptyset\}$. Denote by $FCC(X)$ the set $\{s \in [0,1]^X : |supp(s)| < \infty \wedge \sum_{x \in X} s(x) = 1\}$. We call $FCC(X)$ *the set of formal convex combinations over* X.

As a subset of $[0,1]^X$, $FCC(X)$ inherits both the product topology, τ_p, and the uniform topology, τ_u, generated by the uniform metric ρ_u ($S_u(f, r)$ will denote an open ball with respect to ρ_u of center $f \in [0,1]^X$ and radius $r > 0$). It is routine to show that these two topologies actually coincide on $FCC(X)$.

Proposition 6.2. *Let $X \in \mathbf{V} \backslash \{\emptyset\}$. Then $\tau_p = \tau_u$ on $FCC(X)$.*

Definition 6.3. Let $X \in \mathbf{V} \backslash \{\emptyset\}$. The set $FCC(X)$ endowed with the uniform topology τ_u (equivalently with the product topology τ_p) is called *the space of formal convex combinations over X*. We will denote it simply by $FCC(X)$.

From Proposition 6.2, it follows immediately:

Proposition 6.4. *For every $X \in \mathbf{V} \backslash \{\emptyset\}$, $FCC(X)$ is a metrizable space.*

Let $X \in \mathbf{V} \backslash \{\emptyset\}$. For $n \in \mathbb{N} \cup \{\infty\}$, $[X]^{<n} = \{A \subseteq X : |A| < n\}$. For $A \in P(X)$, $A^{\wedge} = \{s \in FCC(X) : supp(s) \subseteq A\}$; if $A \in [X]^{<\infty}$, $A^{\perp} = \{s \in FCC(X) : supp(s) = A\}$. ($A$ can be regarded as the projection on X of the set $A^{\perp}$.)

We will sometime identify a natural number n with the set of all its predecessors: $n = \{0, 1, \cdots, n-1\}$.

Remark. Let $X \in \mathbf{V} \backslash \{\emptyset\}$.

(1) For every $x \in X$, let $s_x \in FCC(X)$ be defined by

$$s_x(z) = \begin{cases} 1 & \text{if } z = x, \\ 0 & \text{if } z \neq x. \end{cases}$$

s_x is the only element in $FCC(X)$ to have as support $\{x\}$. It is then clear that $x^{\perp} = x^{\wedge} = \{s_x\}$. In the following, we will identify both $x^{\perp}$ and $x^{\wedge}$ with the function s_x.

(2) The subspace topology on $\{x^{\wedge} : x \in X\} \subseteq FCC(X)$ is the discrete one.

(3) The map $\varpi : \{x^{\wedge} : x \in X\} \to X$ defined by $x^{\wedge} \to x$ is a bijection between $\{x^{\wedge} : x \in X\}$ and X and a homeomorphism if and only if X is a discrete space.

(4) For every function $\phi : FCC(X) \to X$, the restriction $\phi \restriction \{x^{\wedge} : x \in X\}$ is continuous. In particular, ϖ is continuous.

(5) For every $A \in P(X)$, the space $(FCC(A), \tau_u)$ is homeomorphic to $A^{\wedge}$ endowed with the subspace topology.

(6) For every $A \in [X]^{<\infty}$, with $|A| = n$, $FCC(|A|)$ is the simplex of dimension n, $P_n = \{\mathbf{t} = (t_0, t_1, \cdots, t_{n-1}) \in [0,1]^n : \sum_{i \in n} t_i = 1\}$. In particular, for $i \in n$, $i^{\wedge}$ is the point in P_n whose coordinates are all 0's except for the i-th one which is 1, usually denoted by $\mathbf{e_i}$.

(7) If $A \in [X]^{<\infty}$, $A^\wedge = \bigcup_{B \subseteq A} B^\perp$.
(8) If $A \in [X]^{<\infty}$, $A^\wedge$ is a closed subset of $([0,1]^X, \tau_u)$, and hence of $FCC(X)$.
(9) If $A \in [X]^{<\infty}$, $A^\wedge$ is compact and metrizable.

7. Formally n-convex spaces

In what follows, we will adopt the convention that $\infty + 1 = \infty$.

Definition 7.1. Let $X \in \mathbf{V} \setminus \{\emptyset\}$. For every $n \in \mathbb{N} \cup \{\infty\}$, let

$$FCC_n(X) = \{s \in FCC(X) : |supp(s)| < n+1\}.$$

It is clear that $FCC(X) = FCC_\infty(X) = \bigcup_{n \in \mathbb{N}} FCC_n(X)$ and that $FCC_1(X) = \{x^\wedge : x \in X\}$ is discrete.

Definition 7.2. Let $X \in \mathbf{V} \setminus \{\emptyset\}$. Given $n \in \mathbb{N} \cup \{\infty\}$, a function $\phi : FCC_n(X) \to X$ and $A \in P(X)$, let

$$conv_\phi^n(A) = \{\phi(s) : s \in A^\wedge \cap FCC_n(X)\}.$$

We call $conv_\phi^n(A)$ the *n-convex hull of A w.r.t. ϕ.* The ∞-convex hull $conv_\phi^\infty(A)$ will be also denoted by $conv_\phi(A)$.

A is called *n-stable (under ϕ)* if $conv_\phi^n(A) \subseteq A$. An ∞-stable subset will be also called *stable*.

Definition 7.3. Let $X \in Top$ and $n \in \mathbb{N} \cup \{\infty\}$. X is *formally n-convex* if there is a function $\pi : FCC_n(X) \to X$ such that

($C1$) $\pi \restriction A^\wedge$ is continuous whenever $A \in [X]^{<n+1}$;
($C2$) $\pi(x^\wedge) = x$ for every $x \in X$ (i.e., π is an extension of ϖ).

The map π is an *n-convex map for X*. A formally ∞-convex space will be called also *formally convex* and an ∞-convex map also *convex map*.

Proposition 7.4. *Every topological space admits a* 1*-convex map and each of its nonempty subsets is* 1*-stable under such a map. In particular, every topological space is formally* 1*-convex.*

Proof: Let $X \in Top$. The map ϖ is a 1-convex map for X (see Remark(4) in section 6). Moreover, if A is a nonempty subset of X, then $\varpi(s) \in A$, whenever $s \in A^\wedge \cap FCC_1(X)$. $\square$

Thus, the class of formally 1-convex spaces coincides with Top. What can we say for $n \geq 2$? To characterize formally n-convex spaces, we need to introduce the notion of "n-wise connectedness."

Definition 7.5. Let $X \in Top$ and $n \geq 2$. We say that X is *n-wise connected* if for every $A \in [X]^{<n+1}$ there exists a continuous function $\theta_A : A^\wedge \to X$ such that $\theta_A(a^\wedge) = a$ for every $a \in A$. A function such as θ_A is a *path through the points of A.*

Definition 7.6. Let $X \in Top$ and $n \geq 2$. We say that X is *compatibly n-wise connected* if it is n-wise connected and for every $A \in [X]^{<n+1}$ the following holds:

($\star$) if $B \subseteq A$, then $\theta_B = \theta_A \restriction B^\wedge$.

The terminology used in the last two definitions is justified by the two propositions below.

Definition 7.7. Let $X \in \mathbf{V}\backslash\{\emptyset\}$. For every $A \in [X]^{<\infty}$, with $|A| = n$, and every enumeration $\{a_0, \cdots, a_{n-1}\}$ of the elements of A, let $H_{a_0,\cdots,a_{n-1}} : A^\wedge \to FCC(n)$ be defined by

$$H_{a_0,\cdots,a_{n-1}}(s) = (s(a_0), \cdots, s(a_{n-1})),$$

whenever $s \in A^\wedge$.

Lemma 7.8. *Let $X \in \mathbf{V}\backslash\{\emptyset\}$. For every $A \in [X]^{<\infty}$, with $|A| = n$, and every enumeration $\{a_0, \cdots, a_{n-1}\}$ of the elements of A, the map $H_{a_0,\cdots,a_{n-1}} : A^\wedge \to FCC(n)$ is a homeomorphism.*

Proposition 7.9. *Let $X \in \mathbf{V}\backslash\{\emptyset\}$. For every $A \in [X]^{<\infty}$, $A^\wedge$ is homeomorphic to the simplex of dimension $|A|$.*

In what follows, let $\gamma : [0, 1] \to FCC(2)$ be defined by

$$\gamma(t) = (t, 1 - t),$$

whenever $t \in [0, 1]$. Note that γ is a homeomorphism of the unit interval, $[0, 1]$, into the simplex of dimension 2, $FCC(2)$.

Lemma 7.10. *Let $X \in Top$. For every $a_0, a_1 \in X$*

(*a*) *if there exists a continuous $\theta_{\{a_0,a_1\}} : \{a_0, a_1\}^\wedge \to X$ such that $\theta_{\{a_0,a_1\}}(a_0^\wedge) = a_0$ and $\theta_{\{a_0,a_1\}}(a_1^\wedge) = a_1$, then $\theta_{\{a_0,a_1\}} \circ H^{-1}_{a_0,a_1} \circ \gamma : [0, 1] \to X$ is continuous and maps 0 into a_1 and 1 into a_0;*

(*b*) *if there exists a continuous* $f_{a_0,a_1} : [0,1] \to X$ *such that* $f_{a_0,a_1}(0) = a_0$ *and* $f_{a_0,a_1}(1) = a_1$*, then* $f_{a_0,a_1} \circ \gamma^{-1} \circ H_{a_1,a_0} : \{a_0, a_1\}^\wedge \to X$ *is continuous and maps* $a_0^\wedge$ *into* a_0 *and* $a_1^\wedge$ *into* a_1*.*

Proposition 7.11. *Let* $X \in Top$*. The following are equivalent:*

(*a*) X *is* 2*-wise connected;*
(*b*) X *is compatibly* 2*-wise connected;*
(*c*) X *is pathwise connected.*

Proof: $(a) \Leftrightarrow (b)$ is clear from the definition. For $(a) \Leftrightarrow (c)$, let $a_0, a_1 \in X$. By Lemma 7.10, there exists a path $\theta_{\{a_0,a_1\}} : \{a_0, a_1\}^\wedge \to X$ through a_0 and a_1 if and only if there exists a continuous path $f : [0,1] \to X$ joining a_0 to a_1. □

We can now give the promised characterization for formally n-convex spaces. First a lemma:

Lemma 7.12. *Let* $X \in Top$ *and* $n \geq 2$*. If for every* $A \in [X]^{<n+1}$*, there exists a path* $\theta_A : A^\wedge \to X$ *through the points of* A *such that* $(\star)$ *of Definition 7.6 is satisfied, then the map* $\pi : FCC_n(X) \to X$ *defined by* $\pi(s) = \theta_{supp(s)}(s)$*, whenever* $s \in FCC_n(X)$*, is an* n*-convex map for* X*.*

Proof: For every $A \in [X]^{<n+1}$, $\pi\restriction A^\wedge = \theta_A$, which is continuous. Also, $\pi(x^\wedge) = x$, whenever $x \in X$. □

Theorem 7.13. *Let* $X \in Top$ *and* $n \geq 2$*. Then* X *is formally* n*-convex if and only if it is compatibly* n*-wise connected.*

Proof: If X is a formally n-convex space and π is an n-convex map for X, then, for every $A \in [X]^{<n+1}$, $\theta_A = \pi\restriction A^\wedge$ is a path through the points of A satisfying $(\star)$ of Definition 7.6.

If X is compatibly n-wise connected, for every $A \in [X]^{<n+1}$, fix $\theta_A : A^\wedge \to X$ to be a continuous path through the points of A so that $(\star)$ of Definition 7.6 is satisfied. By the previous lemma, $\pi : FCC_n(X) \to X$ defined by $\pi(s) = \theta_{supp(s)}(s)$, whenever $s \in FCC_n(X)$, is an n-convex map for X. □

Corollary 7.14. *A topological space is formally* 2*-convex if and only if it is pathwise connected.*

We shall show now that topological vector spaces belong to the class of formally n-convex spaces whenever $n \in \mathbb{N} \cup \{\infty\}$.

Let X be a convex subset of a topological vector space. Define $\pi_X : FCC(X) \to X$ as

$$\pi_X(s) = \sum_{x \in supp(s)} s(x) \cdot x,$$

whenever $s \in FCC(X)$. For every $x \in X$, $\pi_X(x^\wedge) = x$. Also, every convex subset of X is stable under π_X.

Lemma 7.15. *Let X be a convex subset of a topological vector space. For every $n \in \mathbb{N} \cup \{\infty\}$, $\pi_X \restriction FCC_n(X)$ is an n-convex map for X.*

Proof: Let $n \in \mathbb{N} \cup \{\infty\}$, $A = \{x_1, \cdots, x_N\} \subseteq X$ $(N < n+1)$ and $s \in A^\wedge$. Consider a neighborhood V of $\pi(s) = \sum_{i \le N} \alpha_i \cdot x_i$ $(\alpha_1, \cdots, \alpha_N \in [0,1])$. By continuity of the addition and of the scalar multiplication (see [31, section 1.6]), there exist $r > 0$ and neighborhoods $V_1, \cdots, V_N$ of $\alpha_1 \cdot x_1, \cdots, \alpha_N \cdot x_N$, respectively, such that $\sum_{i \le N} \beta_i \cdot x_i \subseteq \sum_{i \le N} V_i \subseteq V$, whenever $|\beta_i - \alpha_i| < r$. Hence, $U = S_u(s, r) \cap A^\wedge$ is a τ_u-open neighborhood of s and $\pi(t) \in \sum_{i \le N} V_i$ for all $t \in U$. $\square$

Proposition 7.16. *Every convex subset of a topological vector space (hence, every normed vector space) is a formally convex space.*

8. Locally $< n$ families and n-paracompactness

Definition 8.1. Let $X \in \mathbf{V} \setminus \{\emptyset\}$. A family $\mathcal{F} = \{f_i \in [0,1]^X : i \in I\}$ is called *normalized* if for every $x \in X$, $\sum_{i \in I} f_i(x) = 1$.

Given a normalized family $\mathcal{F} = \{f_i \in [0,1]^X : i \in I\}$, the family $\{f_i^{-1}((0,1]) : i \in I\}$ is a cover for X.

Definition 8.2. Let $X \in \mathbf{V} \setminus \{\emptyset\}$ and $\mathcal{B} \subseteq P(X)$. Given $n \in \mathbb{N} \cup \{\infty\}$, we say that $\mathcal{B}$ is *point* $< n$ if

$$\forall x \in X, |\{B \in \mathcal{B} : x \in B\}| < n.$$

A normalized family $\mathcal{F} = \{f_i : i \in I\}$ for X is a point $< n$ if the family $\{f_i^{-1}((0,1]) : i \in I\}$ is a point $< n$ cover for X.

Definition 8.3 (Section 5.1 in [12]**).** Let $X \in Top$. A normalized family $\mathcal{F} = \{f_i : i \in I\}$ for X is a *partition of unity* if each f_i is continuous.

Following Definition 3.3.8 in [3], we say that a partition of unity $\{f_i : i \in I\}$ for X is *subordinated* to an open cover $\{U_i : i \in I\}$ of X if for every $i \in I$, $\overline{f_i^{-1}((0,1])} \subseteq U_i$.

Definition 8.4. Let $X \in Top$ and $\mathcal{B} \subseteq P(X)$. Given $n \in \mathbb{N} \cup \{\infty\}$, we say that $\mathcal{B}$ is *locally* $< n$ if

$$\forall x \in X \ \ \exists U_x \in \mathcal{N}(x) : |\{B \in \mathcal{B} : U_x \cap B \neq \emptyset\}| < n.$$

A partition of unity $\mathcal{F} = \{f_i : i \in I\}$ for X is locally $< n$ if the family $\{f_i^{-1}((0,1]) : i \in I\}$ is a locally $< n$ cover for X.

The notions of point $< \infty$ and locally $< \infty$ family coincide with the ones of point-finite and locally finite family, respectively. Moreover, for every $m \in \mathbb{N} \cup \{\infty\}$, $n \geq m$: point $< m \Rightarrow$ point $< n$; locally $< m \Rightarrow$ locally $< n$; locally $< m \Rightarrow$ point $< n$.

Definition 8.5. Let $X \in \mathbf{V} \backslash \{\emptyset\}$ admit a point-finite normalized family, $\mathcal{F} = \{f_x : x \in X\}$, indexed in X. The function $A_{\mathcal{F}} = \prod_{x \in X} f_x : X \to FCC(X)$ which associates to each $a \in X$ the function $A_{\mathcal{F}}(a) \in FCC(X)$ defined by

$$A_{\mathcal{F}}(a)(x) = f_x(a),$$

whenever $x \in X$, is the *evaluation map of the family* $\mathcal{F}$.

If $X \in \mathbf{V} \backslash \{\emptyset\}$ admits a point $< n$ normalized family indexed in X ($n \in \mathbb{N}$), then $A_{\mathcal{F}}(X) \subseteq FCC_{n-1}(X) \subseteq FCC_n(X)$.

Lemma 8.6. *For every $X \in Top$ admitting a point-finite partition of unity indexed in X, $\mathcal{F} = \{f_x : x \in X\}$, the evaluation map $A_{\mathcal{F}} : X \to FCC(X)$ is continuous.*

Proof: $A_{\mathcal{F}}$ is the product of a family of continuous functions. □

Given $X, Y \in \mathbf{V} \backslash \{\emptyset\}$, every $h \in Y^X$ induces a map from $FCC(X)$ into $FCC(Y)$. For $x \in X$, denote by ρ_x the x-th projection map for $[0,1]^X$, that is the map $\rho_x : s \to s(x)$.

For every $y \in Y$, let $h_y : FCC(X) \to [0,1]$ be defined by

$$h_y(s) = \sum_{x \in h^{-1}(y)} \rho_x(s) = \sum_{x \in h^{-1}(y)} s(x)$$

whenever $s \in FCC(X)$. We shall denote by h^* the product map $\prod_{y \in Y} h_y : FCC(X) \to FCC(Y)$.

Lemma 8.7. *Let $X, Y \in \mathbf{V} \backslash \{\emptyset\}$ and $h \in Y^X$. For every $y \in Y$ and every $n \in \mathbb{N}$, $h_y \restriction FCC_n(X)$ is continuous.*

Proof: Fix $y \in Y$ and $n \in \mathbb{N}$. Let $s \in FCC_n(X)$ and $\epsilon > 0$. For every $t \in FCC_n(X)$, $|supp(s) \cup supp(t)| \leq 2n$. Choose $\delta < \frac{\epsilon}{2n}$ and let $H_t = h^{-1}(y) \cap (supp(s) \cup supp(t))$. Then, for every $t \in S_u(s, \delta) \cap FCC_n(X)$, we have $|\sum_{x \in h^{-1}(y)} t(x) - \sum_{x \in h^{-1}(y)} s(x)| = |\sum_{x \in H_t} (t(x) - s(x))| \leq \sum_{x \in H_t} |t(x) - s(x)| < |H_t| \cdot \delta < \epsilon$; thus, $h_y(t) \in (h_y(s) - \epsilon, h_y(s) + \epsilon)$. □

Proposition 8.8. *Let $X, Y \in \mathbf{V} \backslash \{\emptyset\}$ and $h \in Y^X$. For every $n \in \mathbb{N}$, $h^* \restriction FCC_n(X)$ is continuous (and its range is contained in $FCC_n(Y)$).*

Proof: For every $n \in \mathbb{N}$, $h^* \restriction FCC_n(X) = \prod_{y \in Y} (h_y \restriction FCC_n(X))$. □

Definition 8.9. Let $X, Y \in \mathbf{V} \backslash \{\emptyset\}$, $\mathcal{F} = \{f_x : x \in X\}$ be a normalized family indexed in X, and $T : X \Rightarrow Y$ be nonempty valued. T is *$\mathcal{F}$-coherent* if for every $x \in X$, $\bigcap_{z \in f_x^{-1}((0,1])} T(z) \neq \emptyset$.

Definition 8.10. Let $X \in \mathbf{V} \backslash \{\emptyset\}$, Y be a formally n-convex space, and $T : X \Rightarrow Y$ be nonempty valued. T is *n-stable valued* if there exists an n-convex map for Y under which $T(x)$ is n-stable for every $x \in X$.

Proposition 8.11. *Let $X, Y \in Top$, $n \in \mathbb{N} \cup \{\infty\}$, and $T : X \Rightarrow Y$ be nonempty valued. Suppose that X admits a locally $< n+1$ partition of unity indexed in X, $\mathcal{P} = \{p_x : x \in X\}$, Y is formally n-convex, and T is n-stable valued. If T is $\mathcal{P}$-coherent, then T has a continuous selection.*

Proof: For every $x \in X$, choose arbitrarily $y_x \in \bigcap_{z \in p_x^{-1}((0,1])} T(z)$, and define $h \in Y^X$ by $h(x) = y_x$. For every $x \in X$, let $U_x \in \mathcal{N}(x)$ be such that $F_x = \{z \in X : U_x \cap p_z^{-1}((0,1]) \neq \emptyset\}$ has cardinality $m_x < n+1$. Let π be an n-convex map for Y such that $T(x)$ is n-stable for all $x \in X$.

We claim that $\pi \circ h^* \circ A_{\mathcal{P}}$ is a continuous selection for T. Since the U_x's form an open cover for X, it suffices to show that $(\pi \circ h^* \circ A_{\mathcal{P}}) \restriction U_x$ is continuous for every $x \in X$.

Fix $x \in X$. By Lemma 8.6, $A_{\mathcal{P}}$ is continuous. For every $u \in U_x$, $A_{\mathcal{P}}(u) \in F_x^{\wedge}$; hence, $A_{\mathcal{P}}(U_x) \subseteq F_x^{\wedge}$. Since $F_x^{\wedge} \subseteq FCC_{m_x}(X)$, $h^*\lceil A_{\mathcal{P}}(U_x)$ is continuous by Proposition 8.8. Now, for every $s \in A_{\mathcal{P}}(U_x)$, $(h^*\lceil A_{\mathcal{P}}(U_x))(s) \in \{y_z : z \in F_x\}^{\wedge}$; hence, $(h^*\lceil A_{\mathcal{P}}(U_x) \circ A_{\mathcal{P}})(U_x) \subseteq \{y_z : z \in F_x\}^{\wedge}$. Since $\{y_z : z \in F_x\} \in [Y]^{<n+1}$, by definition of n-convex map, $(\pi \circ h^* \circ A_{\mathcal{P}})\lceil U_x = \pi\lceil(h^*\lceil A_{\mathcal{P}}(U_x) \circ A_{\mathcal{P}})(U_x)$ is continuous.

Finally, we check that $\pi \circ h^* \circ A_{\mathcal{P}}$ is a selection for T. For every $x \in X$, $\{y_z : x \in p_z^{-1}((0,1])\} \subseteq T(x)$. Since $(h^* \circ A_{\mathcal{P}})(x) \in \{y_z : x \in p_z^{-1}((0,1])\}^{\wedge}$, $h^*(A_{\mathcal{P}}(x)) \in (T(x))^{\wedge}$. Since $T(x)$ is n-stable under π, $\pi(h^*(A_{\mathcal{P}}(x))) \in T(x)$. □

Definition 8.12. Let X be a T_1 space. For every $n \in \mathbb{N} \cup \{\infty\}$, X is called *n-paracompact* if every open cover of X has a locally $< n+1$ partition of unity subordinated to it.

For every $m \in \mathbb{N}$, m-paracompactness implies n-paracompactness for all $n \geq m$, and n-paracompactness implies paracompactness.

The proof of the following result is routine. As in [11, section 1.6] and [12, section 7.1], *dimX* is the "covering dimension" of a normal space X.

Proposition 8.13. *Let X be a T_1 space. Then*

(1) *X is ∞-paracompact if and only if it is paracompact;*
(2) *X is 1-paracompact if and only if it is ultraparacompact;*
(3) *if X is n-paracompact for $n > 1$, then it is paracompact and $dimX \leq n-1$; the reverse holds if X is compact.*

Corollary 8.14. *Let $n \in \mathbb{N}$. Every finite n-dimensional CW-complex is $(n+1)$-paracompact.*

Proof: Every finite CW-complex is a Hausdorff compact space. □

So, for every $n \in \mathbb{N}$, the Euclidean n-sphere S^n is $(n+1)$-paracompact. Every compact 2-manifold is a 3-paracompact space [4]; meanwhile, every finite graph provides an example of 2-paracompact space (for more about CW-complexes, see [18, section IX.3]).

[4]Every compact 2-manifold can be endowed with the structure of a finite, 2-dimensional CW-complex via triangulation.

Corollary 8.15. *For every $n, m \in \mathbb{N}$ such that $n \leq m$ and $m \geq 1$, the Menger's cube M_n^m is $(n+1)$-paracompact.*

Proof: M_n^m is an n-dimensional compact subspace of $\mathbb{R}^m$ (see [11, chapter 1, section 11]). □

M_0^1 is the Cantor space, 2^ω, and it is, of course, $1-paracompact$ (i.e., ultraparacompact). All 1-dimensional[5] Menger's cubes, in particular M_1^2 and M_1^3, are 2-paracompact.

Lemma 8.16. *Let X be an n-paracompact space for some $n \in \mathbb{N} \cup \{\infty\}$. Let $Y \in Top$ and $T : X \Rightarrow Y$ be nonempty valued. If T is coherent, then there exists a locally $< n+1$ partition of unity $\mathcal{P}$ for X such that T is $\mathcal{P}$-coherent.*

Proof: For every $x \in X$, let $W_x \in \mathcal{N}(x)$ be such that $\bigcap_{z \in W_x} T(z) \neq \emptyset$. $\mathcal{W} = \{W_x : x \in X\}$ is an open cover for X. Since X is n-paracompact, there is a locally $< n+1$ partition of unity $\mathcal{P} = \{p_x : x \in X\}$ subordinated to $\mathcal{W}$; i.e., for every $x \in X$, $\overline{p_x^{-1}((0,1])} \subseteq W_x$. Hence, for every $x \in X$, $\bigcap_{z \in p_x^{-1}((0,1])} T(z) \neq \emptyset$. □

9. The key result: coherence implies existence of continuous selections

Proposition 8.11 and Lemma 8.16 yield the key result of this paper.

Theorem 9.1. *Let $n \in \mathbb{N} \cup \{\infty\}$. Let X be an n-paracompact space, Y be a formally n-convex space, and $T : X \Rightarrow Y$ be nonempty n-stable valued. If T is coherent, then T has a continuous selection.*

Corollary 9.2. *Let X be an ultraparacompact space, $Y \in Top$, and $T : X \Rightarrow Y$ be a nonempty valued multifunction. If T is coherent, then T has a continuous selection.*

Proof: By Proposition 7.4, Y is formally 1-convex and T is 1-stable valued. □

Corollary 9.3 ((b) $\Rightarrow$ (a) of Proposition 3.9). *Let X be an ultraparacompact space, (Y, d) be a metric space, $T : X \Rightarrow Y$ be a*

[5]The construction of M_1^2 and M_1^3 is illustrated by figures 12 and 13 in [11, section I.11].

nonempty valued multifunction, and $\epsilon > 0$*. If* ϵT *is coherent, then* ϵT *has a continuous selection.*

Corollary 9.4. *Let* X *be a paracompact space,* Y *be a convex subset of a topological vector space, and* $T : X \Rightarrow Y$ *be a nonempty convex valued multifunction. If* T *is coherent, then* T *has a continuous selection.*

Corollary 9.5 (**Lemma 2.4.3 in** [36]). *Let* X *be a paracompact space,* Y *be a convex subset of a topological vector space, and* $T : X \Rightarrow Y$ *be nonempty convex valued. If* T *has open fibers, then* T *has a continuous selection.*

Proof: By Proposition 4.1, T is coherent. □

Corollary 9.6 ((**b**) $\Rightarrow$ (**a**) **of Proposition 3.8**). *Let* X *be a paracompact space and* $(Y, \|\cdot\|)$ *be a normed vector space. Let* $T : X \Rightarrow Y$ *be nonempty convex valued and* $\epsilon > 0$*. If* ϵT *is coherent, then* ϵT *has a continuous selection.*

Combined with Corollary 4.5, Theorem 9.1 leads to the following from which both of Michael's approximation lemmas may be derived.

Proposition 9.7. *Let* $n \in \mathbb{N} \cup \{\infty\}$*. Let* X *be an* n*-paracompact space,* Y *be a formally* n*-convex space, and* $T : X \Rightarrow Y$ *be nonempty valued. Suppose that for every* $\epsilon > 0$*,* ϵT *is* n*-stable valued. If* T *is l.s.c., then for every* $\epsilon > 0$*, there exists a continuous* $f \in Y^X$ *such that* $f \leq \epsilon T$*.*

Corollary 9.8 (**Lemma 3.6**). *Let* X *be a paracompact space,* Y *be a normed linear space, and* $T : X \Rightarrow Y$ *be nonempty convex valued. If* T *is l.s.c., then for every* $\epsilon > 0$*, there exists a continuous* $f \in Y^X$ *such that* $f \leq \epsilon T$*.*

Proof: By Proposition 7.16, Y is formally convex. Since in normed linear spaces convexity is preserved under enlargements, all enlargements of T are convex valued multifunctions. Moreover, all convex subsets of Y are stable (under the canonical convex map π_Y; see Lemma 7.15). Hence, for every $\epsilon > 0$, ϵT is stable valued. □

Corollary 9.9 (**Lemma 3.7**). *Let* X *be an ultraparacompact space,* Y *be a metric space, and* $T : X \Rightarrow Y$ *be nonempty valued. If* T *is* *l.s.c., then for every $\epsilon > 0$, there exists a continuous $f \in Y^X$ such that $f \leq \epsilon T$.*

Proof: By Proposition 7.4, each ϵT is 1-stable valued. □

10. Some applications

Theorem 9.1 can also have concrete applications. We shall describe some of them.

Definition 10.1. Let $X \in Top$ and $\mathcal{F} \subseteq P(X)$. $\mathcal{F}$ is a *pairwise-pathwise-connected family*, or shortly a *PPC family*, if

$$\forall F, F' \in \mathcal{F},\ F \cap F' \text{is pathwise connected.}$$

$\mathcal{F}$ is a *collectionwise-pathwise-connected family*, or shortly a *CPC family*, if

$$\forall \mathcal{F}' \subseteq \mathcal{F},\ \bigcap \mathcal{F}' \text{is pathwise connected.}$$

PPC and CPC families exist; for instance, every family of convex subsets is PPC and CPC. Of course, every CPC family is a PPC family; for the reverse, we need to consider further hypotheses.

Example 3. Let K be the complete bipartite graph $K_{2,3}$ ([35, chapter 2]). Let B be one of its two black vertices and W_1, W_2, W_3 its white vertices. For $i = 1, 2, 3$, let BW_i be the edge joining B and W_i. The family $\{K \backslash BW_i : i = 1, 2, 3\}$ is PPC, but not CPC.

Proposition 10.2. *Let X be a connected graph admitting a cut-set C of cardinality ≥ 3. The family $\{X \backslash e : e \in C\}$ is PPC, but not CPC.*

Proof: $\bigcap_{e \in C} X \backslash e = X \backslash C$ is a disconnected graph, since C is a disconnecting set. At the same time, by the minimality of C, $(X \backslash e) \cap (X \backslash e')$ is a connected graph, whenever $e, e' \in C$. □

Proposition 10.2 shows how for connected graphs the existence of at least one cut-set of cardinality ≥ 3 implies that PPC $\not\Leftrightarrow$ CPC. On the other hand, if no cut-set of cardinality ≥ 3 exists, then there may be equivalence between the PPC and the CPC family notion. This is the case, for example, when the given graph is a tree or a cycle.

Proposition 10.3. *Let X be either a tree or a cycle graph. For every $\mathcal{F} \subseteq P(X)$, the following are equivalent:*

(a) *$\mathcal{F}$ is PPC;*
(b) *$\mathcal{F}$ is CPC.*

Proof: $(a) \Rightarrow (b)$: Fix $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| > 2$ and let $a, b \in \bigcap \mathcal{F}'$.

If X is a tree, then for every $F, F' \in \mathcal{F}'$, $F \cap F'$ must contain the minimal path joining a to b, and consequently, so does $\bigcap \mathcal{F}'$.

If X is a cycle graph, let P_1 and P_2 denote the two paths connecting a and b. Fix arbitrarily $F \in \mathcal{F}'$ and for $i = 1, 2$, let $\mathcal{H}_i = \{H \in \mathcal{F}' : P_i \subseteq F \cap H\}$. If either $\mathcal{H}_1 = \emptyset$ or $\mathcal{H}_2 = \emptyset$, then $P_2 \in \bigcap \mathcal{F}'$ or $P_1 \in \bigcap \mathcal{F}'$, respectively. If both $\mathcal{H}_i \neq \emptyset$, then $F = X$. Consequently[6], either $\mathcal{H}_1 = \{X\}$ or $\mathcal{H}_2 = \{X\}$. Thus, either $P_2 \in \bigcap \mathcal{F}'$ or $P_1 \in \bigcap \mathcal{F}'$, respectively. □

A more general condition for PPC $\Rightarrow$ CPC is given below.

Definition 10.4. Let $X \in Top$. $F \in P(X)$ is called *uniquely pathwise connected* if for every $a, b \in F$ and for every pair $f, g \in F^{[0,1]}$ of one-to-one continuous paths joining a and b, we have

$$f([0,1]) = g([0,1]).$$

Proposition 10.5. *Let $X \in Top$ and $\mathcal{F} \subseteq P(X)$ be such that for every $F \in \mathcal{F}$, F is uniquely pathwise connected. The following are equivalent:*

(a) *$\mathcal{F}$ is PPC;*
(b) *$\mathcal{F}$ is CPC.*

Proof: Fix $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| > 2$ and let $a, b \in \bigcap \mathcal{F}'$. Fix $F \in \mathcal{F}'$. Since $\mathcal{F}$ is PPC, for every $G \in \mathcal{F}' \backslash \{F\}$, there exists a continuous path θ_G in $F \cap G$ from a to b. Without loss of generality, all θ_G's are one-to-one so that $\theta_G([0,1]) = \theta_{G'}([0,1])$ for $G, G' \in \mathcal{F}' \backslash \{F\}$. □

Remark 10.6. If $\mathcal{F}$ is a PPC family consisting of uniquely pathwise connected sets, $\forall F, F' \in \mathcal{F}$, $F \cap F'$ is uniquely pathwise connected (that is, $\mathcal{F}$ is pairwise-uniquely-pathwise-connected, or PUPC). Nevertheless, the assumptions of Proposition 10.5 cannot be weakened in these terms.

[6]Suppose that $\mathcal{H}_1 \neq \{X\}$. Then there exists $H' \in \mathcal{H}$ such that $H' \neq X$. If there would exist $G' \in \mathcal{G}$ with $G' \neq X$, $H' \cap G'$ would be disconnected, as H' cannot contain all the points of P_2, and G' cannot contain all the points of P_1. Hence, for all $G \in \mathcal{G}$, $G = X$.

Example 4. Let X be the graph consisting of two vertices v_1, v_2 joined by three edges e_1, e_2, e_3. For $i = 1, 2, 3$, let $\overline{e}_i = e_i \cup \{v_1, v_2\}$. Consider the subgraphs $F = \overline{e}_1 \cup \overline{e}_3$, $G = \overline{e}_1 \cup \overline{e}_2$, and $H = \overline{e}_2 \cup \overline{e}_3$. Since $F \cap G = \overline{e}_1$, $F \cap H = \overline{e}_3$, $H \cap G = \overline{e}_2$, and $F \cap G \cap H = \{v_1, v_2\}$, the family $\{F, G, H\}$ is PUPC, but not CPC.

Lemma 10.7. *Let X be a pathwise connected space and $\mathcal{F} \subseteq P(X)$ be a CPC family. Then there exists a 2-convex map under which each $F \in \mathcal{F}$ is 2-stable.*

Proof: For every $x \in X$, let $\theta_x : x^\wedge \to X$ be define by $\theta_x(x^\wedge) = x$.

For every $A \in P(X)$ with $|A| = 2$, let $\{a_0^A, a_1^A\}$ be an enumeration of A and $f_A \in X^{[0,1]}$ be a continuous path from a_0^A to a_1^A. If $\mathcal{F}_A = \{F \in \mathcal{F} : A \subseteq F\} \neq \emptyset$, choose f_A such that $f_A([0,1]) \subseteq \bigcap \mathcal{F}_A$ (note that $A \subseteq \bigcap \mathcal{F}_A$ and $\bigcap \mathcal{F}_A$ is pathwise connected by assumption). Define $\theta_A = f_A \circ \gamma^{-1} \circ H_{a_1^A, a_0^A}$.

By lemmas 7.10(b) and 7.12, $\pi : FCC_2(X) \to X$ defined by $\pi(s) = \theta_{supp(s)}(s)$, for $s \in FCC_2(X)$, is a 2-convex map for X.

Also, for every $F \in \mathcal{F}$ and every $s \in F^\wedge \cap FCC_2(X)$, we have $\pi(s) \in F$. Indeed, let $supp(s) = \{a_0, a_1\}$. Since $\mathcal{F}_{supp(s)} \neq \emptyset$, $f_{supp(s)}([0,1]) \subseteq \bigcap \mathcal{F}_{supp(s)} \subseteq F$. Hence, $\pi(s) = \theta_{\{a_0,a_1\}}(s) = f_{supp(s)}(\gamma^{-1}(H_{a_1^A, a_0^A}(s))) \subseteq f_{supp(s)}([0,1]) \subseteq F$. □

Theorem 10.8. *Let X be a 2-paracompact space, Y be a pathwise connected space, and $T : X \Rightarrow Y$ be nonempty valued. If T is coherent and $\{T(x) : x \in X\}$ is a CPC family, then T has a continuous selection.*

Proof: By the lemma above, Y is formally 2-convex and T is 2-stable valued. By Theorem 9.1, T has a continuous selection. □

Corollary 10.9. *Let X be a 2-paracompact space and Y be a pathwise connected space. Every coherent, nonempty uniquely pathwise connected valued multifunction $T : X \Rightarrow Y$ such that $\{T(x) : x \in X\}$ is a PPC family, has a continuous selection.*

Proof: By Proposition 10.5, $\{T(x) : x \in X\}$ is a CPC family. □

Remark 10.10. Corollary 10.9 can be applied in particular when X is a finite 1-dimensional CW-complex (for instance, a finite graph) or a Menger's cube M_n^m with $m \geq 1$ and $n = 0, 1$.

Corollary 10.11. *Let Y be a pathwise connected space and $T : [0,1] \Rightarrow Y$ be a coherent, nonempty uniquely pathwise connected valued multifunction such that $\{T(x) : x \in X\}$ is a PPC family. Then, there exists a continuous path $f_T \in Y^{[0,1]}$ such that $f_T \leq T$.*

11. Our Selection Theorem

Inspired by Michael's classical proofs [21], we isolate the following lemmas.

Lemma 11.1. *Let $X \in Top$, (Y, d) be a complete metric space and $T : X \Rightarrow Y$ be nonempty closed valued. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions from X to Y such that*

(a) $\forall n \in \mathbb{N}$, $2^{-n} f_n$ *is coherent;*
(b) $\forall n \in \mathbb{N}, f_n \leq 2^{-n}T$*;*
(c) $\forall n \in \mathbb{N}, f_{n+1} \leq 2^{-n+1} f_n$.

Then T has a continuous selection, namely $f = lim_{n\to\infty} f_n$.

Proof: We show the continuity of the limit f. Let $\epsilon > 0$ and fix $x_0 \in X$. There exists N such that for $n \geq N$, $d(f_n(x), f(x)) < \frac{\epsilon}{3}$ for every $x \in X$. Choose $n \geq N$ such that $2^{-n} < \frac{\epsilon}{9}$. By (a), there exist $U \in \mathcal{N}(x_0)$ and $y \in Y$ such that $y \in S_d(f_n(x), 2^{-n})$, whenever $x \in U$. Then, for $x \in U$, $d(f(x), f(x_0)) \leq d(f(x), f_n(x)) + d(f_n(x), y) + d(y, f_n(x_0)) + d(f_n(x_0), f(x_0)) < \epsilon$. □

Lemma 11.2. *Let $X \in Top$, (Y, d) be a metric space, and $T : X \Rightarrow Y$ be nonempty valued. Suppose that*

(i) *there exists a continuous $f \in Y^X$ such that $f \leq 2^{-1}T$;*
(ii) $\forall n \geq 2$, $\forall$ *continuous $f \leq 2^{-n+1}T$, there exists a continuous $g_{n,f} \in Y^X$ such that $g_{n,f} \leq 2^{-n}(T \cap 2^{-n+1} f)$.*

Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq Y^X$ such that

(a) $\forall n \in \mathbb{N}$, $2^{-n} f_n$ *is coherent;*
(b) $\forall n \in \mathbb{N}, f_n \leq 2^{-n}T$*;*
(c) $\forall n \in \mathbb{N}, f_{n+1} \leq 2^{-n+1} f_n$.

Proof: By (i), there exists a continuous $f_1 \in Y^X$ such that $f_1 \leq 2^{-1}T$. By (ii), there exists a continuous $g_{2,f_1} \in Y^X$ such that $g_{2,f_1} \leq 2^{-2}(T \cap 2^{-1} f_1)$. By Proposition 5.4, both $2^{-1} f_1$ and $2^{-2} g_{2,f_1}$ are coherent. Note also that $2^{-2}(T \cap 2^{-1} f_1) \leq 2^{-2}T$ and $2^{-2}(T \cap$

$2^{-1}f_1) \leq 2^{-2}(2^{-1}f_1) \leq (2^{-2}+2^{-1})f_1 \leq 2^0 f_1$; hence, the functions f_1 and g_{2,f_1} satisfy (a), (b), and (c). Let $f_2 = g_{2,f_1}$.

Suppose that $f_1, \cdots, f_{n-1}$ have been constructed. At the stage n, use the inequality $f_{n-1} \leq 2^{-n+1}T$ and assumption (ii) to establish the existence of a continuous function $g_{n,f_{n-1}} \in Y^X$ such that $g_{n,f_{n-1}} \leq 2^{-n}(T \cap 2^{-n+1}f_{n-1})$. Check that $f_1, \cdots, f_{n-1}$ and $f_n = g_{n,f_{n-1}}$ satisfy (a), (b), and (c). □

From the lemma above, we can easily derive the following more general result.

Lemma 11.3. *Let $X \in Top$, (Y, d) be a metric space, and $T : X \Rightarrow Y$ be nonempty valued. Suppose that*

(i) *$\forall \epsilon > 0$, there exists a continuous $f \in Y^X$ such that $f \leq \epsilon T$;*
(ii) *$\forall \epsilon > 0$, $\forall$ continuous $f \leq \epsilon T$, there exists a continuous function $g \in Y^X$ such that $g \leq \frac{\epsilon}{2}(T \cap \epsilon f)$.*

Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq Y^X$ such that

(a) *$\forall n \in \mathbb{N}$, $2^{-n}f_n$ is coherent;*
(b) *$\forall n \in \mathbb{N}, f_n \leq 2^{-n}T$;*
(c) *$\forall n \in \mathbb{N}, f_{n+1} \leq 2^{-n+1}f_n$.*

We also use the following fact (see [21, Proof of Theorem 1]).

Lemma 11.4. *Let $X \in Top$, (Y, d) be a metric space, and $\epsilon > 0$. Let $T : X \Rightarrow Y$ be nonempty valued and $f \in Y^X$ be such that $f \leq \epsilon T$. If T is l.s.c. and f is continuous, then $T \cap \epsilon f$ is l.s.c..*

We are now ready to state our main result.

Theorem 11.5. *Let $n \in \mathbb{N} \cup \{\infty\}$. Let X be an n-paracompact space, (Y, d) be a complete metric formally n-convex space, and $T : X \Rightarrow Y$ be nonempty closed valued. Suppose that*

(i) *$\forall \epsilon > 0$, ϵT is n-stable valued;*
(ii) *$\forall \epsilon > 0$, $\forall f \in Y^X$, if $f \leq \epsilon T$ then $\forall \delta > 0$, $\delta(T \cap \epsilon f)$ is n-stable valued.*

If T is l.s.c., then T has a continuous selection.

Proof: By Proposition 9.7, (i) of Lemma 11.3 holds. By Lemma 11.4 and again Proposition 9.7, (ii) of Lemma 11.3 also holds. Apply Lemma 11.3 and Lemma 11.1. □

When $n = \infty$ and $n = 1$, we get Theorem 3.2 (Theorem 1 in [21]) and Theorem 3.3 (Theorem 2 in [21]), respectively.

Corollary 11.6. *Let X be a paracompact space, (Y, d) be a complete metric formally convex space, and $T : X \Rightarrow Y$ be nonempty closed valued. Suppose that*

(*i*) *$\forall \epsilon > 0$, ϵT is stable valued;*
(*ii*) *$\forall \epsilon > 0$, $\forall f \in Y^X$, if $f \leq \epsilon T$ then $\forall \delta > 0$, $\delta(T \cap \epsilon f)$ is stable valued.*

If T is l.s.c., then T has a continuous selection.

Corollary 11.7. *Let X be a paracompact space, Y be a Banach space, and $T : X \Rightarrow Y$ be nonempty, closed and convex valued. If T is l.s.c., then T has a continuous selection.*

Proof: Reasoning as in Corollary 9.8, we may check that both (*i*) and (*ii*) of Corollary 11.6 are satisfied. □

Corollary 11.8. *Let X be an ultraparacompact space, Y be a complete metric space, and $T : X \Rightarrow Y$ be nonempty closed valued. If T is l.s.c., then T has a continuous selection.*

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