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WHITNEY PRESERVING MAPS ONTO DECOMPOSITION SPACES

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ABSTRACT. We prove that if X has a continuous decomposition $\mathcal{A}$ into terminal continua, then the projection $\pi : X \longrightarrow X/\mathcal{A}$ is a Whitney preserving map.

1. INTRODUCTION

In [1], the author introduced the notion of Whitney preserving functions and proved that every Whitney preserving map $f : X \longrightarrow [0,1]$, from a continuum X containing a dense arc component onto the unit interval $[0,1]$, is a homeomorphism. Example 3 in [1] shows the necessity of X having a dense arc component; in this paper, the author generalizes Example 3 of [1] by proving that, if X has a continuous decomposition $\mathcal{A}$ into terminal continua, then the projection $\pi : X \longrightarrow X/\mathcal{A}$ is a Whitney preserving map, where $X/\mathcal{A}$ is the decomposition space. This result is combined with the construction given in [4] to prove that for any one-dimensional continuum Y, there exists a one-dimensional continuum X, different from Y, such that there exists a Whitney preserving map from X to Y.

A *continuum* is a nonempty compact connected metric space. Throughout this paper, $X, Y,$ and Z will denote continua, unless otherwise specified. $C(X)$ denotes the *hyperspace of subcontinua*

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of X; the topology for $C(X)$ is given by the Hausdorff metric (see [2]). The unit interval $[0,1]$ will be denoted by I.

A continuous function $f : X \longrightarrow Y$ is called *weakly confluent* if for each subcontinuum C of Y there is a component A of $f^{-1}(C)$ such that $f(A) = C$.

A collection $\mathcal{A}$ of nonempty compact subsets of X is called a *set-theoretic decomposition of* X provided the elements of $\mathcal{A}$ are mutually disjoint and $\bigcup \mathcal{A} = X$. If $\mathcal{A}$ is a set-theoretic decomposition of X, then $X/\mathcal{A}$ will denote the decomposition space obtained by identifying all the points of each member of $\mathcal{A}$.

Definition 1.1. A continuous function $\mu : C(X) \longrightarrow \mathbb{R}$ is called a *Whitney map* for $C(X)$ if it satisfies the following two conditions:

(i) $\mu(\{x\}) = 0$ for all $x \in X$, and
(ii) $\mu(A) < \mu(B)$ whenever $A \subseteq B$ and $A \neq B$.

Let $t \in [0, \mu(X)]$; the set $\mu^{-1}(t)$ is called a *Whitney level of* $C(X)$.

Definition 1.2. An *order arc* α in $C(X)$ is an arc in $C(X)$ such that if $A, B \in \alpha$, then either $A \subseteq B$ or $B \subseteq A$.

It is well known that for any two elements $A, B \in C(X)$ such that $A \subseteq B$ and $A \neq B$, there exists an order arc in $C(X)$ from A to B (see [2, Theorem 14.6]). For more about order arcs and the structure of $C(X)$, see [2] or [6].

Definition 1.3. Let X be a continuum and let $\mathcal{A}$ be a subset of $C(X)$. We define the set $C_{\mathcal{A}}(X)$ as

$$C_{\mathcal{A}}(X) = \{B \in C(X) : B \supseteq A \text{ for some } A \in \mathcal{A}\}.$$

Remark. Note that if the set $\mathcal{A}$ is a Whitney level, say $\mathcal{A} = \mu^{-1}(t)$ for some Whitney map μ and some t, then the set $C_{\mathcal{A}}(X)$ is the "top" $\mu^{-1}([t, \mu(X)])$.

Definition 1.4. Let $f : X \longrightarrow Y$ be a continuous function. The *induced map* $\widehat{f} : C(X) \longrightarrow C(Y)$ is given by

$$\widehat{f}(A) = f(A), \text{ for all } A \in C(X).$$

We will make use of the following known results about induced maps.

(a) If $f : X \longrightarrow Y$ is a continuous function, then $\hat{f}$ is continuous, and

(b) f is weakly confluent if and only if $\hat{f}$ is onto.

Definition 1.5. A continuous function $f : X \longrightarrow Y$ is called a *Whitney preserving map* if there exist two Whitney maps

$$\mu : C(X) \longrightarrow \mathbb{R} \text{ and } \nu : \mathrm{C(Y)} \longrightarrow \mathbb{R} \text{ such that}$$

$$\text{for all } t \in [0, \mu(X)],\ \widehat{f}(\mu^{-1}(t)) = \nu^{-1}(s) \text{ for all } s \in [0, \nu(Y)].$$

In other words, a map f is Whitney preserving if there exist two Whitney maps for which the induced map $\widehat{f}$ maps Whitney levels of $C(X)$ onto Whitney levels of $C(Y)$. In this case, we say that f is $\mu, \nu-$ *Whitney preserving.*

Proposition 1.6. *Let $f : X \longrightarrow Y$ be a continuous function, and let $\mu : C(X) \longrightarrow \mathbb{R}$ and $\nu : \mathrm{C(Y)} \longrightarrow \mathbb{R}$ be two Whitney maps. Suppose there exist $t \in [0, \mu(X)]$ and $s \in [0, \nu(Y)]$ such that $\widehat{f}(\mu^{-1}(t)) \subseteq \nu^{-1}(s)$. If f is weakly confluent, then $\widehat{f}(\mu^{-1}(t)) = \nu^{-1}(s)$.*

Proof: Let $t \in [0, \mu(X)]$ and $s \in [0, \nu(Y)]$ such that $\widehat{f}(\mu^{-1}(t)) \subseteq \nu^{-1}(s)$. To prove the equality, we will show that the reverse contention holds.

Let $C \in \nu^{-1}(s)$. Since f is weakly confluent, $\widehat{f}$ is onto; hence, there exists $B \in C(X)$ such that

$$f(B) = C. \tag{1}$$

Now let $x \in B$ and let α be an order arc from $\{x\}$ to X such that $B \in \alpha$. (This can be done by constructing, first, an order arc from $\{x\}$ to B and then another from B to X.)

Let $A = \alpha \cap \mu^{-1}(t)$. Note that either

$$B \subseteq A \text{ or } A \subseteq B. \tag{2}$$

By hypothesis, since $A \in \mu^{-1}(t)$, $\nu(\widehat{f}(A)) = s$; hence, by (1) and (2), either $C \subseteq f(A)$ or $f(A) \subseteq C$. Therefore, since $\nu(\widehat{f}(A)) = s$ and $\nu(C) = s$, $f(A) = C$. This proves $\nu^{-1}(s) \subseteq \widehat{f}(\mu^{-1}(t))$. □

The following corollary is an immediate consequence of Proposition 1.6. We leave it without a proof.

Corollary 1.7. *Let $f : X \longrightarrow Y$ be a weakly confluent map, and let $\mu : C(X) \longrightarrow \mathbb{R}$ and $\nu : C(Y) \longrightarrow \mathbb{R}$ be two Whitney maps. If for all $t \in [0, \mu(X)]$ there exists $s \in [0, \nu(Y)]$ such that $\widehat{f}(\mu^{-1}(t)) \subseteq \nu^{-1}(s)$, then f is μ, ν-Whitney preserving.*

Example 1.8. Every homeomorphism is a Whitney preserving map.

To prove this, let $h : X \longrightarrow Y$ be a homeomorphism. Let $\mu : C(X) \longrightarrow \mathbb{R}$ be any Whitney map. We define $\nu : C(Y) \longrightarrow \mathbb{R}$ as $\nu(K) = \mu(\widehat{h^{-1}}(K))$ for all $K \in C(Y)$. The continuity of $\widehat{h}$ and μ and the fact that h is a homeomorphism imply that ν is a Whitney map for $C(Y)$.

Next we prove that h is μ, ν-Whitney preserving. Take two continua A and B in any Whitney level $\mu^{-1}(t)$ of $C(X)$. By the definition of ν, we have that

$$\nu(\widehat{h}(A)) = \mu(\widehat{h^{-1}}(\widehat{h}(A))) = \mu(h^{-1}(h(A))) = \mu(A) = t \text{ and}$$

$$\nu(\widehat{h}(B)) = \mu(\widehat{h^{-1}}(\widehat{h}(B))) = \mu(h^{-1}(h(B))) = \mu(B) = t.$$

Hence, $\nu(\widehat{h}(A)) = \nu(\widehat{h}(B))$, implying $\widehat{h}(\mu^{-1}(t)) \subseteq \nu^{-1}(s)$. Therefore, by Corollary 1.7, h is μ, ν-Whitney preserving, since h (being a homeomorphism) is weakly confluent.

Proposition 1.9. *Let $g : X \longrightarrow Y$ and let $f : Y \longrightarrow Z$ be two continuous functions. If g is a μ, λ-Whitney preserving map and f is a λ, ν-Whitney preserving map, then $f \circ g$ is a μ, ν-Whitney preserving map.*

Proof: Let $t \in [0, \mu(X)]$; since g is μ, λ-Whitney preserving, and $\widehat{g}(\mu^{-1}(t)) = \lambda^{-1}(r)$ for some $r \in [0, \lambda(Y)]$, and since f is λ, ν-Whitney preserving, $\widehat{f}(\lambda^{-1}(r)) = \nu^{-1}(s)$ for some $s \in [0, \nu(Z)]$. Therefore, $\widehat{g \circ f}(\mu^{-1}(t)) = \nu^{-1}(s)$ for some $s \in [0, \nu(Z)]$. Hence, $g \circ f$ is a μ, ν-Whitney preserving map. □

2. Main Results

Definition 2.1. Let X be a continuum. A subcontinuum A of X is called a *terminal continuum* if for every subcontinuum B of X such that $A \cap B \neq \emptyset$, then either $A \subseteq B$ or $B \subseteq A$.

Proposition 2.2. *Let X be a continuum and let $\mu : C(X) \longrightarrow \mathbb{R}$ be a Whitney map for $C(X)$. If there is $t > 0$ such that $\mu^{-1}(t)$ is a set-theoretic decomposition of X, then every element of $\mu^{-1}(t)$ is a terminal continuum.*

Proof: Let $A \in \mu^{-1}(t)$. To prove that A is a terminal continuum, take $B \in C(X)$ such that $A \cap B \neq \emptyset$.

Assume first that $\mu(B) \leq t$. Let $x \in B$ be any point, and let α be an order arc from $\{x\}$ to $A \cup B$ such that $B \in \alpha$. Now, since $A \subseteq A \cup B$, $t = \mu(A) \leq \mu(A \cup B)$; hence, there exists $C \in \alpha$ such that $\mu(C) = t$; that is, $C \in \mu^{-1}(t)$. Since α is an order arc and $B \in \alpha$, we have that $B \subseteq C$. Therefore, $C \cap A \neq \emptyset$, but $\mu^{-1}(t)$ is a decomposition of X; hence, $C = A$. This implies $B \subseteq A$.

Now assume $\mu(B) > t$. Take a point x in $A \cap B$. Let α be an order arc from $\{x\}$ to B. Again, since $\mu(B) > t$, there exists $C \in \alpha$ such that $\mu(C) = t$ and $C \subseteq B$. Since $x \in C$, $A \cap C \neq \emptyset$. Then, since $\mu^{-1}(t)$ is a decomposition, $A = C$. Hence, $A \subseteq B$. □

Theorem 2.3. *Let $f : X \longrightarrow Y$ be a μ, ν-Whitney preserving map and let $s_0 = \max \left\{ s \in [0, \mu(X)] : \widehat{f}(\mu^{-1}(s)) = \nu^{-1}(0) \right\}$. If $s_0 > 0$, then $\mu^{-1}(s_0)$ is a continuous decomposition of X and every element of $\mu^{-1}(s_0)$ is a terminal continuum.*

Proof: Let $s_0 = \max \left\{ s \in [0, \mu(X)] : \widehat{f}(\mu^{-1}(s)) = \nu^{-1}(0) \right\}$. Assume $s_0 > 0$, since $\mu^{-1}(s_0)$ is a Whitney level, we have that $\bigcup \mu^{-1}(s_0) = X$. Now let $A, B \in \mu^{-1}(s_0)$, and assume that $A \cap B \neq \emptyset$. Then $A \cup B$ is a continuum with $A \subseteq A \cup B$, and $f(A) \subseteq f(A \cup B) = f(A) \cup f(B)$. Then, since $f(A)$ and $f(B)$ are points and $f(A \cup B)$ is connected, we have that $f(A \cup B)$ is a point. Hence, by the definition of s_0, $A \cup B \in \mu^{-1}(s_0)$. Therefore, $A = A \cup B$; that is, $A = B$. This shows that $\mu^{-1}(s_0)$ is a set-theoretic decomposition of X. Now, since $\mu^{-1}(s_0)$ is compact in $C(X)$, $\mu^{-1}(s_0)$ is a continuous decomposition of X.

By Proposition 2.2, every element of $\mu^{-1}(s_0)$ is a terminal continuum. □

Remark. Notice that the previous theorem says that the largest Whitney level that is mapped to the singletons is a continuous decomposition of X into terminal continua, which does not imply that the set $\left\{ f^{-1}(y) : y \in Y \right\}$ is a continuous decomposition into terminal continua; for instance, the elements $f^{-1}(y)$ may not be connected.

Lemma 2.4. *Let X be a continuum. Assume X has a continuous decomposition $\mathcal{A}$ into nondegenerate terminal continua. Then, for* *every* $t > 0$*, there exists a Whitney map* $\mu : C(X) \longrightarrow \mathbb{R}$ *such that* $\mathcal{A} = \mu^{-1}(t)$.

Proof: Let X be a continuum such that X has a continuous decomposition $\mathcal{A}$ into nondegenerate terminal continua, and let $t > 0$.

Let $m : \mathcal{A} \longrightarrow \mathbb{R}$ be the constant function such that $m(A) = t$ for all $A \in \mathcal{A}$. Then, from [2, 16.10, p. 132] and the fact that $\mathcal{A}$ is closed in $C(X)$, m can be extended to a Whitney map $\mu : C(X) \longrightarrow \mathbb{R}$. Therefore, by the definition of m,

$$\mathcal{A} \subseteq \mu^{-1}(t). \tag{3}$$

Now we prove the reverse contention. Let $B \in \mu^{-1}(t)$. Since $\mathcal{A}$ is a decomposition of X, there exists $A \in \mathcal{A}$ such that $A \cap B \neq \emptyset$. This and the fact that A is a terminal continuum imply that either $A \subseteq B$ or $B \subseteq A$. On the other hand, $\mu(A) = \mu(B) = t$; hence, $A = B$, implying

$$\mu^{-1}(t) \subseteq \mathcal{A}. \tag{4}$$

Combining (3) and (4), we have $\mathcal{A} = \mu^{-1}(t)$. □

Theorem 2.5. *Let* X *be a continuum such that* X *has a continuous decomposition* $\mathcal{A}$ *into nondegenerate terminal continua. Then the projection* $\pi : X \longrightarrow X/\mathcal{A}$ *is a Whitney preserving map.*

Proof: Let $t_0 > 0$. By Lemma 2.4, there exists a Whitney map $\mu : C(X) \longrightarrow \mathbb{R}$ such that $\mathcal{A} = \mu^{-1}(t_0)$. Define $\nu : C(X/\mathcal{A}) \longrightarrow \mathbb{R}$ as

$$\nu(B) = \mu(\pi^{-1}(B)) - t_0 \text{ for all } B \in C(X/\mathcal{A}).$$

We prove that π is a μ, ν-Whitney preserving map, but first we prove that ν is a Whitney map. Observe that, since $\mathcal{A}$ is a continuous decomposition into terminal continua, π is open and monotone. Hence, ν is a continuous function. Let $\{y\}$ be a singleton of $C(X/\mathcal{A})$. Then $\pi^{-1}(y) \in \mathcal{A}$ and $\mu(\pi^{-1}(y)) = t_0$. Therefore, $\nu(y) = 0$.

Now let $C, D \in C(X/\mathcal{A})$ such that $C \subseteq D$ and $C \neq D$. Then $\pi^{-1}(C) \subseteq \pi^{-1}(D)$ and $\pi^{-1}(C) \neq \pi^{-1}(D)$. Since π is monotone, $\pi^{-1}(C), \pi^{-1}(D)$ are subcontinua of X. Therefore, $\mu(\pi^{-1}(C)) < \mu(\pi^{-1}(D))$, implying $\nu(C) < \nu(D)$. That is, ν is a Whitney map for $C(X/\mathcal{A})$.

To prove that π is a μ,ν-Whitney preserving map, first, observe that for any subcontinuum A of X such that $\mu(A) \leq t_0$, $\widehat{\pi}(A)$ is a singleton of $C(X/\mathcal{A})$. That is, $\widehat{\pi}$ maps every Whitney level $\mu^{-1}(t)$ with $t \leq t_0$ to the Whitney level $\nu^{-1}(0)$. Now let A be such that $\mu(A) > t_0$. Then, since every element of $\mathcal{A}$ is a terminal continuum, $\pi^{-1}(\pi(A)) = A$. Therefore, if A and B belong to the same Whitney level $\mu^{-1}(t)$, $t > t_0$, then $\widehat{\pi}(A)$ and $\widehat{\pi}(B)$ belong to the same Whitney level $\nu^{-1}(t - t_0)$. This proves that for every t there exists s such that $\widehat{\pi}(\mu^{-1}(t)) \subseteq \nu^{-1}(s)$. Then, by Corollary 1.7, π is a Whitney preserving map. □

Remark. Notice that, since all the elements of $\mathcal{A}$ are terminal continua, the function $\widehat{\pi}_{|\mu^{-1}([t_0,\mu(X)])} : \mu^{-1}([t_0, \mu(X)]) \longrightarrow C(X/\mathcal{A})$ is a one-to-one function. Also, since π is monotone, $\widehat{\pi}_{|\mu^{-1}([t_0,\mu(X)])}$ is onto, implying that $\mu^{-1}([t_0, \mu(X)])$ is homeomorphic to $C(X/\mathcal{A})$.

Therefore, using the Remark following Definition 1.3 and the fact, in this case, that $\mathcal{A}$ is the Whitney level $\mu^{-1}(t_0)$, we can summarize the previous observation into the following corollary.

Corollary 2.6. *Let X be a continuum. If $\mathcal{A}$ is a continuous decomposition of X into terminal continua, then $C_{\mathcal{A}}(X)$ is homeomorphic to $C(X/\mathcal{A})$.*

Relative to Theorem 2.5 and the Remark following the proof of Theorem 2.3, we have the following corollary.

Corollary 2.7. *Let $f : X \longrightarrow Y$ be a continuous function. If f is a monotone open map such that $f^{-1}(y)$ is a nondegenerate terminal continuum for all $y \in Y$, then f is a Whitney preserving map.*

Proof: Note that in this case, the set $\mathcal{A} = \left\{f^{-1}(y) : y \in Y\right\}$ is a continuous decomposition into terminal continua. In this case, the natural projection onto $X/\mathcal{A}$ coincides with f. Hence, by Theorem 2.5, f is Whitney preserving. □

In [4], Wayne Lewis proved the following theorem. (See also [5, Theorem 76, p. 117].)

Theorem 2.8 (W. Lewis [4], [5]). *For every one-dimensional continuum M there exists a one-dimensional continuum $\widehat{M}$ with a continuous decomposition G into pseudo-arcs such that $\widehat{M}/G$ is homeomorphic to M.*

Remark. Lewis's construction of $\widehat{M}$ has the property that the elements of G are terminal continua. (See Construction of $\widehat{M}$, [4, p. 93] and paragraph above Question 1, [4, p. 98].)

Combining Theorem 2.5 and Theorem 2.8, we obtain the following theorem, which yields a lot of examples of Whitney preserving maps.

Theorem 2.9. *For every one-dimensional continuum M there exists a one-dimensional continuum $\widehat{M}$ (other than M) such that there is a Whitney preserving map $f : \widehat{M} \longrightarrow M$.*

Proof: Let M be a one-dimensional continuum. By Theorem 2.8, there exists a one-dimensional continuum $\widehat{M}$ with a continuous decomposition G into pseudo-arcs such that $\widehat{M}/G$ is homeomorphic to M. Furthermore, $\widehat{M}$ can be constructed in such a way that the elements of G are terminal continua. We will consider that this is the case; see Remark above.

Let $h : \widehat{M}/G \longrightarrow M$ be a homeomorphism. From Example 1.8, h is Whitney preserving and, from the proof of Example 1.8, given any Whitney map $\mu' : C(\widehat{M}/G) \longrightarrow \mathbb{R}$, there exists a Whitney map $\nu : C(M) \longrightarrow \mathbb{R}$ such that h is a μ', ν-Whitney preserving map.

Note that, since the elements of G are terminal continua, $\widehat{M}$ and G satisfy the hypothesis of Theorem 2.5. Therefore, the projection $\pi : \widehat{M} \longrightarrow \widehat{M}/G$ is a μ, λ-Whitney preserving map, for some Whitney maps μ and λ. Hence, from the previous paragraph and Proposition 1.9, the function $h \circ \pi : \widehat{M} \longrightarrow M$ is a μ, ν-Whitney preserving map, for some Whitney map ν. □

We end this paper with an example, showing the necessity in Theorem 2.5 of $\mathcal{A}$ to be a continuous decomposition and the elements of $\mathcal{A}$ to be terminal continua. In other words, we show a continuum Γ with an upper semi-continuous decomposition $\mathcal{A}$, such that the projection is not Whitney preserving. Furthermore, there is not a Whitney preserving map, at all, from Γ to $\Gamma/\mathcal{A}$.

Example 2.10. Let Γ be the *V-lambda-continuum* (see Figure 1). For a construction of Γ, see [3, pp. 191-192].

The subcontinua of Γ which are maximal with respect to being an arc will be called *traunches*. Notice that there are three types

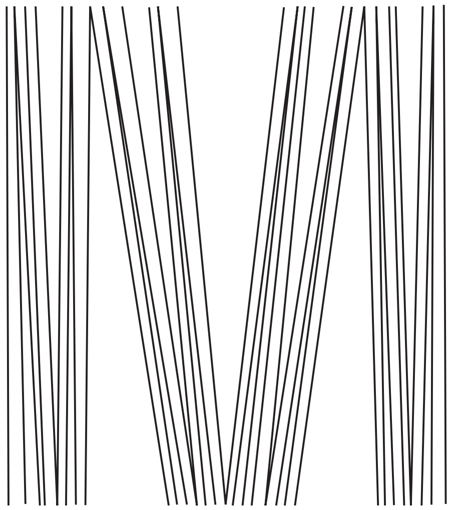

FIGURE 1. Γ

of traunches. The traunches having the form of V will be called V-traunches, the traunches having the form Λ will be called Λ-traunches, and the traunches which are straight lines will be called I-traunches. The set of traunches form an upper semi-continuous decomposition of X into nondegenerate continua. Let $\mathcal{A}$ denote this decomposition.

It is known that the quotient space $\Gamma/\mathcal{A}$ is homeomorphic to the unit interval I. Since homeomorphisms are Whitney preserving, (see Example 1.8), to prove that there is not a Whitney preserving map from Γ to $\Gamma/\mathcal{A}$, we will prove that there is not a Whitney preserving map from Γ to the unit interval I, rather than to $\Gamma/\mathcal{A}$.

Suppose, to the contrary, that there exists a $\mu, \nu-$Whitney preserving map $f : \Gamma \longrightarrow I$. If this is the case, then we prove that the image of every traunch is constant.

First, we show that f is constant for all I-traunches. Let $A = [a, b]$ be an I-traunch. Then there exists a sequence of Λ-traunches $\{A_n\}_{n\in\mathbb{N}}$ such that

$$A_n \to A. \tag{5}$$

Denote each Λ-traunch A_n, of this sequence, by $[a_n, b_n, c_n]$, where a_n and c_n are the end points of A_n, and b_n is the bending point of A_n.

Because $A_n \to A$, we have that

$$a_n \to a, c_n \to a, \text{ and } b_n \to b. \tag{6}$$

Assume, without loss of generality, that $f_{|A_n}$ is nondegenerate for all n; otherwise, by continuity, $f(A)$ would be a point. Then for every n, by Lemma 10 and Lemma 12 of [1],

$$f_{|A_n} \text{ is a homeomorphism.} \tag{7}$$

Hence, the image $f(A_n)$ is an arc for all n, and the images $f(a_n)$ and $f(c_n)$, of the end points a_n and c_n, are end points of $f(A_n)$. Therefore, by continuity and (6), $f(a_n) \to f(a)$ and $f(c_n) \to f(a)$. This and (7) imply that $f(A)$ is a point.

Now let $[a, b, c]$ be any Λ-traunch, where b denotes the bending point. Note that $[a, b]$ and $[b, c]$ are limits of I-traunches. Therefore, by continuity, $f([a, b])$ and $f([b, c])$ are points. Hence, since $[a, b, c] = [a, b] \cup [b, c]$ and $[a, b] \cap [b, c] \neq \emptyset$, $f([a, b, c])$ is a point. This shows that the image of any Λ-traunch is a point. A similar argument can be used to prove that the image of any V-traunch is a point. Therefore, we have proved so far that if f is a $\mu, \nu-$Whitney preserving map, then the image of every traunch is a point.

Next, let $s_0 = \max \left\{ s \in [0, \mu(X)] : \widehat{f}(\mu^{-1}(s)) = \nu^{-1}(0) \right\}$. The previous argument implies that $s_0 > 0$. Hence, by Theorem 2.3, $\mu^{-1}(s_0)$ is a continuous decomposition of Γ into nondegenerate terminal continua. On the other hand, the only terminal continua of Γ are the I-traunches. Then, since the images of the Λ-traunches and V-traunches are degenerate, $\mu^{-1}(s_0)$ contains elements which are not terminal continua, a contradiction. Therefore, there is not a Whitney preserving map from Γ to I, implying that there is not a Whitney preserving map from Γ to $\Gamma/\mathcal{A}$.

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