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MAPPINGS IN PONOMAREV-SYSTEMS

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Abstract. Let $(f, M, X, \mathcal{P})$ be a *Ponomarev*-system. We prove that f is a pseudo-sequence-covering (resp., sequentially-quotient) mapping iff $\mathcal{P}$ is a $qcsf$-network (resp., cs^*f-network) of X. As an application of these results, f is a pseudo-sequence-covering s-mapping iff it is a sequentially-quotient s-mapping, and where "s-" can not be omitted.

1. Introduction

In 1960, V. I. Ponomarev [12] proved that every first countable space can be characterized as an open image of a subspace of a Baire's zero-dimensional space. Recently Shou Lin [9] generalized "Ponomarev's method" to establish a system $(f, M, X, \mathcal{P})$, called a *Ponomarev*-system, obtaining the following results [9], [10], [14].

Theorem 1.1. *For a Ponomarev-system $(f, M, X, \mathcal{P})$, the following hold.*

(1) If $\mathcal{P}$ is a point-finite (resp., point-countable) network of X, then f is a compact mapping (resp., s-mapping).

(2) If $\mathcal{P}$ is a point-countable cs^*-network of X, then f is a sequentially-quotient (pseudo-sequence-covering) s-mapping.

Taking Theorem 1.1 into account, the following questions naturally arise.

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Question 1.2. Let $(f, M, X, \mathcal{P})$ be a *Ponomarev*-system.

(1) Can the implications in Theorem 1.1 be reversed?

(2) Can "point-countable" and "s-" in Theorem 1.1(2) be omitted?

(3) More precisely, what is the necessary-and-sufficient condition such that f is a pseudo-sequence-covering (resp., sequentially-quotient) mapping?

In this paper, we generalize cs-network and cs^*-network to introduce $qcsf$-network and cs^*f-network and give two necessary-and-sufficient conditions such that f is pseudo-sequence-covering and f is sequentially-quotient, respectively. By this result, we answer Question 1.2(1) affirmatively and Question 1.2(2) negatively. As an application of these results, f is a pseudo-sequence-covering s-mapping iff it is a sequentially-quotient s-mapping where "s-" can not be omitted.

Throughout this paper, all spaces are assumed to be Hausdorff and all mappings are continuous and onto. $\mathbb{N}$ denotes the set of all natural numbers and $\{x_n\}$ denotes a sequence where the n-th term is x_n. For a sequence $L = \{x_n\}$, $f(L)$ denotes the sequence $\{f(x_n)\}$. Let X be a space and $P \subset X$. A sequence $\{x_n\}$ converging to x in X is eventually in P if $\{x_n : n > k\} \cup \{x\} \subset P$ for some $k \in \mathbb{N}$ and frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a family of subsets of X and let $x \in X$. $\bigcup\mathcal{P}$ and $(\mathcal{P})_x$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$ and the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$, respectively. For a sequence $\{P_n : n \in \mathbb{N}\}$ of subsets of a space X, we abbreviate $\{P_n : n \in \mathbb{N}\}$ to $\{P_n\}$. A point $b = (\beta_n)_{n\in\mathbb{N}}$ of a Tychonoff-product space is abbreviated to (β_n).

2. Main Results

Definition 2.1. Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X, where $\mathcal{P}_x \subset (\mathcal{P})_x$. $\mathcal{P}$ is called a network of X [11], if for every $x \in U$ with U open in X, there exists $P \in \mathcal{P}_x$ such that $x \in P \subset U$, where $\mathcal{P}_x$ is called a network at x in X.

Definition 2.2. Let $\mathcal{P}$ be a network of a space X. Assume that there exists a countable $\mathcal{P}_x \subset \mathcal{P}$ such that $\mathcal{P}_x$ is a network at x in X for every $x \in X$. Put $\mathcal{P} = \{P_\beta : \beta \in \Lambda\}$. For every $n \in \mathbb{N}$, put $\Lambda_n = \Lambda$ and endow Λ_n with a discrete topology. Put $M = \{b =$

$(\beta_n) \in \Pi_{n\in\mathbb{N}}\Lambda_n : \{P_{\beta_n}\}$ *is a network at some point* x_b *in* $X\}$. Then M, which is a subspace of the product space $\Pi_{n\in\mathbb{N}}\Lambda_n$, is a metric space and x_b is unique for every $b \in M$. Define $f : M \longrightarrow X$ by $f(b) = x_b$, then f is a mapping, and $(f, M, X, \mathcal{P})$ is called a *Ponomarev*-system [9], [10], [14].

Definition 2.3. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is called a compact mapping (resp., s-mapping), if $f^{-1}(y)$ is a compact (resp., separable) subset of X for every $y \in Y$.

(2) f is called a sequence-covering mapping [13], if for every convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L) = S$.

(3) f is called a pseudo-sequence-covering mapping [7], if for every convergent sequence S converging to y in Y, there exists a compact subset K of X such that $f(K) = S \bigcup\{y\}$.

(4) f is called a sequentially-quotient mapping [2], if for every convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S.

Remark 2.4. (1) "Pseudo-sequence-covering mapping" in Definition 2.3(3) is also called "sequence-covering mapping" by G. Gruenhage, E. Michael, and Y. Tanaka in [5].

(2) Sequence-covering mapping $\Longrightarrow$ pseudo-sequence-covering mapping $\Longrightarrow$ (if the domain is metric) sequentially-quotient mapping [9].

Definition 2.5. Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is called a cs-network of X [6], if whenever S is a sequence in X converging to $x \in U$ with U open in X, then S is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

(2) $\mathcal{P}$ is called a qcs-network of X [4], if whenever S is a sequence in X converging to $x \in U$ with U open in X, there exists a finite subfamily $\mathcal{P}'$ of $(\mathcal{P})_x$ such that S is eventually in $\bigcup \mathcal{P}' \subset U$;

(3) $\mathcal{P}$ is called a cs^*-network of X [3], if whenever S is a sequence in X converging to $x \in U$ with U open in X, then S is frequently in $P \subset U$ for some $P \in \mathcal{P}$.

(4) $\mathcal{P}$ is called a csf-network of X [4], if whenever S is a sequence converging to a point x in X, there exists a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a network at x in X and S is eventually in P for every $P \in \mathcal{P}_S$, where $\mathcal{P}_S$ is called a csf-network for S in X.

(5) $\mathcal{P}$ is called a $qcsf$-network of X, if whenever S is a sequence converging to x in X, there exists a countable $\mathcal{P}_S \subset (\mathcal{P})_x$ such that for every open neighborhood U of x in X, S is eventually in $\bigcup \mathcal{P}_U \subset U$ for some finite subfamily $\mathcal{P}_U$ of $\mathcal{P}_S$, where $\mathcal{P}_S$ is called a $qcsf$-network for S in X.

(6) $\mathcal{P}$ is called a cs^*f-network of X, if whenever S is a sequence converging to x in X, there exists a countable $\mathcal{P}_S \subset \mathcal{P}$ such that $\mathcal{P}_S$ is a network at x in X and S is frequently in $\bigcap \mathcal{P}'_S$ for any finite subfamily $\mathcal{P}'_S$ of $\mathcal{P}_S$, where $\mathcal{P}_S$ is called a cs^*f-network for S in X.

Remark 2.6. We have the following implications: $(*)$ is known from Remark 2.14; other implications are known from Definition 2.5. In section 3, we will also show that none of these implications can be reversed.

$$\begin{array}{ccccc} csf\text{-network} & \Longrightarrow & qcsf\text{-network} & \overset{(*)}{\Longrightarrow} & cs^*f\text{-network} \\ \Downarrow & & \Downarrow & & \Downarrow \\ cs\text{-network} & \Longrightarrow & qcs\text{-network} & \Longrightarrow & cs^*\text{-network} \end{array}$$

Lemma 2.7. *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system and let $U = (\Pi_{n\in\mathbb{N}}\Gamma_n)\bigcap M$, where $\Gamma_n \subset \Lambda_n$ for every $n \in \mathbb{N}$. Then $f(U) \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$ for every $k \in \mathbb{N}$.*

Proof: Let $b = (\beta_n) \in U$ and let $k \in \mathbb{N}$. Then $\{P_{\beta_n}\}$ is a network at $f(b)$ in X and $\beta_k \in \Gamma_k$. So $f(b) \in P_{\beta_k} \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$. This proves that $f(U) \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$. □

Proposition 2.8. *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then f is a compact mapping (resp., s-mapping) iff $\mathcal{P}$ is a point-finite (resp., point-countable) network of X.*

Proof: By Theorem 1.1, we need only to prove necessities.

We give a proof for only the parenthetic part. If $\mathcal{P}$ is not point-countable, then, for some $x \in X$, there exists an uncountable subset Γ of Λ such that $\Gamma = \{\beta \in \Lambda : x \in P_\beta\}$. Let $\{P_{\beta_n}\}$ be a network at x in X. For every $\beta \in \Gamma$, put $c_\beta = (\gamma_n)$, where $\gamma_1 = \beta$, and $\gamma_n = \beta_{n-1}$ for $n > 1$, then $\{P_{\gamma_n}\}$ is a network at x in X, so $c_\beta \in f^{-1}(x)$. Put $U_\beta = (\{\beta\} \times (\Pi_{n>1}\Lambda_n))\bigcap M$ for every $\beta \in \Gamma$. It is easy to see that $\{U_\beta \bigcap f^{-1}(x) : \beta \in \Gamma\}$ is an uncountable collection of disjoint open subsets of $f^{-1}(x)$. So $f^{-1}(x)$ is not ccc; hence, it is not separable, a contradiction. □

Lemma 2.9. *Let $f : X \longrightarrow Y$ be a mapping, and let $\{y_n\}$ be a sequence converging to y in Y. If $\{B_n\}$ is a decreasing network at some $x \in f^{-1}(y)$ in X, and $\{y_n\}$ is frequently in $f(B_k)$ for every $k \in N$, then there is a sequence $\{x_k\}$ converging to x such that $x_k \in f^{-1}(y_{n_k})$ for every $k \in \mathbb{N}$.*

Proof: For every $k \in N$, $\{y_n\}$ is frequently in $f(B_k)$, so there exists $n_k \in \mathbb{N}$ such that $y_{n_k} \in f(B_k)$. Pick $x_k \in f^{-1}(y_{n_k}) \bigcap B_k$ for every $k \in N$. Without loss of generality, we can assume $n_k < n_{k+1}$ for every $k \in \mathbb{N}$. It is not difficult to prove that $\{x_k\}$ converges to x. □

Theorem 2.10. *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then f is a sequentially-quotient mapping iff $\mathcal{P}$ is a cs^*f-network of X.*

Proof: Necessity. Let f be sequentially-quotient. Whenever $\{x_n\}$ is a sequence converging to x in X, there exists a sequence $\{b_k\}$ converging to b in M such that $f(b_k) = x_{n_k}$ for every $k \in \mathbb{N}$. Let $b = (\beta_i) \in (\Pi_{i\in\mathbb{N}}\Lambda_i) \bigcap M$. Put $\mathcal{P}_S = \{P_{\beta_i} : i \in \mathbb{N}\}$, then $\mathcal{P}_S \subset \mathcal{P}$ is a countable network at x in X. For every finite subset F of $\mathbb{N}$, put $n_0 = \max\{n : n \in F\}$, and put $B = ((\Pi_{i\leq n_0}\{\beta_i\}) \times (\Pi_{i>n_0}\Lambda_i)) \bigcap M$, then B is an open neighborhood of b in M. Thus, $\{b_k\}$ is eventually in B, and so $\{x_{n_k}\}$ is eventually in $f(B)$; that is, $\{x_n\}$ is frequently in $f(B)$. $f(B) \subset P_{\beta_i}$ for every $i \in F$ from Lemma 7, so $f(B) \subset \bigcap\{P_{\beta_i} : i \in F\}$. Thus, $\{x_n\}$ is frequently in $\bigcap\{P_{\beta_i} : i \in F\}$. So $\mathcal{P}_S$ is a cs^*f-network for S in X. This proves that $\mathcal{P}$ is a cs^*f-network of X.

Sufficiency. Let $\mathcal{P}$ be a cs^*f-network of X, and let $S = \{x_n\}$ be a sequence converging to x in X. Then there exists a countable $\mathcal{P}_S = \{P_{\beta_n}\} \subset \mathcal{P}$ such that $\mathcal{P}_S$ is a cs^*f-network for S in X. Put $b = (\beta_n)$, then $b \in f^{-1}(x)$. For every $n \in \mathbb{N}$, put $B_n = ((\Pi_{i\leq n}\{\beta_i\}) \times (\Pi_{i>n}\Lambda_i)) \bigcap M$. Then $\{B_n\}$ is a decreasing neighborhood base at b in Y. It is not difficult to prove that $f(B_n) = \bigcap_{i\leq n} P_{\beta_i}$ for every $n \in \mathbb{N}$. In fact, $f(B_n) \subset \bigcap_{i\leq n} P_{\beta_i}$ from Lemma 2.7. On the other hand, if $y \in \bigcap_{i\leq n} P_{\beta_i}$, then there exists $a = (\alpha_n) \in M$ such that $f(a) = y$; that is, $\{P_{\alpha_n}\}$ is a network at y in X. Put $c = (\gamma_n) \in M$, where $\gamma_i = \beta_i$ for $i \leq n$ and $\gamma_i = \alpha_{i-n}$ for $i > n$, then $c \in B_n$ and $\{P_{\gamma_n}\}$ is a network at y in X. So $y = f(c) \in f(B_n)$. This shows that $\bigcap_{i\leq n} P_{\beta_i} \subset f(B_n)$, and so $f(B_n) = \bigcap_{i\leq n} P_{\beta_i}$. Thus, $\{x_n\}$ is frequently in $f(B_n)$ for every $n \in \mathbb{N}$. By Lemma 3, there exists a sequence $\{b_k\}$ converging to b in M with $b_k \in f^{-1}(x_{n_k})$ for every $k \in \mathbb{N}$. This proves that f is sequentially-quotient. □

Lemma 2.11. *Let X be a product space $\Pi_{n\in\mathbb{N}}\Gamma_n$, where Γ_n is a discrete space for every $n \in \mathbb{N}$. If K is a compact subset of X, then for every $n \in \mathbb{N}$, $p_n(K)$ is finite, where $p_n : X \longrightarrow \Gamma_n$ is a project.*

Proof: If not, then $p_n(K)$ is infinite for some $n \in \mathbb{N}$. Put $U_\beta = (\Pi_{i<n}\Gamma_i)\times\{\beta\}\times(\Pi_{i>n}\Gamma_i))$ for every $\beta \in p_n(K)$, then $\mathcal{U} = \{U_\beta : \beta \in p_n(K)\}$ is a family consisting of infinite open subsets of X, which covers K. It is clear that $\mathcal{U}$ has no proper subfamily covering K. This contradicts that K is compact. □

Lemma 2.12. *Let $\mathcal{P}$ be a qcsf-network of X and $S = \{x_n\}$ be a sequence converging to a point x in a space X. Put $L = \{x_n : n \in \mathbb{N}\}\bigcup\{x\}$, then there exists a countable family $\{\mathcal{P}_n : n \in \mathbb{N}\}$ satisfying conditions (1), (2), and (3) below, where every $\mathcal{P}_n$ is a finite subfamily of $\mathcal{P}$ and covers L.*

(1) If $L \subset U$ with U open in X, then there exists $n \in \mathbb{N}$ such that $L \subset \bigcup\mathcal{P}_n \subset U$.

(2) For every $n \in \mathbb{N}$ and every $P \in \mathcal{P}_n$, $P\bigcap L$ is closed in X.

(3) If $y \in L$ and $P_n \in (\mathcal{P}_n)_y$ for every $n \in \mathbb{N}$, then $\{P_n\}$ is a network at y in X.

Proof: Let P_S be a *qcsf*-network for S in X. Put $\mathcal{E} = \{\mathcal{F} : \mathcal{F}\ is\ a\ finite\ subfamily\ of\ \mathcal{P}_S\ and\ S\ is\ eventually\ in\ \bigcup\mathcal{F}\}$, then $\mathcal{E}$ is countable. For every $\mathcal{F} \in \mathcal{E}$, we construct a countable family $\mathcal{G}_\mathcal{F}$ consisting of finite subfamilies of $\mathcal{P}$ as follows. Put $L(\mathcal{F}) = L-\bigcup\mathcal{F}$, then $L(\mathcal{F})$ is finite. For every $y \in L(\mathcal{F})$, there exists a countable network $\mathcal{P}_y \subset \mathcal{P}$ at y in X such that $y \in P \subset X - (L - \{y\})$ for every $P \in \mathcal{P}_y$. For convenience, if $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n$ are finite families, we denote the set $\{\{A_1, A_2, \cdots, A_n\} : A_i \in \mathcal{A}_i,\ i = 1, 2, \cdots, n\}$ by $\Pi\{\mathcal{A}_i : i = 1, 2, \cdots, n\}$. Put $\mathcal{G}_\mathcal{F} = \{\mathcal{F}\bigcup\mathcal{B} : \mathcal{B} \in \Pi\{\mathcal{P}_y : y \in L(\mathcal{F})\}\}$. Note that $\Pi\{\mathcal{P}_y : y \in L(\mathcal{F})\}$ is countable; thus, we construct a countable family $\mathcal{G}_\mathcal{F}$ consisting of finite subfamilies of $\mathcal{P}$ for every $\mathcal{F} \in \mathcal{E}$. Put $\mathcal{D} = \bigcup\{\mathcal{G}_\mathcal{F} : \mathcal{F} \in \mathcal{E}\}$, then $\mathcal{D}$ is countable and every element of $\mathcal{D}$ is a finite subfamily of $\mathcal{P}$. Put $\mathcal{D} = \{\mathcal{P}_n : n \in \mathbb{N}\}$. It is clear that $\mathcal{P}_n$ covers L for every $n \in \mathbb{N}$. We need only to prove that $\mathcal{D}$ satisfies the following conditions.

(1) If $L \subset U$ with U open in X, then there exists $n \in \mathbb{N}$ such that $L \subset \bigcup\mathcal{P}_n \subset U$.

Let $L \subset U$ with U open in X. Since $x \in U$, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}_S$ such that S is eventually in $\bigcup \mathcal{F} \subset U$. So $\mathcal{F} \in \mathcal{E}$. For every $y \in L(\mathcal{F})$, $\mathcal{P}_y \subset \mathcal{P}$ is a network at y in X, so there exists $P_y \in \mathcal{P}_y$ such that $y \in P_y \subset U$. Note that $\{P_y : y \in L(\mathcal{F})\} \in \Pi\{\mathcal{P}_y : y \in L(\mathcal{F})\}$, so $\mathcal{F} \bigcup \{P_y : y \in L(\mathcal{F})\} \in \mathcal{G}_{\mathcal{F}} \subset \mathcal{D}$; hence, there exists $n \in \mathbb{N}$ such that $\mathcal{F} \bigcup \{P_y : y \in L(\mathcal{F})\} = \mathcal{P}_n$. It is easy to see that $L \subset \bigcup \mathcal{P}_n \subset U$.

(2) For every $n \in \mathbb{N}$ and every $P \in \mathcal{P}_n$, $P \bigcap L$ is closed in X.

Let $n \in \mathbb{N}$, and let $\mathcal{P}_n = \mathcal{F} \bigcup \{P_y : y \in L(\mathcal{F})\} \in \mathcal{G}_{\mathcal{F}}$ for some $\mathcal{F} \in \mathcal{E}$, where $P_y \subset X - (L - \{y\})$ for every $y \in L(\mathcal{F})$. For every $P \in \mathcal{P}_n$, if $P \in \mathcal{F}$, then $x \in P \bigcap L$, so $P \bigcap L$ is closed in X; if $P = P_y$ for some $y \in L(\mathcal{F})$, then $P \bigcap L = P_y \bigcap L = \{y\}$ is closed in X.

(3) If $y \in L$ and $P_n \in (\mathcal{P}_n)_y$ for every $n \in \mathbb{N}$, then $\{P_n\}$ is a network at y in X.

Let $y \in L$, and let $P_n \in (\mathcal{P}_n)_y$ for every $n \in \mathbb{N}$. Let $y \in U$ with U open in X. If $y = x$, then there exists a finite subfamily $\mathcal{F}_1$ of $\mathcal{P}_S$ such that S is eventually in $\bigcup \mathcal{F}_1 \subset U$, so $\mathcal{F}_1 \in \mathcal{E}$. Put $\mathcal{G}_{\mathcal{F}_1} = \{\mathcal{P}_{n_k} : k \in \mathbb{N}\}$, where for every $k \in \mathbb{N}$, $\mathcal{P}_{n_k} = \mathcal{F}_1 \bigcup \mathcal{B}_k$ and $\mathcal{B}_k \in \Pi\{\mathcal{P}_z : z \in L(\mathcal{F}_1)\}$. For every $k \in \mathbb{N}$, $y = x \notin \bigcup \mathcal{B}_k$, so $(\mathcal{P}_{n_k})_y = (\mathcal{F}_1 \bigcup \mathcal{B}_k)_y = \mathcal{F}_1$; hence, $P_{n_k} \in \mathcal{F}_1$. Pick $k_0 \in \mathbb{N}$, then $y \in P_{n_{k_0}} \in \mathcal{F}_1$, so $P_{n_{k_0}} \subset \bigcup \mathcal{F}_1 \subset U$. If $y \neq x$, then $X - \{y\}$ is an open neighborhood of x. So there exists a finite subfamily $\mathcal{F}_2$ of $\mathcal{P}_S$ such that S is eventually in $\bigcup \mathcal{F}_2 \subset X - \{y\}$. It is clear that $\mathcal{F}_2 \in \mathcal{E}$ and $y \in L(\mathcal{F}_2)$. Put $\mathcal{G}_{\mathcal{F}_2} = \{\mathcal{P}_{m_k} : k \in \mathbb{N}\}$, where for every $k \in \mathbb{N}$, $\mathcal{P}_{m_k} = \mathcal{F}_2 \bigcup \mathcal{B}_k$ and $\mathcal{B}_k \in \Pi\{\mathcal{P}_z : z \in L(\mathcal{F}_2)\}$. Since $\mathcal{P}_y$ is a network at y in X, there exists $P_y \in \mathcal{P}_y$ such that $y \in P_y \subset U$. By construction of $\mathcal{G}_{\mathcal{F}_2}$, there exists $k_0 \in \mathbb{N}$ such that $P_y \in \mathcal{B}_{k_0}$. Let $\mathcal{B}_{k_0} = \{P_z : z \in L(\mathcal{F})\}$, then $y \notin P_z$ for every $z \in L(\mathcal{F}) - \{y\}$. Furthermore, $P_{m_{k_0}} \neq P_z$ for every $z \in L(\mathcal{F}) - \{y\}$. So $P_{m_{k_0}} = P_y \subset U$. This proves that $\{P_n\}$ is a network at y in X.

This completes the proof of the lemma. □

Theorem 2.13. *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then f is a pseudo-sequence-covering mapping iff $\mathcal{P}$ is a qcsf-network of X.*

Proof: Necessity. Let f be a pseudo-sequence-covering mapping. Whenever $S = \{x_n\}$ is a sequence in X converging to x in X, put

$L = \{x_n : n \in \mathbb{N}\} \bigcup \{x\}$, then there exists a compact subset K of M such that $f(K) = L$. Note that $f^{-1}(x) \bigcap K \subset \Pi_{n\in\mathbb{N}}\Lambda_n$ is compact, so $p_n(f^{-1}(x) \bigcap K)$ is finite for every $n \in \mathbb{N}$ from Lemma 2.11, where every $p_n : \Pi_{n\in\mathbb{N}}\Lambda_n \longrightarrow \Lambda_n$ is a project. Put $\Gamma = \bigcup\{p_n(f^{-1}(x) \bigcap K) : n \in \mathbb{N}\}$ and $\mathcal{P}_S = \{P_\beta : \beta \in \Gamma\}$, then $\mathcal{P}_S \subset (\mathcal{P})_x$ is countable. Let $x \in U$ with U open in X. We construct U_b for every $b \in f^{-1}(x) \bigcap K$ as follows. Let $b = (\beta_i^b)_{i\in\mathbb{N}} \in f^{-1}(x) \bigcap K$, then $\beta_i^b \in \Gamma$ for every $i \in \mathbb{N}$ and $\{P_{\beta_i^b} : i \in \mathbb{N}\}$ is a network at x in X, so there exists $i_b \in \mathbb{N}$ such that $x \in P_{\beta_{i_b}^b} \subset U$. Put $U_b = ((\Pi_{n<i_b}\Lambda_n) \times \{\beta_{i_b}^b\} \times (\Pi_{n>i_b}\Lambda_n)) \bigcap M$, then U_b is an open neighborhood of b, and $x \in f(U_b) \subset P_{\beta_{i_b}^b} \subset U$ from Lemma 2.7. Since $\{U_b : b \in f^{-1}(x) \bigcap K\}$ covers $f^{-1}(x) \bigcap K$ and $f^{-1}(x) \bigcap K$ is compact, there exists a finite subset $\mathbb{F}$ of $f^{-1}(x) \bigcap K$ such that $\{U_b : b \in \mathbb{F}\}$ covers $f^{-1}(x) \bigcap K$. Put $\mathcal{P}_U = \{P_{\beta_{i_b}^b} : b \in \mathbb{F}\}$, then $\mathcal{P}_U \subset \mathcal{P}_S$ and $\bigcup \mathcal{P}_U = \bigcup\{P_{\beta_{i_b}^b} : b \in \mathbb{F}\} \subset U$. We need only to prove that sequence $\{x_n\}$ converging to x is eventually in $\bigcup \mathcal{P}_U$.

If not, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin \bigcup \mathcal{P}_U$ for every $k \in \mathbb{N}$. That is, $x_{n_k} \notin P_{\beta_{i_b}^b}$ for every $k \in \mathbb{N}$ and every $b \in \mathbb{F}$. We pick $c_k \in K$ such that $f(c_k) = x_{n_k}$ for every $k \in \mathbb{N}$. If $c_k \in U_b$ for some $k \in \mathbb{N}$ and some $b \in \mathbb{F}$, then $x_{n_k} = f(c_k) \in f(U_b) \subset P_{\beta_{i_b}^b}$. This is a contradiction. So $c_k \notin U_b$ for every $k \in \mathbb{N}$ and every $b \in \mathbb{F}$. Thus, $\{c_k : k \in \mathbb{N}\} \subset K - \bigcup\{U_b : b \in \mathbb{F}\}$. Note that $K - \bigcup\{U_b : b \in \mathbb{F}\}$ is a compact metric subspace, $\{c_k\}$ has a subsequence $\{c_{k_i}\}$ converging to a point $c \in K - \bigcup\{U_b : b \in \mathbb{F}\}$. Thus, $c \notin f^{-1}(x)$; that is, $f(c) \neq x$. On the other hand, $\{f(c_{k_i})\}$ converges to $f(c)$ by the continuity of f and $\{f(c_{k_i})\} = \{x_{n_{k_i}}\}$ converges to x. Thus, $f(c) = x$. This is a contradiction. This proves that the sequence $\{x_n\}$ converging to x is eventually in $\bigcup \mathcal{P}_U$.

Sufficiency. Let $\mathcal{P}$ be a $qcsf$-network of X. Whenever $S = \{x_n\}$ is a sequence converging to x in X, put $L = \{x_n : n \in \mathbb{N}\} \bigcup \{x\}$. By Lemma 2.12, there exists a countable family $\{\mathcal{P}_n : n \in \mathbb{N}\}$ satisfying conditions (1), (2), and (3) in Lemma 2.12, where every $\mathcal{P}_n$ is a finite subfamily of $\mathcal{P}$ and covers L. For every $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\beta : \beta \in \Gamma_n\}$, then Γ_n is a finite subset of Λ_n. Put $K = \{(\beta_n) \in \Pi_{n\in\mathbb{N}}\Gamma_n : L \bigcap (\bigcap_{n\in\mathbb{N}} P_{\beta_n}) \neq \emptyset\}$.

Claim 1. $K \subset M$ and $f(K) \subset L$.

Let $b = (\beta_n) \in K$, then $L\bigcap(\bigcap_{n\in\mathbb{N}} P_{\beta_n}) \neq \emptyset$. Pick $y \in L\bigcap(\bigcap_{n\in\mathbb{N}} P_{\beta_n})$, then $y \in L$ and $P_{\beta_n} \in (\mathcal{P}_n)_y$ for every $n \in \mathbb{N}$. By Lemma 2.12(3), $\{P_{\beta_n}\}$ is a network at y in X. So $b \in M$ and $f(b) = y \in L$. This proves that $K \subset M$ and $f(K) \subset L$.

Claim 2. $L \subset f(K)$.

Let $y \in L$. Pick $\beta_n \in \Gamma_n$ for every $n \in \mathbb{N}$ such that $y \in P_{\beta_n}$, then $\{P_{\beta_n}\}$ is a network at y in X. Put $b = (\beta_n)$, then $b \in K$ and $f(b) = y$. This proves that $L \subset f(K)$.

Claim 3. K is a compact subset of M.

Since $K \subset \Pi_{n\in\mathbb{N}}\Gamma_n$ and $\Pi_{n\in\mathbb{N}}\Gamma_n$ is a compact subset of $\Pi_{n\in\mathbb{N}}\Lambda_n$, we need only to prove that K is a closed subset of $\Pi_{n\in\mathbb{N}}\Gamma_n$. Let $b = (\beta_n) \in \Pi_{n\in\mathbb{N}}\Gamma_n - K$. Then $L\bigcap(\bigcap_{n\in\mathbb{N}} P_{\beta_n}) = \emptyset$; that is, $\bigcap_{n\in\mathbb{N}}(L\bigcap P_{\beta_n}) = \emptyset$. By Lemma 2.12(2), $L\bigcap P_{\beta_n}$ is closed in compact subset L for every $n \in \mathbb{N}$. So there exists $n_0 \in \mathbb{N}$ such that $\bigcap_{n\leq n_0}(L\bigcap P_{\beta_n}) = \emptyset$; that is, $L\bigcap(\bigcap_{n\leq n_0} P_{\beta_n}) = \emptyset$. Put $W = (\Pi_{n\leq n_0}\{\beta_n\}) \times (\Pi_{n>n_0}\Gamma_n)$. Then W is open in $\Pi_{n\in\mathbb{N}}\Gamma_n$ and $b \in W$. It is easy to see that $W\bigcap K = \emptyset$. So K is a closed subset of $\Pi_{n\in\mathbb{N}}\Gamma_n$.

By the above three claims, we prove that f is a pseudo-sequence-covering mapping. □

Remark 2.14. The implication $(*)$ holds. In fact, if $\mathcal{P}$ is a *qcsf*-network of a space X, then, for the *Ponomarev*-system $(f, M, X, \mathcal{P})$, f is a pseudo-sequence-covering mapping from Theorem 2.13, so f is a sequentially-quotient mapping from Remark 2.4(2). Thus, $\mathcal{P}$ is a cs^*f-network of X from Theorem 2.10.

3. The other results

In this section, we establish some relationships among point-countable networks and give some examples to show that none of the implications in Remark 2.6 can be reversed.

Proposition 3.1. *Let $\mathcal{P}$ be a point-countable cover of a space X. Then the following are equivalent.*

(1) $\mathcal{P}$ is a qcsf-network of X.
(2) $\mathcal{P}$ is a cs^*f-network of X.
(3) $\mathcal{P}$ is a qcs-network of X.
(4) $\mathcal{P}$ is a cs^*-network of X.

Proof: We need only to prove (4) $\Longrightarrow$ (1) from Remark 2.6.

If $\mathcal{P}$ is a cs^*-network of X, then $\mathcal{P}$ is a point-countable cs^*-network of X. For the *Ponomarev*-system $(f, M, X, \mathcal{P})$, f is a pseudo-sequence-covering mapping from Theorem 1.1. So $\mathcal{P}$ is a $qcsf$-network of X from Theorem 2.13. □

Corollary 3.2. *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system. Then the following are equivalent.*

(1) f is a sequentially-quotient s-mapping.
(2) f is a pseudo-sequence-covering s-mapping.
(3) $\mathcal{P}$ is a point-countable $qcsf$-network of X.
(4) $\mathcal{P}$ is a point-countable cs^*f-network of X.
(5) $\mathcal{P}$ is a point-countable qcs-network of X.
(6) $\mathcal{P}$ is a point-countable cs^*-network of X.

Corollary 3.3 ([8]). *A space is a sequentially-quotient s-image of a metric space iff it is a pseudo-sequence-covering s-image of a metric space.*

Remark 3.4. Corollary 3.2 and Proposition 2.8 answer Question 1.2(1).

The following example gives a negative answer for Question 1.2(2).

Recall the definition of sequential fan S_ω [1]. Let $T_0 = \{a_n : n \in \mathbb{N}\}$ be a sequence converging to $x_0 \notin T_0$, and let T_n be a sequence converging to $a_n \notin T_n$ for every $n \in \mathbb{N}$. Let T be the topological sum of $\{T_n \bigcup \{a_n\} : n \in \mathbb{N}\}$. S_ω is defined as a quotient space obtained from T by identifying all point $a_n \in T$ to the point x_0.

Example 3.5. *There exists a Ponomarev-system $(f, M, X, \mathcal{P})$ such that $\mathcal{P}$ is a cs-network, but $\mathcal{P}$ is not a cs^*f-network, and so f is not sequentially-quotient (pseudo-sequence-covering).*

Proof: Let X be the sequential fan space S_ω, then X has no countable neighborhood base at x_0 [9], where x_0 is the non-isolated point in X. Put $\mathcal{P} = \{U \subset X : U \textit{ is open in } X\} \bigcup \{\{x_0\}\}$, then $\mathcal{P}$ is a network of X, and there exists a countable $\mathcal{P}_x \subset \mathcal{P}$ such that $\mathcal{P}_x$ is a network at x in X for every $x \in X$. Thus, $(f, M, X, \mathcal{P})$ is a Ponomarev-system.

Claim 1. $\mathcal{P}$ is a cs-network of X.

It is clear.

Claim 2. $\mathcal{P}$ is not a cs^*f-network of X.

Let S be a non-trivial sequence converging to x_0 in X. If $\mathcal{P}$ is a cs^*f-network of X, then there exists a countable $\mathcal{P}_S \subset \mathcal{P}$ such that $\mathcal{P}_S$ is a cs^*f-network for S in X. It is clear that every element in $\mathcal{P}_S$ is open in X. Note that $\mathcal{P}_S$ is a countable network at x_0 in X. So $\mathcal{P}_S$ is a countable neighborhood base at x_0 in X. This contradicts that X has no countable neighborhood base at x_0. So $\mathcal{P}$ is not a cs^*f-network of X.

Thus, the proof is complete. □

Are (1) and (2) in Corollary 3.2 equivalent if "s-" is omitted? The answer is negative by the following example. But we do not know if "s-" in Corollary 3.3 can be omitted.

We call a family $\mathcal{D}$ of subsets of a set D *almost disjoint* if $D_1 \bigcap D_2$ is finite whenever $D_1, D_2 \in \mathcal{D},\ D_1 \neq D_2$.

Example 3.6. *There exists a network $\mathcal{P}$ of a space X such that $\mathcal{P}$ is a cs^*f-network of X, but $\mathcal{P}$ is not a qcs-network of X. So, for the Ponomarev-system $(f, M, X, \mathcal{P})$, f is sequentially-quotient, but f is not pseudo-sequence-covering.*

Proof: Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ be endowed with the usual subspace topology of real line $\mathbb{R}$. Using Zorn's Lemma, there exists a family $\mathcal{D}$ of subsets of X such that $\mathcal{D}$ is almost disjoint and maximal with respect to these properties. Then $\mathcal{D}$ must be uncountable; denote it by $\{D_\alpha : \alpha \in \Gamma\}$. We can assume $0 \in D_\alpha$ for every $\alpha \in \Gamma$. Let $\alpha \in \Gamma$ and let D_α be denoted by $\{y_i : i \in \mathbb{N}\} \bigcup \{0\}$. Put $D_\alpha(n) = \{y_i : i \geq n\} \bigcup \{0\}$ for every $n \in \mathbb{N}$, and put $\mathcal{D}_\alpha = \{D_\alpha(n) : n \in \mathbb{N}\}$. Thus, we construct $\mathcal{D}_\alpha$ for every $\alpha \in \Gamma$. Put $\mathcal{P} = (\bigcup\{\mathcal{D}_\alpha : \alpha \in \Gamma\}) \bigcup \{\{1/n\} : n \in \mathbb{N}\}$, then $\mathcal{P}$ is a network of X. It is easy to prove that $\mathcal{D}_\alpha$ is a countable network at 0 in X for every $\alpha \in \Gamma$.

Claim 1. $\mathcal{P}$ is a cs^*f-network of X.

Let $S = \{x_i\}$ be a sequence converging to x in X. Put $L = \{x_i : i \in \mathbb{N}\} \bigcup \{x\}$. Without loss of generality, we can assume S is nontrivial; that is, the limit point $x = 0$ and L is an infinite subset of X. Since $\mathcal{D}$ is a maximal almost disjoint family, there exists a $D_\alpha \in \mathcal{D}$ such that $D_\alpha \bigcap L$ is a nontrivial subsequence of L. Put $\mathcal{P}_S = \mathcal{D}_\alpha$, then $\mathcal{P}_S \subset \mathcal{P}$ is a countable network at 0 in X. Note that $\mathcal{D}_\alpha = \{D_\alpha(n)\}$ is decreasing and $D_\alpha(n)$ is an infinite subset of L for every $n \in \mathbb{N}$. $\bigcap \mathcal{P}'_S$, and so $L \bigcap (\bigcap \mathcal{P}'_S)$ is infinite for any finite subfamily $\mathcal{P}'_S$ of $\mathcal{P}_S$. This shows that S is frequently in $\bigcap \mathcal{P}'_S$ for any finite subfamily $\mathcal{P}'_S$ of $\mathcal{P}_S$. Thus, we prove that $\mathcal{P}$ is a cs^*f-network of X.

Claim 2. $\mathcal{P}$ is not a qcs-network of X.

Let S be the sequence $S = \{1/n\}$ converging to 0 in X. If $\mathcal{P}$ is a qcs-network of X, then S is eventually in $\bigcup \mathcal{P}'$ for some finite subfamily $\mathcal{P}'$ of $\mathcal{P}$. Thus, there exist a finite subfamily $\{D_{\alpha_1}, D_{\alpha_2}, \cdots, D_{\alpha_l}\}$ of $\mathcal{D}$ and $m \in \mathbb{N}$ such that $L = \{1/n : n \geq m\} \bigcup \{0\} \subset \bigcup \{D_{\alpha_i} : i = 1, 2, \cdots, l\}$. Γ is infinite; pick $\alpha \in \Gamma - \{\alpha_i : i = 1, 2, \cdots, l\}$. It is clear that $L' = L \bigcap D_\alpha$ is infinite and $L' \subset L \subset \bigcup \{D_{\alpha_i} : i = 1, 2, \cdots, l\}$. So there exists $i \in \{1, 2, \cdots, l\}$ such that $L' \bigcap D_{\alpha_i}$ is infinite. Thus, $D_\alpha \bigcap D_{\alpha_i}$ is infinite. This contradicts that $\mathcal{D}$ is almost disjoint. So $\mathcal{P}$ is not a qcs-network of X.

Hence, the proof is complete. □

Remark 3.7. There exists a space X with a point-countable cs^*-network $\mathcal{P}$, which has not any point-countable cs-network [9, Example 1.5.6]. So $\mathcal{P}$ is a $qcsf$-network of X, but it is not a cs-network. Combining Example 3.5 and Example 3.6, none of the implications in Remark 2.6 can be reversed.

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