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TWO - PASS MAPS AND INDECOMPOSABILITY OF INVERSE LIMITS OF GRAPHS

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Abstract. In this paper we show that an inverse limit on an n-od using a single two-pass map produces an indecomposable continuum. This extends a well-known result for inverse limits on intervals. Then we show that if G is a graph that is neither an arc nor an n-od, there is a two-pass map f of G onto itself such that $\varprojlim\{G, f\}$ is decomposable.

1. Introduction

The two-pass condition is a simple condition on a bonding map in an inverse limit of intervals using a single bonding map that is sufficient to insure the indecomposability of the inverse limit [3, Theorem 6.3, p. 35]. In general, a two-pass condition on a map f of a graph G is not sufficient to insure that the inverse limit on G using the single map f is indecomposable. In this paper, we define a natural two-pass condition on a map of a graph to itself and show that if the graph is an arc or a simple n-od for some positive integer n then this two-pass condition is sufficient for the indecomposability of the inverse limit. We also show in section 4 that if G is a graph that is neither an arc nor a simple n-od, there is a two-pass map of G onto G so that the inverse limit is decomposable.

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A *continuum* is a compact, connected metric space. A continuum is *decomposable* if it is the union of two of its proper subcontinua and is *indecomposable* otherwise. A *graph* is a continuum that is the union of a finite collection of arcs such that if two arcs of the collection have a point in common, their common part is a common end point of the two arcs. A *branch point* of a graph is a point of the graph that is of order greater than 2 while an *end point* of a graph is a point of order 1. By a *mapping* we mean a continuous transformation. If X and Y are continua, we use the notation $f : X \twoheadrightarrow Y$ to connote that $f[X] = Y$.

If $X_1, X_2, X_3, \ldots$ is a sequence of continua and $f_1, f_2, f_3, \ldots$ is a sequence of mappings such that $f_i : X_{i+1} \to X_i$, the *inverse limit* of the inverse sequence $\{X_i, f_i\}$ is the subset of the product $\prod_{i>0} X_i$ containing the point $(x_1, x_2, x_3, \ldots)$ if and only if $f_i(x_{i+1}) = x_i$. The inverse limit of this inverse limit sequence is denoted $\varprojlim\{X_i, f_i\}$. If X is a continuum and $f : X \to X$ is a mapping, by $\varprojlim\{X, f\}$ we mean the inverse limit of the inverse sequence $\{X_i, f_i\}$ where, for each i, $X_i = X$ and $f_i = f$. We denote by π_i the projection of the inverse limit into the ith factor space X_i.

2. Kuykendall's Theorem

If $X_1, X_2, X_3, \ldots$ is a sequence of continua and $f_1, f_2, f_3, \ldots$ is a sequence of mappings such that $f_i : X_{i+1} \to X_i$ and $j > i$; for convenience, we denote $f_i \circ f_{i+1} \circ \cdots \circ f_{j-1}$ by f_i^j and note that $f_i^j : X_j \to X_i$. In 1972, Daniel P. Kuykendall [4] proved the following theorem characterizing indecomposability in inverse limits of continua.

Theorem 2.1 (Kuykendall). *Suppose (X_1, d_1), (X_2, d_2), $(X_3, d_3), \ldots$ is a sequence of continua and for each i $f_i : X_{i+1} \to X_i$ is a mapping. Then, $\varprojlim\{X_i, f_i\}$ is indecomposable if and only if for each positive number ε and each positive integer i there are a positive integer $j > i$ and three points of X_j such that if H is a subcontinuum of X_j containing two of these three points then $d_i(x, f_i^j[H]) < \varepsilon$ for each point x of X_i.*

The following theorem is a special case of Kuykendall's Theorem. It is easy to prove without using Kuykendall's Theorem so we provide a proof for the sake of completeness. If $f : X \to X$ is a mapping and n is a positive integer, we denote the n-fold composition of f with itself by f^n.

Theorem 2.2. *Suppose X is a continuum, $f : X \to X$ is a mapping, and there are a positive integer n and three points of X such that if C is a subcontinuum of X containing two of these three points then $f^n[C] = X$. Then $\varprojlim\{X, f\}$ is indecomposable.*

Proof: Suppose $M = \varprojlim\{X, f\}$ and $M = H \cup K$ where H and K are proper subcontinua of M. There exists a positive integer N such that $\pi_i H \neq X$ and $\pi_i K \neq X$ for $i \geq N$. By hypothesis there are a positive integer n and three points of X such that if C is a subcontinuum of X containing two of these three points then $f^n[C] = X$. Since $\pi_{N+n}[H] \cup \pi_{N+n}[K] = X$, two of these three points must belong to one of $\pi_{N+n}[H]$ and $\pi_{N+n}[K]$; assume that two of them belong to $\pi_{N+n}[H]$. Then, $\pi_N H = f^n[\pi_{N+n}[H]] = X$, a contradiction to the choice of N. □

3. Two-pass Maps and Indecomposablilty

If G is a graph, subgraphs G_1 and G_2 are said to be *non-overlapping* provided that if p is a point of $G_1 \cap G_2$ then p is an end point of both G_1 and G_2. A map $f : G \to G$ is said to be a *two-pass map* provided there exist two non-overlapping subgraphs G_1 and G_2 of G such that $f[G_i] = G$ for $i = 1, 2$. A *tree* is a graph that contains no simple closed curve.

The following theorem will be of use in this section. In [1, Theorem 2.4] it is couched in slightly different language as a part of a more comprehensive theorem characterizing W-sets in locally connected continua.

Theorem 3.1 (Hatch). *Suppose X is a locally connected continuum and H is a subcontinuum of X having the property that if q is a point of $X - H$ there is a point r_q of H such that if K is a subcontinuum of X containing q and r_q then H is a subset of K. Then, if $f : Y \twoheadrightarrow X$ is a mapping of a continuum Y onto X there is a subcontinuum C of Y such that $f[C] = H$.*

Lemma 3.2 will be helpful in proving Theorem 3.3. A proof could be modeled on the proof of Lemma 1, page 190, of [2]. However, we give a proof using Hatch's Theorem.

Lemma 3.2. *Suppose X is a continuum, T is a tree, and I is an arc lying in T such that if p is a branch point of T that belongs to I then p is an end point of I. If $f : X \twoheadrightarrow T$ is a mapping of X onto T, there is a subcontinuum C of X such that $f[C] = I$.*

Proof: Suppose X is a continuum, $f : X \twoheadrightarrow T$ is a mapping of X onto a tree T, and I is an arc lying in T such that if p is a branch point of T that belongs to I then p is an end point of I. Such an arc I either contains an end point of T or separates T. In either case, if q is a point of $T - I$ there is a point r_q of I such that if K is a subcontinuum of T that contains q and r_q then I is a subset of K. By Hatch's Theorem there is a subcontinuum C of X such that $f[C] = I$. □

Theorem 3.3. *Suppose T is a tree and $f : T \to T$ is a mapping. Suppose H and I are non-overlapping subtrees of T and I is an arc such that if p is a branch point of T that belongs to I then p is an end point of I. If $f[H] = f[I] = T$ then $\varprojlim\{T, f\}$ is indecomposable.*

Proof: Since $f|H : H \twoheadrightarrow T$ is a mapping of a continuum H onto T, by Lemma 3.2 there is a subtree K of H such that $f[K] = I$. Since K is a tree, it follows that there is an arc $\alpha = [x, y]$ lying in K such that $f[\alpha] = I$ and no proper subarc of α is mapped onto I by f. Note that $f^2[\alpha] = T$. If α and I have a point in common, it must be a common end point of the two arcs. Suppose it is x, and z is the other end point of I. Then x, y, and z are three points such that if C is a subcontinuum of T containing two of them then $f^2[C] = T$. By Theorem 2.2 $\varprojlim\{T, f\}$ is indecomposable.

If α and I are mutually exclusive, denote the two end points of I by a and b and notice that one of a and b separates α from the other. Suppose a separates α from b in T, i.e., $T - \{a\}$ is the union of two mutually separated sets, one containing α and the other containing b. Then, x, y, and b are three points of T such that if C is a subcontinuum of T containing two of them, then $f^2[C] = T$. Again, by Theorem 2.2 $\varprojlim\{T, f\}$ is indecomposable. □

Theorem 3.4. *If T is a tree with no more than one branch point and $f : T \to T$ is a two-pass map, then $\varprojlim\{T, f\}$ is indecomposable.*

Proof: Since the theorem is known when T is an arc, we assume that T is an n-od for some positive integer $n \geq 3$. Since f is a two-pass map, there are two non-overlapping subtrees T_1 and T_2 of

T such that $f[T_i] = T$ for $i = 1, 2$. Since T is an n-od, one of T_1 and T_2 is an arc such that if the branch point is in it then the branch point is an end point of it. Thus, T_1 and T_2 satisfy the hypothesis of Theorem 3.3 and, therefore, $\varprojlim\{T, f\}$ is indecomposable. □

4. Two-pass maps that produce decomposable inverse limits

In this section we show that if G is a graph that contains a simple closed curve or has at least two branch points, then there is a two-pass map $f : G \to G$ such that $\varprojlim\{G, f\}$ is decomposable.

The following lemma is easy to prove, but will be useful in the construction of two-pass maps in this section.

Lemma 4.1. *If X is a locally connected continuum, and p and q are points of X, and A is an arc lying in X having end points p and q, then there is a map $\phi : A \twoheadrightarrow X$ of A onto X such that $\phi(p) = p$ and $\phi(q) = q$.*

If G is a graph that is not an arc and is not an n-od, there are four cases:

(1) G is a simple closed curve;
(2) G contains a simple closed curve and has only one branch point;
(3) there exist two branch points v_1 and v_2 of G such that at least one arc lying in G and having end points v_1 and v_2 separates G;
(4) for each two branch points v_1 and v_2 of G if A is an arc lying in G having end points v_1 and v_2 then A does not separate G.

In each of the four cases for the graph G, we identify four subarcs $\alpha_1, \alpha_2, \beta_1,$ and β_2 of G; two subgraphs G_1 and G_2 of G; and a map $f : G \twoheadrightarrow G$ with the following properties:

(a) G_1 and G_2 are proper subgraphs of G and $G = G_1 \cup G_2$;
(b) $f[G_i] = G_i$ for $i = 1, 2$;
(c) $\alpha_i \subset G_i$ and $\beta_i \subset G_i$ for $i = 1, 2$;
(d) $f[\alpha_i] = G_i$ and $f[\beta_i] = G_i$ for $i = 1, 2$;
(e) $\alpha_1 \cup \alpha_2$ and $\beta_1 \cup \beta_2$ are non-overlapping arcs.

The first two properties guarantee that $\varprojlim\{G, f\}$ is decomposable while the final three guarantee that f is a two-pass map.

(1) We begin with the simple closed curve. Let G be a simple closed curve and choose two points p and q of G. Then G is the union of two arcs G_1 and G_2 such that p and q are the two end points of both G_1 and G_2. The arc G_1 is the union of arcs α_1 and β_1 such that p is in α_1 and q is in β_1 and the only point common to α_1 and β_1 is a common end point of the two arcs. The arc G_2 is the union of two arcs α_2 and β_2 such that p is in α_2 and q is in β_2 and the only point common to α_2 and β_2 is a common end point of the two arcs.

By Lemma 4.1, there exist four maps ϕ_1, ϕ_2, ψ_1 and ψ_2 with $\phi_i : \alpha_i \twoheadrightarrow G_i$ and $\psi_i : \beta_i \twoheadrightarrow G_i$ and each of the four maps fixes both end points of its domain. Let $f : G \to G$ be the map such that $f|\alpha_i = \phi_i$ and $f|\beta_i = \psi_i$ for $i = 1, 2$. Then $f[G_i] = G_i$ for $i = 1, 2$ and $\varprojlim\{G, f\} = \varprojlim\{G_1, f|G_1\} \cup \varprojlim\{G_2, f|G_2\}$. It is easy to see that $\varprojlim\{G_1, f|G_1\}$ and $\varprojlim\{G_2, f|G_2\}$ are proper subcontinua of $\varprojlim\{G, f\}$ so this inverse limit is decomposable. Let $\alpha = \alpha_1 \cup \alpha_2$ and $\beta = \beta_1 \cup \beta_2$. Then α and β are non-overlapping arcs lying in G such that $f[\alpha] = f[\beta] = G$.

(2) Next, we consider a graph G with a single branch point but G contains a simple closed curve. Let S be a simple closed curve contained in G. Then S is the union of two arcs, A and B, having common end points p and q where q is the branch point of G. The arc A is the union of two arcs α_1 and β_1 such that p is in α_1, q is in β_1, and the only point common to α_1 and β_1 is a common end point of the two arcs. The arc B is the union of two arcs α_2 and β_2 such that p is in α_2, q is in β_2, and the only point common to α_2 and β_2 is a common end point of the two arcs. Let G_1 be the subgraph $A \cup (G - S)$ and $G_2 = B$.

By Lemma 4.1, there are mappings $\phi_i : \alpha_i \twoheadrightarrow G_i$ and $\psi_i : \beta_i \twoheadrightarrow G_i$ for $i = 1, 2$ and each of the four mappings fixes both end points of its domain. Let $f : G \twoheadrightarrow G$ be given by $f(x) = x$ for x not in S and $f|\alpha_i = \phi_i$ and $f|\beta_i = \psi_i$ for $i = 1, 2$. Then $f[G_i] = G_i$ for $i = 1, 2$ and $\varprojlim\{G, f\} = \varprojlim\{G_1, f|G_1\} \cup \varprojlim\{G_2, f|G_2\}$. It is easy to see that $\varprojlim\{G_1, f|G_1\}$ and $\varprojlim\{G_2, f|G_2\}$ are proper subcontinua of $\varprojlim\{G, f\}$ so this inverse limit is decomposable. Let $\alpha = \alpha_1 \cup \alpha_2$

and $\beta = \beta_1 \cup \beta_2$. Then α and β are non-overlapping arcs lying in G such that $f[\alpha] = f[\beta] = G$.

(3) Suppose G is a graph with branch points v_1 and v_2 and A is an arc lying in G with end points v_1 and v_2 such that $G - A$ is not connected. Assume $G - A = H \cup K$ where H and K are mutually separated with v_1 in $\overline{H}$. We consider several cases. First, assume that v_2 is not in $\overline{H}$. Choose an arc α_1 lying in $\overline{H}$ such that v_1 is the only branch point of G in α_1 and $\alpha_1 \cap A = \{v_1\}$. Since v_2 is a branch point of G not in $\overline{H}$, v_2 is in $\overline{K}$. Choose an arc β_2 lying in $\overline{K}$ such that v_2 is the only branch point of G in β_2 and $\beta_2 \cap A = \{v_2\}$. Since v_1 and v_2 are branch points, there exist arcs α_2 and β_1 containing v_1 and v_2, respectively, and no other branch point of G, and such that $\alpha_1 \cap \alpha_2 = \{v_1\}$ and $\beta_1 \cap \beta_2 = \{v_2\}$ and $(\alpha_1 \cup \alpha_2) \cap (\beta_1 \cup \beta_2) = \emptyset$. If $\overline{H} - (\alpha_1 \cup \alpha_2)$ is not connected, let C_H be the closure of the union of the components of $\overline{H} - (\alpha_1 \cup \alpha_2)$ that have a limit point in $\alpha_2 - \{v_1\}$ and let C_H be empty if there are no such components. If $\overline{K} - (\beta_1 \cup \beta_2)$ is not connected, let C_K be the closure of the union of the components of $\overline{K} - (\beta_1 \cup \beta_2)$ that have a limit point in $\beta_1 - \{v_2\}$ and let C_K be empty if there are no such components. Let $G_1 = \overline{(\overline{H} - (\alpha_2 \cup C_H))} \cup A \cup (\beta_1 \cup C_K)$ and $G_2 = \overline{(\overline{K} - (\beta_1 \cup C_K))} \cup A \cup (\alpha_2 \cup C_H)$.

Next, the construction of subgraphs G_1 and G_2 is similar in the case where $G - A$ is the union of mutually separated sets H and K where v_2 is in $\overline{K}$ and v_1 is not in $\overline{K}$ or where v_1 is in $\overline{K}$ and v_2 is not in $\overline{K}$. Finally, in a case where $\{v_1, v_2\}$ is a subset of $\overline{H} \cap \overline{K}$, there are arcs α_1 and β_1 containing a single branch point of G and lying in $\overline{H}$ such that $\alpha_1 \cap A = \{v_1\}$ and $\beta_1 \cap A = \{v_2\}$, and there are arcs α_2 and β_2 containing a single branch point of G and lying in $\overline{K}$ such that $\alpha_2 \cap A = \{v_1\}$ and $\beta_2 \cap A = \{v_2\}$. Let $G_1 = H \cup A$ and $G_2 = K \cup A$.

By Lemma 4.1, there are mappings $\phi_i : \alpha_i \twoheadrightarrow G_i$ and $\psi_i : \beta_i \twoheadrightarrow G_i$ for $i = 1, 2$ such that each of the four mappings fixes both end points of its domain. Let $f : G \twoheadrightarrow G$ be given by $f(x) = x$ for x not in $\alpha_1 \cup \alpha_2 \cup \beta_1 \cup \beta_2$ and $f|\alpha_i = \phi_i$ and $f|\beta_i = \psi_i$ for $i = 1, 2$. Then $f[G_i] = G_i$ for $i = 1, 2$ and $\varprojlim\{G, f\} = \varprojlim\{G_1, f|G_1\} \cup \varprojlim\{G_2, f|G_2\}$. It is easy to see that $\varprojlim\{G_1, f|G_1\}$ and $\varprojlim\{G_2, f|G_2\}$ are proper subcontinua of $\varprojlim\{G, f\}$ so this inverse limit is decomposable. Let $\alpha = \alpha_1 \cup \alpha_2$ and $\beta = \beta_1 \cup \beta_2$.

Then α and β are non-overlapping arcs lying in G such that $f[\alpha] = f[\beta] = G$.

(4) Finally, suppose G is a graph with at least two branch points, v_1 and v_2, and A is an arc lying in G having end points v_1 and v_2. Since we are considering the case that no arc joining two branch points separates G, $G - A$ is connected and we may assume that A contains no branch point of G other than v_1 and v_2. There is an arc B lying in G such that $A \cap B = \{v_1, v_2\}$. Then $G_1 = A$ and $G_2 = \overline{G - A}$ are graphs whose union is G, and B is a subset of G_2. There exist four arcs, α_i and β_i for $i = 1, 2$, such that α_1 and β_1 are mutually exclusive arcs lying in A, and α_2 and β_2 are mutually exclusive arcs lying in B, and $\alpha_1 \cap \alpha_2 = \{v_1\}$, $\beta_1 \cap \beta_2 = \{v_2\}$. Further, α_i and β_i and $i = 1, 2$ may be chosen so that none of them contains a branch point of G that is not one of its end points.

By Lemma 4.1, there are mappings $\phi_i : \alpha_i \twoheadrightarrow G_i$ and $\psi_i : \beta_i \twoheadrightarrow G_i$ for $i = 1, 2$ such that each of the four mappings fixes both end points of its domain. Let $f : G \twoheadrightarrow G$ be given by $f(x) = x$ for x not in $\alpha_1 \cup \alpha_2 \cup \beta_1 \cup \beta_2$ and $f|\alpha_i = \phi_i$ and $f|\beta_i = \psi_i$ for $i = 1, 2$. Then $f[G_i] = G_i$ for $i = 1, 2$ and $\varprojlim\{G, f\} = \varprojlim\{G_1, f|G_1\} \cup \varprojlim\{G_2, f|G_2\}$. Again, it is easy to see that $\varprojlim\{G_1, f|G_1\}$ and $\varprojlim\{G_2, f|G_2\}$ are proper subcontinua of $\varprojlim\{G, f\}$ so this inverse limit is decomposable. Let $\alpha = \alpha_1 \cup \alpha_2$ and $\beta = \beta_1 \cup \beta_2$. Then α and β are non-overlapping arcs lying in G such that $f[\alpha] = f[\beta] = G$.

5. An Example

We end this article with an observation about our definition of "non-overlapping." In considering "two-pass" conditions on a map f of a tree G that imply that $\varprojlim\{G, f\}$ is indecomposable, one might consider defining two subgraphs to be non-overlapping by simply requiring that if the graphs intersect then their intersection is degenerate. Certainly this is sufficient for arcs. It is also sufficient for the simple triod because if two subgraphs of a simple triod intersect at a single point, then one of the graphs is an arc such that if it contains the branch point then the branch point is an end point of the arc. This is the condition in Theorem 3.3 that yields indecomposability of an inverse limit on the graph with a map that throws both of the subgraphs onto the graph. However, this condition is not sufficient even for a simple 4-od, as we see by the following example.

Example 5.1. *If G is a simple 4-od, there exist a map $f : G \twoheadrightarrow G$ and two arcs α and β lying in G such that $\alpha \cap \beta$ is degenerate, $f[\alpha] = f[\beta] = G$, and $\varprojlim\{G, f\}$ is decomposable.*

Denote the four arms of G by $\alpha_1, \alpha_2, \beta_1$, and β_2. Let $G_1 = \alpha_1 \cup \beta_1$ and $G_2 = \alpha_2 \cup \beta_2$. By Lemma 4.1, there are mappings $\phi_i : \alpha_i \twoheadrightarrow G_i$ and $\psi_i : \beta_i \twoheadrightarrow G_i$ for $i = 1, 2$ such that each of the four mappings fixes both end points of its domain. Let $f : G \to G$ be the map such that $f|\alpha_i = \phi_i$ and $f|\beta_i = \psi_i$ for $i = 1, 2$. Then $f[G_i] = G_i$ for $i = 1, 2$ and $\varprojlim\{G, f\} = \varprojlim\{G_1, f|G_1\} \cup \varprojlim\{G_2, f|G_2\}$. Since $\varprojlim\{G_1, f|G_1\}$ and $\varprojlim\{G_2, f|G_2\}$ are proper subcontinua of $\varprojlim\{G, f\}$, this inverse limit is decomposable. Let $\alpha = \alpha_1 \cup \alpha_2$ and $\beta = \beta_1 \cup \beta_2$. Then α and β are arcs lying in G such that $f[\alpha] = f[\beta] = G$ and the only point common to α and β is the branch point of G.

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