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WHEN ARE LOCAL CONNECTIVITY FUNCTIONS CONNECTIVITY?

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Abstract. Suppose X is a continuum, then X is Peano if and only if every real valued local connectivity function defined on X is a connectivity function if and only if every real valued local connectivity function defined on X is a Darboux function. This gives a partial answer to a question of J. Stallings. We also characterize the continua for which every local connectivity function has connected graph, answering a question of S. B. Nadler, Jr.

1. Introduction

In his 1959 paper, J. Stallings asked for conditions under which a local connectivity function $f: X \to Y$ was a connectivity function [6, #5, p. 262]. We give an answer to this question by characterizing continua for which every real valued local connectivity function is connectivity.

Theorem 1. *Let X be a continuum. The following are equivalent*

(a1) *Every real valued local connectivity function defined on X is connectivity.*
(b1) *Every real valued local connectivity function defined on X is Darboux.*
(c1) *X is Peano.*

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The results presented here are a direct result of a visit I made to West Virginia University in the summer of 2003. I would like to thank Georgia Southern University and West Virginia University for their support. I would also like thank Bruce McLean and Sam Nadler for stimulating discussions.

Until now, it was known only that (a1) is true for peripherally connected locally unicoherent spaces ([1] and [6]) and hereditarily arcwise connected spaces [5].

We also complete work begun by Sam B. Nadler, Jr. [5] on characterizing continua with the property that every real valued local connectivity function has connected graph. We prove the following characterization of these continua:

Theorem 2. *Let X be a continuum. The following conditions are equivalent.*

(a2) *Every real valued local connectivity function defined on X has connected range.*
(b2) *X is continuum chainable.*
(c2) *Every real valued local connectivity function defined on X has connected graph.*

For the interested reader, the continua for which every real valued function with connected graph is connectivity have been characterized by Nadler [4].

2. Preliminaries

A continuum is a nonempty compact connected metric space. A space is a Peano continuum provided that it is a locally connected continuum. Suppose X is a metric space with metric d. Given nonempty sets $A, B \subseteq X$ we define $\mathrm{d}(A, B) = \inf(\{d(x, y)\colon x \in A \;\&\; y \in B\})$. The diameter of a nonempty set $A \subseteq X$ is defined by $\mathrm{diam}(A) = \sup\{d(x, y)\colon x, y \in A\}$. For a set $A \subseteq X$ we write $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\mathrm{bd}(A)$ for the topological closure, interior, and boundary of A in X, respectively.

Given $\delta > 0$, we say a continuum is a δ-continuum provided that its diameter is less than δ. Given a space X, a point $p \in X$, and $\delta > 0$, we define the δ-component of $x \in X$ to be the set of all points x in X such that there is a finite collection of δ-continua $\mathcal{C}$ such that $x, p \in \bigcup\mathcal{C}$ and $\bigcup\mathcal{C}$ is connected. We say an indexed collection of sets $\mathcal{C} = \{C_1, \ldots, C_n\}$ is a δ-chain provided that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| < 2$. If $x \in C_1$ and $y \in C_n$, we say $\mathcal{C}$ is a chain from x to y. If $C \in \mathcal{C}$, we say C is a link of $\mathcal{C}$. We define the length of a chain to be the number of links it contains, i.e., its cardinality.

If $\mathcal{C}$ is a chain such that $\text{diam}(C) < \delta$ for every $C \in \mathcal{C}$, we say $\mathcal{C}$ is a δ-chain.

A continuum X such that for every $\delta > 0$ and $x, y \in X$ there is a δ-chain $\{C_1, \dots C_n\}$ of continua such that $x \in C_1$ and $y \in C_n$ is said to be continuum chainable [2].

Let X and Y be topological spaces. We say $f\colon X \to Y$ is Darboux provided that $f[C]$ is connected for every connected $C \subseteq X$. We say $f\colon X \to Y$ is connectivity provided that the graph of $f|C$ is connected for every connected $C \subseteq X$. If there is an open cover $\mathcal{U}$ of X such that $f|U$ is connectivity for every $U \in \mathcal{U}$, then we say that f is a local connectivity function. In general, we will identity a function with its graph and use the same symbol to describe both objects.

We denote the real line by $\mathbb{R}$.

3. Proof of Theorem 2

Lemma 3. *If $x, y \in X$ are in the same δ-component of a space X, then there is a δ-chain of continua from x to y.*

Proof: Since x and y are in the same δ-component, there is a finite collection $\mathcal{A}$ of δ-continua such that $x, y \in \bigcup \mathcal{A}$ and $\bigcup \mathcal{A}$ is connected.

By 8.13 of [3], we may index $\mathcal{A}$ to form a weak chain $\{A_1, \dots, A_n\}$ from x to y, i.e., $x \in A_1$, $y \in A_n$, and $A_i \cap A_{i+1} \neq \emptyset$ for each $1 \leq i \leq n-1$. Let $j_1 = \max\{i \colon x \in A_i\}$. If $y \in A_{j_1}$, then $\{A_{j_1}\}$ is the desired δ-chain. Otherwise, begin an inductive construction. Assume we have defined $\{j_1, \dots, j_k\}$ so that $j_1 < j_2 < \dots < j_k$, $y \notin A_{j_k}$, and $j_l = \max\{i > j_{l-1} \colon A_i \cap A_{j_{l-1}} \neq \emptyset\}$ for all $1 < l \leq k$. To see the inductive step, let $j_{k+1} = \max\{i > j_k \colon A_i \cap A_{j_k} \neq \emptyset\}$. Notice that the set we are taking the maximum over is nonempty since $j_k < n-1$ and $A_{j_k} \cap A_{j_k+1} \neq \emptyset$. Clearly, $j_k < j_{k+1}$. If $y \in A_{j_{k+1}}$ the construction stops. Since $\mathcal{A}$ is finite, we know there is an L such that $y \in A_{j_L}$

It is immediate from the construction that $\{A_{j_1}, \dots, A_{j_L}\}$ is a δ-chain of continua from x to y. □

Proof of Theorem 2: We show that (a2) implies (b2). Suppose X is not continuum chainable. Let $x, y \in X$ be such that there is no δ-chain from x to y. By Lemma 3, x and y are in different

δ-components of X. Let A be the δ-component of x. The characteristic function of A is easily checked to be local connectivity (just cover X with open $\delta/2$ balls), but the range of f is not connected since A is a proper subset of X.

We show that (b2) implies (c2). Suppose X is continuum chainable and that $f\colon X \to \mathbb{R}$ is local connectivity. Let $\mathcal{U}$ be a cover of X by open sets such that $f|U$ is connectivity for every $U \in \mathcal{U}$. Let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. Fix $p \in X$. For every $x \in X$ there is a δ-chain $\mathcal{C}_x$ of continua from p to x. Since $f|C$ is connected for each $C \in \mathcal{C}$, $f|\bigcup\mathcal{C}$ is connected. Finally, $f = \bigcup_{x\in X} \mathcal{C}_x$ is connected since $p \in \bigcap \mathcal{C}_x$ for all $x \in X$.

That (c2) implies (a2) is obvious. □

4. Proof of Theorem 1

We first note that (a1) implies (b1) is obvious from the definitions.

Lemma 4. *Suppose X is a continuum such that every real valued local connectivity function is Darboux. For every connected set $C \subseteq X$, $x \in \mathrm{cl}(C)$, and $\delta > 0$ there is a connected set E of diameter less than δ such that $E \cap C \neq \emptyset$ and $x \in E$.*

Proof: Suppose there is a connected set C, an $x \in \mathrm{cl}(C)$, and a $\delta > 0$ such that $C \cap E = \emptyset$ for every connected set E of diameter less than δ containing x. Let U be an open set of diameter less than δ with $x \in U$. Let W be an open set such that $x \in W \subseteq \mathrm{cl}(W) \subseteq U$. Let T be the component of x in U. Define a function f on X so that $f(x) = 1$, $f[\mathrm{bd}(W) \cap T] = \{0\}$, $f|(\mathrm{cl}(W) \cap T)$ is continuous, and $f[X \setminus (\mathrm{cl}(W) \cap T)] = \{0\}$. Since the restriction of f to any component of U is continuous and $f|(X \setminus \mathrm{cl}(W))$ is constant, f is a local connectivity function. Since $\mathrm{diam}(T) < \delta$ and $x \in T$ and T is connected, $T \cap C = \emptyset$. So, $C \cup \{x\}$ is connected and $f[C \cup \{x\}] = \{0, 1\}$. Thus, f is a local connectivity function which is not Darboux. □

Proof that (b1) implies (c1): By way of contradiction, assume that X is not Peano and condition (b1) holds. Let p be a point where X is not connected im kleinen. Let U be an open set such that $p \in U$ and no neighborhood of p contained in U is connected. Let K be the component of p in U. Let $W \subseteq U$ be a compact neighborhood of p. Since X is not connected im kleinen at p, there is a sequence of points $\{p_n\}_{n\in\omega}$ in $\mathrm{int}(W) \setminus K$ converging to p. Let L_n denote the component of W that contains p_n. Since $p_n \notin K$ for every n, we may assume that $L_n \cap L_k \neq \emptyset$ if and only if $n = k$. Clearly, $L_n \cap K = \emptyset$ for all n. Since each L_n must bump the boundary of W, we may assume there is a $\epsilon > 0$ such that $\mathrm{diam}(L_n) > \epsilon$ for all n. Moreover, we may assume that the sequence $\{L_n\}_{n\in\omega}$ converges to some continuum L. Clearly, $L \subseteq W$, $p \in L$, and $\mathrm{diam}(L) \geq \epsilon$.

Let $\delta < \epsilon/8$. For each $n \in \omega$ we may find, by Theorem 2, a δ-chain $\mathcal{C}_n$ of continua from p_n to p. Moreover, we assume $\mathcal{C}_n$ has minimal length among all such chains. For $A \subseteq \omega$ we let $S_A = \bigcup_{n\in A}(\bigcup \mathcal{C}_n)$. Notice that S_A is connected.

We claim that $L \subseteq \mathrm{cl}(S_A)$ for all infinite $A \subseteq \omega$. Let $x \in L$ and $\gamma > 0$. Clearly, $x \in \mathrm{cl}(S_A \bigcup(\bigcup_{n\in A} L_n))$, and $S_A \bigcup(\bigcup_{n\in A} L_n)$ is connected. By Lemma 4, there is a connected set T of diameter strictly less than $\min\{\gamma, \mathrm{d}(x, X \setminus W)\}$ such that $x \in T$ and $T \cap (S_A \bigcup(\bigcup_{n\in A} L_n)) \neq \emptyset$. Notice that $T \cap \bigcup_{n\in A} L_n = \emptyset$; otherwise, p_n and p would be in the same component of W for some n. Thus, $T \cap S_A \neq \emptyset$. So, $\mathrm{d}(x, S_A) < \gamma$. Since γ was arbitrary, $x \in \mathrm{cl}(S_A)$.

We claim that $q \in S_A$ for every infinite $A \subseteq \omega$ and $q \in L$ such that $\mathrm{d}(p, q) > \epsilon/2$. By the previous claim, we have $q \in \mathrm{cl}(S_A)$. Let $\{\gamma_n\}_{n\in\omega}$ be a sequence of real numbers such that $\lim \gamma_n = 0$ and $0 < \gamma_n < \epsilon/8$. By Lemma 4, there is for every γ_n a connected set of T_n of diameter γ_n such that $q \in T_n$ and $T_n \cap S_A \neq \emptyset$. Let $T = \bigcup_{n\in\omega} T_n$. Let $B \subseteq A$ be the set of all $k \in A$ such that $T \cap \bigcup \mathcal{C}_k \neq \emptyset$.

Assume B is infinite. For each $k \in B$ let $\mathcal{C}_k^*$ be the shortest subchain of $\mathcal{C}_k$ from p_k to any point in T. Since $\mathrm{d}(p, T) > 3\epsilon/8$ and the last link of $\mathcal{C}_k^*$ has diameter less than $\epsilon/8$, $\mathcal{C}_k^*$ is at least two links shorter than $\mathcal{C}_k$. Let $Q = T \cup (\bigcup(\bigcup_{k\in B} \mathcal{C}_k^*))$. Notice Q is connected. Since $p \in \mathrm{cl}(Q)$, there is a connected set R such that $p \in R$, $R \cap Q \neq \emptyset$, and $\mathrm{diam}(R) < \epsilon/8$. Since $\mathrm{d}(p, T) > 3\epsilon/8$, it follows that for some $k \in B$, $R \cap (\bigcup \mathcal{C}_k^*) \neq \emptyset$. Now $p_k, p \in \bigcup(\mathcal{C}_k^* \cup \{R\})$ and $\bigcup(\mathcal{C}_k^* \cup \{R\})$ is connected. Arguing as in Lemma 3, $\mathcal{C}_k^* \cup \{R\}$ contains a subchain $\mathcal{D}$ from p_k to p. However, the length of $\mathcal{D}$ must be at least one less than the length of $\mathcal{C}_k$, contradicting minimality.

We may now assume B is finite. Since $T_n \cap S_A \neq \emptyset$ for all n, there is a $k \in B$ such that $T_n \cap \bigcup \mathcal{C}_k \neq \emptyset$ for infinitely many $n \in \omega$.

Since $\lim \operatorname{diam}(T_n) = 0$, $q \in \operatorname{cl}(\bigcup \mathcal{C}_k)$. However, $\bigcup \mathcal{C}_k$ is closed. So, $q \in \bigcup \mathcal{C}_k \subseteq S_A$, establishing the claim.

Let $q \in L$ and $\mathrm{d}(p, q) > \epsilon/2$. By the preceding claim, we may find an infinite $A \subseteq \omega$ such that $q \in \bigcup \mathcal{C}_k$ for every $k \in A$. For each $k \in A$ let $\mathcal{C}_k^*$ be the shortest subchain of $\mathcal{C}_k$ from p_k to q. Since $\mathrm{d}(p, q) > \epsilon/2$ and the last link of $\mathcal{C}_k^*$ has diameter less than $\epsilon/8$, $\mathcal{C}_k^*$ is at least two links shorter than $\mathcal{C}_k$. Let $Q = \bigcup_{k \in A} \bigcup \mathcal{C}_k^*$. Since $p \in \operatorname{cl}(Q)$ and Q is connected, there is a connected set R such that $p \in R$, $R \cap Q \neq \emptyset$, and $\operatorname{diam}(R) < \epsilon/8$. It follows that for some $k \in A$, $p, p_k \in \bigcup(\mathcal{C}_k^* \cup \{R\})$ and $\bigcup(\mathcal{C}_k^* \cup \{R\})$ is connected. By Lemma 3, $\mathcal{C}_k^* \cup \{R\}$ contains a subchain $\mathcal{D}$ from p_k to p. However, the length of $\mathcal{D}$ would be at least one less than the length of $\mathcal{C}_k$, contradicting minimality. □

Proof that (c1) implies (a1): Let X be Peano and $f \colon X \to \mathbb{R}$ be local connectivity. By way of contradiction, assume $C \subseteq X$ is connected and $f|C$ is not connected. Let $W, V \subseteq X \times \mathbb{R}$ be open sets such that $f|C \subseteq V \cup W$, $V \cap f|C \neq \emptyset$, $W \cap f|C \neq \emptyset$, and $V \cap W \cap f|C = \emptyset$.

By way of contradiction, assume that for every $x \in C$ there is an open $O_x \subseteq X$ such that $f|(O_x \cap C) \subseteq V$ or $f|(O_x \cap C) \subseteq W$. Now $\{x \colon f|(O_x \cap C) \subseteq W\}$ and $\{x \colon f|(O_x \cap C) \subseteq V\}$ is a partition of C into two nonempty disjoint open sets contradicting that C is connected. So there is a $p \in C$ such that for every neighborhood N of p, we have $f|(N \cap C) \cap V \neq \emptyset$ and $f|(N \cap C) \cap W \neq \emptyset$. Assume that $(p, f(p)) \in W$.

Let U_1 be an open neighborhood of p such that $\operatorname{cl}(U_1)$ is a Peano continuum and $f|\operatorname{cl}(U_1)$ is connectivity. Since $\operatorname{cl}(U_1)$ is Peano, $\operatorname{cl}(U_1)$ is semi-locally-connected by [3, 8.44(d)]. So, we can find an open U_2 such that $p \in U_2 \subseteq \operatorname{cl}(U_2) \subseteq U_1$ such that $\operatorname{cl}(U_1) \setminus U_2$ has finitely many components $E_1, \ldots, E_n$.

Let $L = (C \cap U_2) \cup (\operatorname{cl}(U_1) \setminus U_2)$. Suppose L_x is a quasicomponent of x in L. By way of contradiction, assume that $L_x \cap (\operatorname{cl}(U_1) \setminus U_2) = \emptyset$. For every $w \in \operatorname{cl}(U_1) \setminus U_2$ there exist L-open sets S_w and T_w such that $L = S_w \cup T_w$, $S_w \cap T_w = \emptyset$, and $w \in S_w$ and $x \in T_x$. Since $\operatorname{cl}(U_1) \setminus U_2$ is compact, there are $w_1, \ldots w_k \in \operatorname{cl}(U_1) \setminus U_2$ such that $\operatorname{cl}(U_1) \setminus U_2 \subseteq \bigcup_{i=1}^k S_{w_i}$, $x \in \bigcap_{i=1}^k T_{w_i}$, $(\bigcup_{i=1}^k S_{w_i}) \cap (\bigcap_{i=1}^k T_{w_i}) = \emptyset$, and $(\bigcup_{i=1}^k S_{w_i}) \cup (\bigcap_{i=1}^k T_{w_i}) = L$. Now $(C \setminus \operatorname{cl}(U_1)) \cup (C \cap \bigcup_{i=1}^k S_{w_i})$ and $C \cap \bigcap_{i=1}^k T_{w_i}$ form a partition of C into two disjoint C-open sets, contradicting that C is connected. Thus, $L_x \cap (\mathrm{cl}(U_1) \setminus U_2) \neq \emptyset$ for every $x \in L$.

Since each quasicomponent of L has nonempty intersection with $\mathrm{cl}(U_1) \setminus U_2$ and $\mathrm{cl}(U_1) \setminus U_2$ has finitely many components, we conclude that L has only finitely many quasicomponents. It follows that L has finitely many components. Let D be the component of p in L. Notice that $D \setminus U_2$ has finitely many components and is compact.

Let $M = (D \setminus U_2) \cup (D \cap \pi_X[f \cap V])$. Suppose M_x is a quasicomponent of x in M. By way of contradiction, assume that $M_x \cap (D \setminus U_2) = \emptyset$. For every $w \in D \setminus U_2$ there exist M-open sets S_w and T_w such that $M = S_w \cup T_w$, $S_w \cap T_w = \emptyset$, and $w \in S_w$ and $x \in T_w$. Since $D \setminus U_2$ is compact, there are $w_1, \ldots w_k \in D \setminus U_2$ such that $D \setminus U_2 \subseteq \bigcup_{i=1}^k S_{w_i}$, $x \in \bigcap_{i=1}^k T_{w_i}$, $(\bigcup_{i=1}^k S_{w_i}) \cap (\bigcap_{i=1}^k T_{w_i}) = \emptyset$, and $(\bigcup_{i=1}^k S_{w_i}) \cup (\bigcap_{i=1}^k T_{w_i}) = M$. Now $(D \setminus U_2) \cup (\bigcup_{i=1}^k S_{w_i})$ and $\bigcap_{i=1}^k T_{w_i}$ form a partition of M into two disjoint M-open sets. Let $A = \bigcap_{i=1}^k T_{w_i}$. Notice that $\mathrm{cl}_D(A) \subseteq U_2$ since $D \setminus U_2 \subseteq \bigcup_{i=1}^k S_{w_i}$. In particular, $\mathrm{cl}_D(A) \subseteq C$. We get a contradiction by showing that $f|A$ is closed and open in $f|D$.

We first show that $f|A$ is closed in $f|D$. Let $\{(x_n, f(x_n))\}_{n\in\omega}$ be a sequence of points in $f|A$ and $(x, f(x)) \in f|D$ be such that $\lim(x_n, f(x_n)) = (x, f(x))$. Since $A \subseteq U_2$, $A \subseteq \pi_X[f \cap V]$. So, $(x_n, f(x_n)) \in V$ for every $n \in \omega$. Since $\mathrm{cl}_D(A) \subseteq C$ and $f \cap V$ is closed in $f|C$, we have $(x, f(x)) \in V$. Now $x \in \mathrm{cl}_D(A) \cap \pi_X[f \cap V] \subseteq \pi_X[f \cap V] \cap D \subseteq M$. So, $x \in \mathrm{cl}_M(A)$. Since A is M-closed, we have $x \in A$. So, $(x, f(x)) \in f|A$. Thus, $f|A$ is closed.

We now show that $f|A$ open in $f|D$. Suppose $x \in A$ and $\{z_n\}_{n\in\omega}$ is a sequence of points in $f|D$ such that $\lim(z_n, f(z_n)) = (x, f(x))$. Since $A \subseteq U_2$, we may assume that $z_n \in C$ for all $n \in \omega$ and that $x \in \pi_X[f \cap V]$. Since $x \in \pi_X[f \cap V]$ and $f \cap V$ is open in $f|C$, $(z_n, f(z_n)) \in V$ for almost all $n \in \omega$. Now all but finitely many $z_n \in \pi_X[f \cap V] \cap U_2 \cap D \subseteq M$. Since A is open in M, we must have $z_n \in A$ for almost all $n \in \omega$. Thus, $f|A$ is open in $f|D$.

Thus, $M_x \cap (D \setminus U_2) \neq \emptyset$ for every $x \in D$.

Since each quasicomponent of M has nonempty intersection with $D \setminus U_2$ and $D \setminus U_2$ has finitely many components, we conclude that M has only finitely many quasicomponents. It follows that M has finitely many components.

Since every neighborhood of p has nonempty intersection with $\pi_X[f \cap V]$, there is a component E of M such that $p \in \mathrm{cl}(E)$. Now $E \cup \{p\}$ is connected, $f|(E \cap U_2) \subseteq V$, and $(p, f(p)) \in W$. Thus, $f|(E \cup \{p\})$ is not connected. However, $E \subseteq U_1$ so $f|(E \cup \{p\})$ is connected, a contradiction. □

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